GENERA IN NORMAL EXTENSIONS

ROBERT GOLD
Let $K/F$ be a finite normal extension of algebraic number fields and let $C_K$ be the ideal class group of $K$. There are two fundamentally different ways to define the principal genus of $C_K$ with respect to $F$. Classically the principal genus is described by norm residue symbols. By the modern definition it is the class group of the maximal unramified extension of $K$ which is the composite of $K$ with an abelian extension of $F$. It is shown here that the two definitions are equivalent.

Let $F$ be a finite algebraic number field and $K$ a finite normal extension of $F$ with $G = \text{Gal}(K/F)$. Let $\mathcal{H}$ be the Hilbert class field of $K$ and let $C_K$ be the ideal class group of $K$. By class field theory the fields lying between $K$ and $\mathcal{H}$ are in one-one correspondence with the subgroups of $C_K$. (See [3] or [4] for the class field theory involved.) Let $L$ be the genus field for $K/F$. As defined by Fröhlich ([1]), $L$ is the composite of $K$ with the maximal abelian extension of $F$ in $K$. Calling this maximal abelian extension $E$, we have $\mathcal{H} \supseteq L = KE \supseteq K$, $E \supseteq F$ and $K \cap E$ is the maximal abelian extension of $F$ in $K$. The subgroup of $C_K$ corresponding to $L$ is the principal genus of $G_K$. Gauss’s definition of the principal genus is based on arithmetic characters. In [2] we showed that when $G$ is abelian the two definitions are equivalent. Here we will show that in fact they are equivalent for any $G$.

Let $C_F$ be the ideal class group of $F$ and let $N_{K/F}$ be the norm map on ideal class groups. Let $\mathcal{H}$ be the Hilbert class field of $F$ and $\mathcal{N}C_K$ the kernel of the norm map. Then the subgroup $\mathcal{N}C_K$ of $C_K$ corresponds to the extension $KF$ of $K$. Clearly $L \supseteq K\mathcal{H}$ and, letting $H$ denote the principal genus of $C_K$, we see that $\mathcal{N}C_K \supseteq H$.

We now proceed to describe the characters in Gauss’s definition. Let $P_1, \ldots, P_t$ be the primes of $K$, finite or infinite, ramified in $K/F$. For each $i$ choose a prime $\mathfrak{p}_i$ in $\mathcal{H}$ such that $\mathfrak{p}_i \cap K = P_i$. This allows a consistent choice of primes in each subfield $k$ by $P_{k,i} = \mathfrak{p}_i \cap k$. And we will denote the completed localization of $k$ at $P_{k,i}$ by $k_i$. In particular we have the chain $\mathcal{H}_i \supseteq \mathcal{L}_i \supseteq K_i, E_i \supseteq F_i$ of local fields. For an ideal $\mathfrak{a}$ of a field $k$ let $[\mathfrak{a}]$ denote the ideal class of $\mathfrak{a}$. Now let $\mathfrak{a}$ be an ideal of $K$ such that $[\mathfrak{a}] \in \mathcal{N}C_K$. Thus $N_{K/F}(\mathfrak{a})$ is a principal ideal of $F$, say $N_{K/F}(\mathfrak{a}) = (a)$, $a \in F$. For each $i$ we have a norm residue symbol $((K_i/F_i)/a)$ which we will also write $((a, K/F)/P_i)$ or most simply $\mathcal{L}_i(a)$. This symbol is an element
of the local group $\text{Gal}(K/F_t)$ modulo its commutator. We may identify $\text{Gal}(K/F_t)$ with the decomposition group $Z_i$ of $F_t$ in $K/F$. Thus we have a homomorphism $\chi : F^\times \to \prod_{i=1}^t Z_i^{ab}$ by $\chi(a) = (\chi_i(a), \ldots, \chi_t(a))$. Let $U_F$ denote the units of $F$, $P_F$ the principal ideals of $F$, and $S = \chi(U_F)$. Then $\chi$ induces a homomorphism which we'll also denote by $\chi : P_F \to \prod_{i=1}^t Z_i^{ab}/S$. Let $(a) \in P_F$ and $(a) = (N_{K/F}(b)) = N_{K/F}(b))$. Then $a = \varepsilon \cdot N_{K/F}(b)$ for some $\varepsilon \in U_F$ and $\chi_i(a) = \chi_i(\varepsilon)\chi_i(N_{K/F}(b)) = \chi_i(\varepsilon)$ for $i = 1, \ldots, t$ since $\chi_i(N_{K/F}(b)) = (K_i/F_i)(N_{K/F}(b))$ and every global norm is a local norm everywhere. It follows that $\chi : P_F \to \prod Z_i^{ab}/S$ vanishes on $N_{K/F}(P_K) \subseteq P_F$. Note that $N_{K/F} : \mathcal{N}_K \to P_F/N(P_K)$ since if $[\mathfrak{A}] = [\mathfrak{B}] \in \mathcal{N}_K$ then $\mathfrak{A} = (\alpha)\mathfrak{B}$ and $N(\mathfrak{A}) = N((\alpha))N(\mathfrak{B}) \in P_F$. Now we can define $f = \chi \circ N_{K/F} : C_K \to P_F/N(P_K) = \prod_{i=1}^t Z_i^{ab}/S$. The formal statement of the equivalence of the two definitions of principal genus is given by

**Theorem.** Let $K/F$ be a finite normal extension of number fields and let $H$ be the principal genus in the sense of Fröhlich. Let $f : \mathcal{N}_K \to \prod_{i=1}^t Z_i^{ab}/S$ be the modified product of local norm residue symbols described above. Then $H = \text{Ker}(f)$.

**Proof.** First we show that $\text{Ker}(f) \subseteq H$. Let $P$ be a prime ideal of $K$, $P \neq P_i$, $i = 1, \ldots, t$; $[P] \in \text{Ker}(f)$, and $P$ of absolute degree 1. Since $[P] \in \text{Ker}(f)$ and $P$ is of degree 1, $N_{K/F}(P) = \mathfrak{p} = (\rho)$ where $\mathfrak{p} = P \cap F$ and $\rho \in F$. Moreover $\rho$ may be chosen so that $\chi_i(\rho) = 1$, $i = 1, \ldots, t$, since $\rho$ times any unit of $F$ generates $\mathfrak{p}$ and $[P] \in \text{Ker}(f)$ implies $\chi(\rho) = \chi(\varepsilon)$ for some $\varepsilon \in U_F$.

Let $M_i$ be the maximal abelian extension of $F_i$ in $L_i$. So $K_i \cap M_i$ is the maximal abelian extension of $F_i$ in $K_i$. Then

$$
\left( \frac{M_i/F_i}{\rho} \right)_{M_i/K_i} = \left( \frac{M_i \cap K_i/F_i}{\rho} \right) = \frac{K_i/F_i}{\rho} = \chi_i(\rho) = 1.
$$

The second equality here follows from the fact that $N_{K_i/F_i}(K_i) = N_{K_i/K_i}(K_i \cap M_i)$. Therefore

$$
\left( \frac{M_i/F_i}{\rho} \right) \in \text{Gal}(M_i/M_i \cap K_i) \subseteq \text{Gal}(M_i/F_i).
$$

Since $P \neq P_i$, any $i, \rho$ is a $P_i$-unit for each $i$. Thus $((M_i/F_i)/\rho) \in T(M_i/F_i) \subseteq \text{Gal}(M_i/F_i)$ where $T$ is the inertia group of the local extension. So we have

$$
\left( \frac{M_i/F_i}{\rho} \right) \in T(M_i/F_i) \cap \text{Gal}(M_i/M_i \cap K_i) = T(M_i/M_i \cap K_i).
$$

**Lemma.** $L_i/F_i$ be a normal extension of local fields and $M_i$ the
The lemma, to be proved below, implies that \(T(M_i/M_i \cap K_i) = \{1\}\) and therefore \(T(M_i/F_i) = 1\). Since \(M_i \supseteq E_i\), it follows that \(T(E_i/F_i, \rho) = 1 = T(\rho, E/F, \rho_i)\) for all \(i\). So we have \(E/F\) abelian, \(\rho \in F\), and \(T(\rho, E/F, \rho_i) = 1\) for \(\rho = F \cap P_i, i = 1, \ldots, t\). Since \(K \supseteq E\), the \(\{\rho_i\}\) includes all primes of \(F\) ramified in \(E/F\). For every unramified prime of \(F\) at which \(\rho\) is a unit the norm residue symbol is 1. The only undetermined symbol is \(T(\rho, E/F, \rho)\). By the product formula for norm residue symbols, the product of all symbols is 1. Hence we must have \(T(\rho, E/F, \rho) = 1\). Recall that \(T(\rho) = \rho\), i.e. \(\rho\) is a prime element at \(\rho\), and \(\rho\) is unramified in \(E/F\). Hence \(T(\rho, E/F, \rho)\) generates the decomposition group of \(\rho\) in \(E/F\). We conclude that \(\rho\) is completely decomposed in \(E/F\). It follows by standard arguments that \(P\) is completely decomposed in \(L/K\) since \(L = KE\). The subgroup of \(C_K\) corresponding to a subfield \(k\) of \(K\) can be characterized as the classes of all prime ideals of \(K\) which are completely decomposed in \(k/K\). Thus \([P] \in H\) since \(H\) corresponds to \(L\).

Now we show that \(\text{Ker}(f) \supseteq H\). Let \(P\) be a prime of \(K\) of absolute degree 1 with \([P] \in H\). Let \(N_{K/F}(P) = \rho = (\rho), \rho \in F\) and as above let \(P_i, i = 1, \ldots, t\) be the primes of \(K\) ramified in \(K/F\). We may assume also \(P \neq P_i\) for any \(i\). Since \([P] \in H, P\) is completely decomposed in \(L/K\). Say, \(P = Q_1 \cdots Q_s\) so that \(N_{L/F}(Q_i) = (\rho)\). Let \(m\) be a divisor of \(F\) divisible by high powers of all \(P_i\) and prime to \(P\). Since \(E\) is the maximal abelian extension of \(F\) in \(L\) and in \(\bar{K}\) the norm limitation theorem implies that

\[(*) \quad N_{E/F}(I_m(E)) \cdot S_m(F) = N_{L/F}(I_m(L)) \cdot S_m(F) = N_{E/F}(I_m(\bar{K})) \cdot S_m(F)\]

where \(I_m(k)\) is the group of ideals of \(k\) relatively prime to \(m\) and \(S_m(k)\) is the ideal ray (Strahl) mod \(m\).

We have noted that \((\rho) = N_{L/F}(Q)\) with \(Q_i \in I_m(L)\). It follows from (*) that we can write \((\rho) = N_{\bar{K}/F}(\mathfrak{a}) \cdot (\alpha)\) where \(\mathfrak{a} \in I_m(\bar{K})\) and \((\alpha) \in S_m(F)\). The norm from \(\bar{K}\) to \(K\) of any ideal of \(\bar{K}\) is a principal ideal of \(K\). Let \(N_{\bar{K}/K}(\mathfrak{a}) = (\alpha), \alpha \in K\). So \((\rho) = (\alpha)N_{\bar{K}/F}(\mathfrak{a}) = (\alpha)N_{\bar{K}/F}(N_{E/K}(\mathfrak{a})) = (\alpha)(N_{K/F}(\alpha))\) or \(\varepsilon\rho = (\alpha)N_{K/F}(\alpha)\) for some unit \(\varepsilon \in U_F\).

Therefore

\[\left(\frac{\varepsilon\rho, K/F}{P_i}\right) = \left(\frac{\alpha, K/F}{P_i}\right) \cdot \left(\frac{N_{K/F}(\alpha), K/F}{P_i}\right).\]

Since a global norm is certainly a local norm \(((N_{K/F}(\alpha), K/F)/P_i) = 1\). Also since \(\alpha \in F, \alpha \equiv 1(m)\) and \(m\) is divisible by high powers of the
we see that \(((M_i \cap K_i/F_i)/\alpha) = 1\). And therefore
\[
\left( \frac{K_i/F_i}{\alpha} \right) = \left( \frac{\alpha, K/F_i}{P_i} \right) = 1.
\]
Thus \(((\varepsilon_p, K/F_i)/P_i) = 1\) for all \(i\). In other words \(\chi(\rho) = \chi(\varepsilon^{-1})\), which gives \([P] \in \text{Ker}(f)\).

**Proof of the lemma.** Let \(T(L/F)\) be the inertia subgroup of \(\text{Gal}(L/F)\). The quotient \(\text{Gal}(L/F)/T(L/F)\) is a cyclic group, hence \(T(L/F)\) contains the commutator subgroup of \(\text{Gal}(L/F)\), which is \(\text{Gal}(L/M)\). Thus \(L/M\) is totally ramified. Letting \(e\) denote the ramification index, we have \(e(L/K \cap M) \geq [L: M] \geq [K: K \cap M]\). This last inequality follows from the fact that \(L \supseteq KM\) and, since \(M/K \cap M\) is galois, \([KM: M] = [K: K \cap M]\). Since \(L/K\) is unramified, \(e(L/K \cap M) \leq [K: K \cap M]\). Therefore \(e(L/K \cap M) = [K: K \cap M] = [L: M] = e(L/M)\) and so \(e(M/K \cap M) = 1\).

**References**


Received July 17, 1975.

Ohio State University
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)  J. DUGUNDJI
University of California Department of Mathematics
Los Angeles, California 90024

R. A. BEAUMONT  D. GILBARG AND J. MILGRAM
University of Washington
Seattle, Washington 98105

ASSOCIATE EDITORS

E. F. BECKENBACH  B. H. NEUMANN  F. WOLF  K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY  STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA  UNIVERSITY OF HAWAII
MONTANA STATE UNIVERSITY  UNIVERSITY OF TOKYO
UNIVERSITY OF NEVADA  UNIVERSITY OF UTAH
NEW MEXICO STATE UNIVERSITY  WASHINGTON STATE UNIVERSITY
OREGON STATE UNIVERSITY  UNIVERSITY OF WASHINGTON
UNIVERSITY OF OREGON  *  *  *
OSAKA UNIVERSITY  AMERICAN MATHEMATICAL SOCIETY

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan
Joseph Anthony Ball and Arthur R. Lubin, *On a class of contractive perturbations of restricted shifts* ......................................................... 309
Joseph Becker and William C. Brown, *On extending higher derivations generated by cup products to the integral closure* ........................................ 325
Andreas Blass, *Exact functors and measurable cardinals* .................................................. 335
Joseph Eugene Collison, *A variance property for arithmetic functions* ......................... 347
Craig McCormack Cordes, *Quadratic forms over nonformally real fields with a finite number of quaternion algebras* .................. 357
Freddy Delbaen, *Weakly compact sets in $H^1$* .................................................. 367
G. D. Dikshit, *Absolute Nörlund summability factors for Fourier series* .................. 371
Edward Richard Fadell, *Nielsen numbers as a homotopy type invariant* ................ 381
Josip Globevnik, *Analytic extensions of vector-valued functions* .......................... 389
Robert Gold, *Genera in normal extensions* ................. 397
Solomon Wolf Golomb, *Formulas for the next prime* ................. 401
Robert L. Griess, Jr., *The splitting of extensions of $SL(3, 3)$ by the vector space $F_3^3$* .................................................. 405
Thomas Alan Keagy, *Matrix transformations and absolute summability* ............. 411
Kazuo Kishi, *Analytic maps of the open unit disk onto a Gleason part* ............. 417
Kwangil Koh, Jiang Luh and Mohan S. Putcha, *On the associativity and commutativity of algebras over commutative rings* ............. 423
James C. Lillo, *Asymptotic behavior of solutions of retarded differential difference equations* .................................................. 431
John Alan MacBain, *Local and global bifurcation from normal eigenvalues* ........ 445
Anna Maria Mantero, *Sets of uniqueness and multiplicity for $L^p$* ..................... 467
J. F. McClendon, *Embedding metric families* .................................................. 481
L. Robbiano and Giuseppe Valla, *Primary powers of a prime ideal* ....................... 491
Wolfgang Ruess, *Generalized inductive limit topologies and barrelledness properties* .................................................. 499
Judith D. Sally, *Bounds for numbers of generators of Cohen-Macaulay ideals* ........ 517
Selma Schirmer, *Mappings of polyhedra with prescribed fixed points and fixed point indices* .................................................. 521
Cho Wei Sit, *Quotients of complete multipartite graphs* ........................................ 531
S. Szanjer and Zbigniew Zielezny, *Solvability of convolution equations in $\mathcal{H}_p'$, $p > 1$* .................................................. 539
Mitchell Herbert Taibleson, *The existence of natural field structures for finite dimensional vector spaces over local fields* ........ 545
William Yslas Vélez, *A characterization of completely regular fields* ............. 553
P. S. Venkatesan, *On right unipotent semigroups* ........................................ 555
Kenneth S. Williams, *A rational octic reciprocity law* ........................................ 563
Robert Ross Wilson, *Lattice orderings on the real field* .................................... 571
Harvey Eli Wolff, *$V$-localizations and $V$-monads. II* ........................................ 579