

Pacific Journal of Mathematics

GENERA IN NORMAL EXTENSIONS

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Let K/F be a finite normal extension of algebraic number fields and let C_K be the ideal class group of K . There are two fundamentally different ways to define the principal genus of C_K with respect to F . Classically the principal genus is described by norm residue symbols. By the modern definition it is the class group of the maximal unramified extension of K which is the composite of K with an abelian extension of F . It is shown here that the two definitions are equivalent.

Let F be a finite algebraic number field and K a finite normal extension of F with $G = \text{Gal}(K/F)$. Let \bar{K} be the Hilbert class field of K and let C_K be the ideal class group of K . By class field theory the fields lying between K and \bar{K} are in one-one correspondence with the subgroups of C_K . (See [3] or [4] for the class field theory involved.) Let L be the genus field for K/F . As defined by Fröhlich ([1]), L is the composite of K with the maximal abelian extension of F in \bar{K} . Calling this maximal abelian extension E , we have $\bar{K} \supseteq L = KE \supseteq K$, $E \supseteq F$ and $K \cap E$ is the maximal abelian extension of F in K . The subgroup of C_K corresponding to L is the principal genus of C_K . Gauss's definition of the principal genus is based on arithmetic characters. In [2] we showed that when G is abelian the two definitions are equivalent. Here we will show that in fact they are equivalent for any G .

Let C_F be the ideal class group of F and let $N_{K/F}: C_K \rightarrow C_F$ be the norm map on ideal class groups. Let \bar{F} be the Hilbert class field of F and ${}_N C_K$ the kernel of the norm map. Then the subgroup ${}_N C_K$ of C_K corresponds to the extension $K\bar{F}$ of K . Clearly $L \supseteq K\bar{F}$ and, letting H denote the principal genus of C_K , we see that ${}_N C_K \supseteq H$.

We now proceed to describe the characters in Gauss's definition. Let P_1, \dots, P_t be the primes of K , finite or infinite, ramified in K/F . For each i choose a prime \bar{P}_i in \bar{K} such that $\bar{P}_i \cap K = P_i$. This allows a consistent choice of primes in each subfield k by $P_{k,i} = \bar{P}_i \cap k$. And we will denote the completed localization of k at $P_{k,i}$ by k_i . In particular we have the chain $\bar{K}_i \supseteq L_i \supseteq K_i$, $E_i \supseteq F_i$ of local fields. For an ideal \mathfrak{A} of a field k let $[\mathfrak{A}]$ denote the ideal class of \mathfrak{A} . Now let \mathfrak{A} be an ideal of K such that $[\mathfrak{A}] \in {}_N C_K$. Thus $N_{K/F}(\mathfrak{A})$ is a principal ideal of F , say $N_{K/F}(\mathfrak{A}) = (a)$, $a \in F$. For each i we have a norm residue symbol $((K_i/F_i)/a)$ which we will also write $((a, K/F)/P_i)$ or most simply $\chi_i(a)$. This symbol is an element

of the local group $\text{Gal}(K_i/F_i)$ modulo its commutator. We may identify $\text{Gal}(K_i/F_i)$ with the decomposition group Z_i of P_i in K/F . Thus we have a homomorphism $\chi_i: F^\times \rightarrow Z_i/[Z_i, Z_i] = Z_i^{ab}$. Let $\chi: F^\times \rightarrow \prod_{i=1}^t Z_i^{ab}$ by $\chi(a) = (\chi_1(a), \dots, \chi_t(a))$. Let U_F denote the units of F , P_F the principal ideals of F , and $S = \chi(U_F)$. Then χ induces a homomorphism which we'll also denote by $\chi: P_F = F^\times/U_F \rightarrow \prod_{i=1}^t Z_i^{ab}/S$. Let $(a) \in P_F$ and $(a) = (N_{K/F}(b)) = N_{K/F}((b))$. Then $a = \varepsilon \cdot N_{K/F}(b)$ for some $\varepsilon \in U_F$ and $\chi_i(a) = \chi_i(\varepsilon)\chi_i(N_{K/F}(b)) = \chi_i(\varepsilon)$ for $i = 1, \dots, t$ since $\chi_i(N_{K/F}(b)) = ((K_i/F_i)/N_{K/F}(b))$ and every global norm is a local norm everywhere. It follows that $\chi: P_F \rightarrow \prod Z_i^{ab}/S$ vanishes on $N_{K/F}(P_K) \subseteq P_F$. Note that $N_{K/F}: {}_N C_K \rightarrow P_F/N(P_K)$ since if $[\mathfrak{A}] = [\mathfrak{B}] \in {}_N C_K$ then $\mathfrak{A} = (\alpha)\mathfrak{B}$ and $N(\mathfrak{A}) = N((\alpha))N(\mathfrak{B}) \in P_F$. Now we can define $f = \chi \circ N_{K/F}: {}_N C_K \rightarrow P_F/N(P_K) \rightarrow \prod Z_i^{ab}/S$. The formal statement of the equivalence of the two definitions of principal genus is given by

THEOREM. *Let K/F be a finite normal extension of number fields and let H be the principal genus in the sense of Fröhlich. Let $f: {}_N C_K \rightarrow \prod_{i=1}^t Z_i^{ab}/S$ be the modified product of local norm residue symbols described above. Then $H = \text{Ker}(f)$.*

Proof. First we show that $\text{Ker}(f) \subseteq H$. Let P be a prime ideal of K , $P \neq P_i$, $i = 1, \dots, t$; $[P] \in \text{Ker}(f)$, and P of absolute degree 1. Since $[P] \in \text{Ker}(f)$ and P is of degree 1, $N_{K/F}(P) = \mathfrak{p} = (\rho)$ where $\mathfrak{p} = P \cap F$ and $\rho \in F$. Moreover ρ may be chosen so that $\chi_i(\rho) = 1$, $i = 1, \dots, t$, since ρ times any unit of F generates \mathfrak{p} and $[P] \in \text{Ker}(f)$ implies $\chi(\rho) = \chi(\varepsilon)$ for some $\varepsilon \in U_F$.

Let M_i be the maximal abelian extension of F_i in L_i . So $K_i \cap M_i$ is the maximal abelian extension of F_i in K_i . Then

$$\left(\frac{M_i/F_i}{\rho}\right)\Big|_{M_i \cap K_i} = \left(\frac{M_i \cap K_i/F_i}{\rho}\right) = \left(\frac{K_i/F_i}{\rho}\right) = \chi_i(\rho) = 1.$$

The second equality here follows from the fact that $N_{K_i/F_i}(K_i) = N_{K_i \cap M_i/F_i}(K_i \cap M_i)$. Therefore

$$\left(\frac{M_i/F_i}{\rho}\right) \in \text{Gal}(M_i/M_i \cap K_i) \subseteq \text{Gal}(M_i/F_i).$$

Since $P \neq P_i$, any i, ρ is a P_i -unit for each i . Thus $((M_i/F_i)/\rho) \in T(M_i/F_i) \subseteq \text{Gal}(M_i/F_i)$ where T is the inertia group of the local extension. So we have

$$\left(\frac{M_i/F_i}{\rho}\right) \in T(M_i/F_i) \cap \text{Gal}(M_i/M_i \cap K_i) = T(M_i/M_i \cap K_i)$$

LEMMA. *L_i/F_i be a normal extension of local fields and M_i the*

maximal abelian extension of F_i in L_i . If $L_i \supseteq K_i \supseteq F_i$ and L_i/K_i is unramified, then $M_i/M_i \cap K_i$ is unramified.

The lemma, to be proved below, implies that $T(M_i/M_i \cap K_i) = \{1\}$ and therefore $((M_i/F_i)/\rho) = 1$. Since $M_i \supseteq E_i$ it follows that $((E_i/F_i)/\rho) = 1 = ((\rho, E/F)/P_i)$ for all i . So we have E/F abelian, $\rho \in F$, and $((\rho, E/F)/\mathfrak{p}_i) = 1$ for $\mathfrak{p}_i = F \cap P_i$, $i = 1, \dots, t$. Since $\bar{K} \supseteq E$, the $\{\mathfrak{p}_i\}$ includes all primes of F ramified in E/F . For every unramified prime of F at which ρ is a unit the norm residue symbol is 1. The only undetermined symbol is $((\rho, E/F)/\mathfrak{p})$. By the product formula for norm residue symbols, the product of all symbols is 1. Hence we must have $((\rho, E/F)/\mathfrak{p}) = 1$. Recall that $(\rho) = \mathfrak{p}$, i.e. ρ is a prime element at \mathfrak{p} , and \mathfrak{p} is unramified in E/F . Hence $((\rho, E/F)/\mathfrak{p})$ generates the decomposition group of \mathfrak{p} in E/F . We conclude that \mathfrak{p} is completely decomposed in E/F . It follows by standard arguments that P is completely decomposed in L/K since $L = KE$. The subgroup of C_K corresponding to a subfield k of \bar{K} can be characterized as the classes of all prime ideals of K which are completely decomposed in k/K . Thus $[P] \in H$ since H corresponds to L .

Now we show that $\text{Ker}(f) \supseteq H$. Let P be a prime of K of absolute degree 1 with $[P] \in H$. Let $N_{K/F}(P) = \mathfrak{p} = (\rho)$, $\rho \in F$ and as above let P_i , $i = 1, \dots, t$ be the primes of K ramified in K/F . We may assume also $P \neq P_i$ for any i . Since $[P] \in H$, P is completely decomposed in L/K . Say, $P = Q_1 \cdots Q_g$ so that $N_{L/F}(Q_i) = (\rho)$. Let \mathfrak{m} be a divisor of F divisible by high powers of all P_i and prime to P . Since E is the maximal abelian extension of F in L and in \bar{K} the norm limitation theorem implies that

$$(*) \quad N_{E/F}(I_{\mathfrak{m}}(E)) \cdot S_{\mathfrak{m}}(F) = N_{L/F}(I_{\mathfrak{m}}(L)) \cdot S_{\mathfrak{m}}(F) = N_{\bar{K}/F}(I_{\mathfrak{m}}(\bar{K})) \cdot S_{\mathfrak{m}}(F)$$

where $I_{\mathfrak{m}}(k)$ is the group of ideals of k relatively prime to \mathfrak{m} and $S_{\mathfrak{m}}(k)$ is the ideal ray (Strahl) mod \mathfrak{m} .

We have noted that $(\rho) = N_{L/F}(Q)$ with $Q_i \in I_{\mathfrak{m}}(L)$. It follows from (*) that we can write $(\rho) = N_{\bar{K}/F}(\mathfrak{A}) \cdot (\alpha)$ where $\mathfrak{A} \in I_{\mathfrak{m}}(\bar{K})$ and $(\alpha) \in S_{\mathfrak{m}}(F)$. The norm from \bar{K} to K of any ideal of \bar{K} is a principal ideal of K . Let $N_{\bar{K}/K}(\mathfrak{A}) = (a)$, $a \in K$. So $(\rho) = (\alpha)N_{\bar{K}/F}(\mathfrak{A}) = (\alpha)N_{K/K}(N_{E/K}(\mathfrak{A})) = (\alpha)(N_{K/F}(a))$ or $\varepsilon\rho = \alpha \cdot N_{K/F}(a)$ for some unit $\varepsilon \in U_F$. Therefore

$$\left(\frac{\varepsilon\rho, K/F}{P_i} \right) = \left(\frac{\alpha, K/F}{P_i} \right) \cdot \left(\frac{N_{K/F}(a), K/F}{P_i} \right).$$

Since a global norm is certainly a local norm $((N_{K/F}(a), K/F)/P_i) = 1$. Also since $\alpha \in F$, $\alpha \equiv 1(\mathfrak{m})$ and \mathfrak{m} is divisible by high powers of the

P_i we see that $((M_i \cap K_i/F_i)/\alpha) = 1$. And therefore

$$\left(\frac{K_i/F_i}{\alpha}\right) = \left(\frac{\alpha, K/F}{P_i}\right) = 1.$$

Thus $((\varepsilon\rho, K/F)/P_i) = 1$ for all i . In other words $\chi(\rho) = \chi(\varepsilon^{-1})$, which gives $[P] \in \text{Ker}(f)$.

Proof of the lemma. Let $T(L/F)$ be the inertia subgroup of $\text{Gal}(L/F)$. The quotient $\text{Gal}(L/F)/T(L/F)$ is a cyclic group, hence $T(L/F)$ contains the commutator subgroup of $\text{Gal}(L/F)$, which is $\text{Gal}(L/M)$. Thus L/M is totally ramified. Letting e denote the ramification index, we have $e(L/K \cap M) \geq [L: M] \geq [K: K \cap M]$. This last inequality follows from the fact that $L \supseteq KM$ and, since $M/K \cap M$ is galois, $[KM: M] = [K: K \cap M]$. Since L/K is unramified, $e(L/K \cap M) \leq [K: K \cap M]$. Therefore $e(L/K \cap M) = [K: K \cap M] = [L: M] = e(L/M)$ and so $e(M/K \cap M) = 1$.

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