Pacific Journal of Mathematics

GENERA IN NORMAL EXTENSIONS

Robert Gold

Vol. 63, No. 2 April 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 63, No. 2, 1976

GENERA IN NORMAL EXTENSIONS

ROBERT GOLD

Let K/F be a finite normal extension of algebraic number fields and let C_K be the ideal class group of K. There are two fundamentally different ways to define the principal genus of C_K with respect to F. Classically the principal genus is described by norm residue symbols. By the modern definition it is the class group of the maximal unramified extension of K which is the composite of K with an abelian extension of F. It is shown here that the two definitions are equivalent.

Let F be a finite algebraic number field and K a finite normal extension of F with $G = \mathrm{Gal}(K/F)$. Let $\bar{K}$ be the Hilbert class field of K and let C_K be the ideal class group of K. By class field theory the fields lying between K and $\bar{K}$ are in one-one correspondence with the subgroups of C_K. (See [3] or [4] for the class field theory involved.) Let L be the genus field for K/F. As defined by Fröhlich ([1]), L is the composite of K with the maximal abelian extension of F in $\bar{K}$. Calling this maximal abelian extension E, we have $\bar{K} \supseteqq L = KE \supseteqq K$, $E \supseteqq F$ and $K \cap E$ is the maximal abelian extension of F in K. The subgroup of C_K corresponding to L is the principal genus of C_K. Gauss's definition of the principal genus is based on arithmetic characters. In [2] we showed that when G is abelian the two definitions are equivalent. Here we will show that in fact they are equivalent for any G.

Let C_F be the ideal class group of F and let $N_{K/F}\colon C_K \to C_F$ be the norm map on ideal class groups. Let $\bar{F}$ be the Hilbert class field of F and ${}_NC_K$ the kernel of the norm map. Then the subgroup ${}_NC_K$ of C_K corresponds to the extension $K\bar{F}$ of K. Clearly $L \supseteqq K\bar{F}$ and, letting H denote the principal genus of C_K, we see that ${}_NC_K \supseteqq H$.

We now proceed to describe the characters in Gauss's definition. Let $P_1, \cdots, P_t$ be the primes of K, finite or infinite, ramified in K/F. For each i choose a prime $\bar{P}_i$ in $\bar{K}$ such that $\bar{P}_i \cap K = P_i$. This allows a consistent choice of primes in each subfield k by $P_{k,i} = \bar{P}_i \cap k$. And we will denote the completed localization of k at $P_{k,i}$ by k_i. In particular we have the chain $\bar{K}_i \supseteqq L_i \supseteqq K_i$, $E_i \supseteqq F_i$ of local fields. For an ideal $\mathfrak{A}$ of a field k let $[\mathfrak{A}]$ denote the ideal class of $\mathfrak{A}$. Now let $\mathfrak{A}$ be an ideal of K such that $[\mathfrak{A}] \in {}_NC_K$. Thus $N_{K/F}(\mathfrak{A})$ is a principal ideal of F, say $N_{K/F}(\mathfrak{A}) = (a)$, $a \in F$. For each i we have a norm residue symbol $((K_i/F_i)/a)$ which we will also write $((a, K/F)/P_i)$ or most simply $\chi_i(a)$. This symbol is an element of the local group $\mathrm{Gal}(K_i/F_i)$ modulo its commutator. We may identify $\mathrm{Gal}(K_i/F_i)$ with the decomposition group Z_i of P_i in K/F. Thus we have a homomorphism $\chi_i\colon F^\times \to Z_i/[Z_i, Z_i] = Z_i^{ab}$. Let $\chi\colon F^\times \to \prod_{i=1}^t Z_i^{ab}$ by $\chi(a) = (\chi_1(a), \cdots, \chi_t(a))$. Let U_F denote the units of F, P_F the principal ideals of F, and $S = \chi(U_F)$. Then χ induces a homomorphism which we'll also denote by $\chi\colon P_F = F^\times/U_F \to \prod_{i=1}^t Z_t^{ab}/S$. Let $(a) \in P_F$ and $(a) = (N_{K/F}(b)) = N_{K/F}((b))$. Then $a = \varepsilon\cdot N_{K/F}(b)$ for some $\varepsilon \in U_F$ and $\chi_i(a) = \chi_i(\varepsilon)\chi_i(N_{K/F}(b)) = \chi_i(\varepsilon)$ for $i = 1, \cdots, t$ since $\chi_i(N_{K/F}(b)) = ((K_i/F_i)/N_{K/F}(b))$ and every global norm is a local norm everywhere. It follows that $\chi\colon P_F \to \prod Z_i^{ab}/S$ vanishes on $N_{K/F}(P_K) \subseteqq P_F$. Note that $N_{K/F}\colon {}_N C_K \to P_F/N(P_K)$ since if $[\mathfrak{A}] = [\mathfrak{B}] \in {}_N C_K$ then $\mathfrak{A} = (\alpha)\mathfrak{B}$ and $N(\mathfrak{A}) = N((\alpha))N(\mathfrak{B}) \in P_F$. Now we can define $f = \chi \circ N_{K/F}\colon {}_N C_K \to P_F/N(P_K) \to \prod Z_i^{ab}/S$. The formal statement of the equivalence of the two definitions of principal genus is given by

THEOREM. *Let K/F be a finite normal extension of number fields and let H be the principal genus in the sense of Fröhlich. Let $f\colon {}_N C_K \to \prod_{i=1}^t Z_i^{ab}/S$ be the modified product of local norm residue symbols described above. Then* $H = \mathrm{Ker}(f)$.

Proof. First we show that $\mathrm{Ker}(f) \subseteqq H$. Let P be a prime ideal of K, $P \neq P_i$, $i = 1, \cdots, t$; $[P] \in \mathrm{Ker}(f)$, and P of absolute degree 1. Since $[P] \in \mathrm{Ker}(f)$ and P is of degree 1, $N_{K/F}(P) = \mathfrak{p} = (\rho)$ where $\mathfrak{p} = P \cap F$ and $\rho \in F$. Moreover ρ may be chosen so that $\chi_i(\rho) = 1$, $i = 1, \cdots t$, since ρ times any unit of F generates $\mathfrak{p}$ and $[P] \in \mathrm{Ker}(f)$ implies $\chi(\rho) = \chi(\varepsilon)$ for some $\varepsilon \in U_F$.

Let M_i be the maximal abelian extension of F_i in L_i. So $K_i \cap M_i$ is the maximal abelian extension of F_i in K_i. Then

$$\left(\frac{M_i/F_i}{\rho}\right)\bigg|_{M_i \cap K_i} = \left(\frac{M_i \cap K_i | F_i}{\rho}\right) = \left(\frac{K_i | F_i}{\rho}\right) = \chi_i(\rho) = 1\ .$$

The second equality here follows from the fact that $N_{K_i/F_i}(K_i) = N_{K_i \cap M_i/F_i}(K_i \cap M_i)$. Therefore

$$\left(\frac{M_i/F_i}{\rho}\right) \in \mathrm{Gal}\,(M_i/M_i \cap K_i) \subseteqq \mathrm{Gal}\,(M_i/F_i)\ .$$

Since $P \neq P_i$, any i, ρ is a P_i-unit for each i. Thus $((M_i/F_i)/\rho) \in T(M_i/F_i) \subseteqq \mathrm{Gal}\,(M_i/F_i)$ where T is the inertia group of the local extension. So we have

$$\left(\frac{M/_iF_i}{\rho}\right) \in T(M_i/F_i) \cap \mathrm{Gal}\,(M_i/M_i \cap K_i) = T(M/_iM_i \cap K_i)$$

LEMMA. *L_i/F_i be a normal extension of local fields and M_i the* *maximal abelian extension of* F_i *in* L_i. *If* $L_i \supseteqq K_i \supseteqq F_i$ *and* L_i/K_i *is unramified, then* $M_i/M_i \cap K_i$ *is unramified.*

The lemma, to be proved below, implies that $T(M_i/M_i \cap K_i) = \{1\}$ and therefore $((M_i/F_i)/\rho) = 1$. Since $M_i \supseteqq E_i$ it follows that $((E_i/F_i)/\rho) = 1 = ((\rho, E/F)/P_i)$ for all i. So we have E/F abelian, $\rho \in F$, and $((\rho, E/F)/\mathfrak{p}_i) = 1$ for $\mathfrak{p}_i = F \cap P_i$, $i = 1, \cdots, t$. Since $\bar{K} \supseteqq E$, the $\{\mathfrak{p}_i\}$ includes all primes of F ramified in E/F. For every unramified prime of F at which ρ is a unit the norm residue symbol is 1. The only undetermined symbol is $((\rho, E/F)/\mathfrak{p})$. By the product formula for norm residue symbols, the product of all symbols is 1. Hence we must have $((\rho, E/F)/\mathfrak{p}) = 1$. Recall that $(\rho) = \mathfrak{p}$, i.e. ρ is a prime element at $\mathfrak{p}$, and $\mathfrak{p}$ is unramified in E/F. Hence $((\rho, E/F)/\mathfrak{p})$ generates the decomposition group of $\mathfrak{p}$ in E/F. We conclude that $\mathfrak{p}$ is completely decomposed in E/F. It follows by standard arguments that P is completely decomposed in L/K since $L = KE$. The subgroup of C_K corresponding to a subfield k of $\bar{K}$ can be characterized as the classes of all prime ideals of K which are completely decomposed in k/K. Thus $[P] \in H$ since H corresponds to L.

Now we show that $\operatorname{Ker}(f) \supseteqq H$. Let P be a prime of K of absolute degree 1 with $[P] \in H$. Let $N_{K/F}(P) = \mathfrak{p} = (\rho)$, $\rho \in F$ and as above let P_i, $i = 1, \cdots, t$ be the primes of K ramified in K/F. We may assume also $P \neq P_i$ for any i. Since $[P] \in H$, P is completely decomposed in L/K. Say, $P = Q_1 \cdots Q_g$ so that $N_{L/F}(Q_1) = (\rho)$. Let $\mathfrak{m}$ be a divisor of F divisible by high powers of all P_i and prime to P. Since E is the maximal abelian extension of F in L and in $\bar{K}$ the norm limitation theorem implies that

$$(*) \quad N_{E/F}(I_{\mathfrak{m}}(E)) \cdot S_{\mathfrak{m}}(F) = N_{L/F}(I_{\mathfrak{m}}(L)) \cdot S_{\mathfrak{m}}(F) = N_{\bar{K}/F}(I_{\mathfrak{m}}(\bar{K})) \cdot S_{\mathfrak{m}}(F)$$

where $I_{\mathfrak{m}}(k)$ is the group of ideals of k relatively prime to $\mathfrak{m}$ and $S_{\mathfrak{m}}(k)$ is the ideal ray (Strahl) mod $\mathfrak{m}$.

We have noted that $(\rho) = N_{L/F}(Q)$ with $Q_i \in I_{\mathfrak{m}}(L)$. It follows from (*) that we can write $(\rho) = N_{\bar{K}/F}(\mathfrak{A}) \cdot (\alpha)$ where $\mathfrak{A} \in I_{\mathfrak{m}}(\bar{K})$ and $(\alpha) \in S_{\mathfrak{m}}(F)$. The norm from $\bar{K}$ to K of any ideal of $\bar{K}$ is a principal ideal of K. Let $N_{\bar{K}/K}(\mathfrak{A}) = (a)$, $a \in K$. So $(\rho) = (\alpha) N_{\bar{K}/F}(\mathfrak{A}) = (\alpha) N_{K/K}(N_{E/K}(\mathfrak{A})) = (\alpha)(N_{K/F}(a))$ or $\varepsilon\rho = \alpha \cdot N_{K/F}(a)$ for some unit $\varepsilon \in U_F$. Therefore

$$\left(\frac{\varepsilon\rho,\ K/F}{P_i}\right) = \left(\frac{\alpha,\ K/F}{P_i}\right) \cdot \left(\frac{N_{K/F}(a),\ K/F}{P_i}\right).$$

Since a global norm is certainly a local norm $((N_{K/F}(a), K/F)/P_i) = 1$. Also since $\alpha \in F$, $\alpha \equiv 1(\mathfrak{m})$ and $\mathfrak{m}$ is divisible by high powers of the

P_i we see that $((M_i \cap K_i/F_i)/\alpha) = 1$. And therefore

$$\left(\frac{K_i/F_i}{\alpha}\right) = \left(\frac{\alpha,\ K/F}{P_i}\right) = 1 .$$

Thus $((\varepsilon\rho,\ K/F)/P_i) = 1$ for all i. In other words $\chi(\rho) = \chi(\varepsilon^{-1})$, which gives $[P] \in \operatorname{Ker}(f)$.

Proof of the lemma. Let $T(L/F)$ be the inertia subgroup of $\operatorname{Gal}(L/F)$. The quotient $\operatorname{Gal}(L/F)/T(L/F)$ is a cyclic group, hence $T(L/F)$ contains the commutator subgroup of $\operatorname{Gal}(L/F)$, which is $\operatorname{Gal}(L/M)$. Thus L/M is totally ramified. Letting e denote the ramification index, we have $e(L/K \cap M) \geqq [L\colon M] \geqq [K\colon K \cap M]$. This last inequality follows from the fact that $L \supseteqq KM$ and, since $M/K \cap M$ is galois, $[KM\colon M] = [K\colon K \cap M]$. Since L/K is unramified, $e(L/K \cap M) \leqq [K\colon K \cap M]$. Therefore $e(L/K \cap M) = [K\colon K \cap M] = [L\colon M] = e(L/M)$ and so $e(M/K \cap M) = 1$.

REFERENCES

1. A. Fröhlich, *The genus field and genus group in finite number fields, I, II*, Mathematika, **6** (1959), 40–46, 142–146.
2. R. Gold, *Genera in abelian extensions*, Proc. Amer. Math. Soc., **47** (1975), 25–28.
3. H. Hasse, *Vorlesungen über Klassenkörpertheorie*, Physica-Verlag, Würzburg, 1967.
4. G. Janusz, *Algebraic Number Fields*, Academic Press, New York, 1973.

Received July 17, 1975.

OHIO STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 63, No. 2 April, 1976