THE SPLITTING OF EXTENSIONS OF SL(3, 3) BY THE VECTOR SPACE $F_3^3$.  

ROBERT L. GRIESS, JR.
THE SPLITTING OF EXTENSIONS OF SL(3, 3) 
BY THE VECTOR SPACE \( F_3 \)

ROBERT L. GRIESS, JR.

We give two proofs that \( H^2(\text{SL}(3, 3), F_3) = 0 \). This result has appeared in a paper by Sah, [6], but our methods are relatively elementary, i.e., we require only elementary homological algebra and do a group-theoretic analysis of an extension of \( \text{SL}(3, 3) \) by \( F_3 \) to show that the extension splits. The starting point is to notice that the vector space is a free module for \( F_3 \), where \( x \) has Jordan canonical form

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

We then can exploit the vanishing of \( H^i(\langle x \rangle, F_3) \) for \( i = 1, 2 \).

For elementary linear algebra, we refer to [2] and for cohomology of groups, we refer to [1], [4], [5] or [6]. Group theoretic notation is standard and follows [3]. Let \( V \) be a 3-dimensional \( F_3 \)-vector space and let \( \text{SL}(3, 3) \) be the associated special linear group. Let \( v_1, v_2, v_3 \) be a basis for \( V \). Define, for \( i, j \in \{1, 2, 3\}, i \neq j, \) and \( t \in F_3 \), \( x_{ij}(t) \in \text{SL}(3, 3) \) by

\[
x_{ij}(t); v_k \mapsto \begin{cases} 
v_k & k \neq i \\
v_i + tv_j, k = i
\end{cases}
\]

Inspection of the Jordan canonical form shows that all \( x_{ij}(t), t \neq 0 \), are conjugate in \( \text{GL}(3, 3) = \{\pm I\} \times \text{SL}(3, 3) \), hence in \( \text{SL}(3, 3) \).

Set \( G = \text{SL}(3, 3) \). We let

\[
1 \longrightarrow V \longrightarrow G^* \overset{\pi}{\longrightarrow} G \longrightarrow 1
\]

be an arbitrary extension of \( G \) by \( V \) with the above action. We will show (*) is split. We use the convention that \( u^* \in G^* \) is a representative (arbitrary, unless otherwise specified) for \( u \in G \).

The alternate proof of splitting (given later) is much neater than the first version. The methods are quite different, however, and it seems worthwhile to give two proofs.

**Lemma 1.** Let \( x = x_{13}(1)x_{23}(1)x_{13}(-1) \). Then \( C_0(x) = \langle x, x_{13}(1) \rangle \). If \( t \in G \) is an involution which inverts \( x \), then \( t \) centralizes \( x_{13}(1) \).

**Proof.** The first statement is elementary linear algebra. Namely, \( x \) has a cyclic vector in \( V \), so that any transformation which commutes with \( x \) is a polynomial in \( x \). Since \( x \) has minimal polynomial of
degree 3, its full commuting algebra is all matrices of the shape
\[
\begin{pmatrix}
a & b & c \\
0 & a & b \\
0 & 0 & a
\end{pmatrix}
, \quad a, b, c \in F_3.
\]
The first statement is now clear. As for the second, it suffices to display an element \( t \) with the required properties, e.g.
\[
t = \begin{pmatrix}
-1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]
The lemma is proven.

**Lemma 2.** \( x_{12}(1)x_{23}(1)x_{13}(-1) \) and all its conjugates are represented in \( G^* \) by elements of order 3. Any two such representatives are conjugate by an element of \( V \).

**Proof.** The Jordan canonical form for \( x = x_{12}(1)x_{23}(1)x_{13}(-1) \) indicates that \( V \) is a free \( F_3\langle x \rangle \)-module. So \( H^i(\langle x \rangle, V) = 0 \) for \( i \geq 1 \). Both statements follow.

**Lemma 3.** Each \( x_{ij}(t) \) is represented in \( G^* \) by an element of order 3, which commutes with an involution of \( G^* \).

**Proof.** We may assume \( i = 1, j = 3, t = 1 \). Let
\[
x = x_{12}(1)x_{23}(1)x_{13}(-1),
\]
and let \( x^* \in G^* \) represent \( x \), \( |x^*| = 3 \). Again by Lemma 2, a Frattini—like argument shows that \( N_{G^*}(\langle x^* \rangle)^* = V \cdot N_{G^*}(\langle x^* \rangle) \). Choose \( y \in N_{G^*}(\langle x^* \rangle) \) with \( y^* = x_{13}(1) \). Then \( C_{G^*}(x^*) = \langle x^*, y, v_3 \rangle \) is abelian. Let \( t \in N_{G^*}(\langle x^* \rangle) \) be an involution inverting \( x^* \). Then by Lemma 1, \( t^* \) centralizes \( x_{13}(1) \) and inverts \( v_3 \). By Fittings theorem,
\[
C_{G^*}(x^*) = \langle y_1 \rangle \times \langle x, v_3 \rangle
\]
where \( \langle y_1 \rangle = C_{G^*}(\langle x^*, t \rangle) \). Clearly \( |y_1| = 3 \) and \( 1 \neq y_i \in \langle x_{13}(1) \rangle \). This proves the lemma.

**Lemma 4.** If \( t \) is an involution of \( G^* \), \( C_{G^*}(t) \) has a Sylow 3-subgroup isomorphic to \( Z_3 \times Z_3 \).

**Proof.** Since \( G \) has one class of involutions, so does \( G^* \). So, we apply Lemma 3 to see that \( t \) centralizes an element of order 3
outside $V$. Since $|C_v(t)| = 3$ and $C_o(t^\pi) \cong \text{GL}(2, 3)$, we are done by the Frattini argument namely, $\langle t \rangle \in \text{Syl}_4(V\langle t \rangle)$ and $V\langle t \rangle \vartriangleleft H$, where $H$ is the preimage in $G^*$ of $C_o(t^\pi)$.

In what follows, let $R = N_G(\langle v_3 \rangle)$ and $Q = 0_3(R)$. Then

\[
R^* = \begin{pmatrix}
A & a \\
0 & b \\
0 & c \\
\end{pmatrix} \quad A \in \text{GL}(2, 3), \; a, b \in F_3, \; c = (\det A)^{-1}
\]

\[
Q^* = \begin{pmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 1 \\
\end{pmatrix} \quad a, b \in F_3.
\]

Let

\[
h = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

and let denote images under $R \to R/\langle v_3 \rangle$. Let $h^* \in R$ be an involution representing $h$.

**Lemma 5.** $Q$ is inverted by $h^*$. Also, $Q$ is elementary abelian and $Q$ is extra special of order $3^5$, exponent 3, with center $\langle v_3 \rangle$.

**Proof.** The first statement is clear since $h$ inverts $Q^*$ and $\langle v_3, v_3 \rangle$. Therefore, $Q$ is abelian. From Lemma 3, we get that $Q$ is elementary and the action of members of $Q^*$ on $V$ implies that $Q$ is extra special. Since $Q^*$ is generated by elements of order 3, by Lemma 3 again, $Q$ has exponent 3.

We now require a technical result for studying automorphisms of $Q$. Since automorphisms commute with commutation, we have a homorphism (which is actually onto) $\text{Aut}(Q) \to \text{Sp}_0(4, 3)$, the group of similitudes of a nondegenerate alternating bilinear form from $F_3^4$ to $F_3$ (a similitude preserves the form up to a scalar multiple; we have $|\text{Sp}_0(4, 3)| = |F_3^4| = 2$, where $\text{Sp}(4, 3)$ is the symplectic group, i.e. the group preserving the form).

**Lemma 6.** Let $M$ be a 4-dimensional $F_3$-vector space supporting a nondegenerate alternating form $(,)$ and let $\text{Sp}_0(4, 3), \text{Sp}(4, 3)$ be the associated group of similitudes, resp. symplectic group. Let $I$ be a maximal totally isotropic subspace and let $K$ be its (global) stabilizer in $\text{Sp}_0(4, 3)$. Then

(i) $\dim I = 2$

(ii) If $J$ is a maximal totally isotropic subspace complementing
In $M$ we may choose a basis $a_1, b_1$ for $I$, $a_2, b_2$ for $J$ so that $(a_i, b_i) = \delta_{i3}$ and $(a_i, a_j) = (b_i, b_j) = 0$. With respect to the basis $\{a_i, a_2, b_1, b_2\}$ for $V$, elements of $K$ have the shape

\[
\begin{pmatrix}
A & B \\
0 & c^t A^{-1}
\end{pmatrix},
\]

$A \in \text{GL}(2, 3)$, $B$ a symmetric $2 \times 2$ matrix, $c \in F_3^*; c = 1$ if and only if the matrix lies in $\text{Sp}(4, 3)$. In this notation, $0_3(K)$ consists of those matrices with $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and if $L$ is the set of matrices with $B = 0$, $L$ complements $0_3(K)$ in $K$.

(iii) If $Y \in \text{Syl}_3(L)$, $0_3(K)$ is a free $F_3 Y$-module.

(iv) Any subgroup of $K$ meeting $0_3(K)$ trivially stabilizes a maximal totally isotropic subspace which complements $I$, and is in fact conjugate to a subgroup of $L$.

Proof. Statements (i) and (ii) are straightforward. To prove (iii), we may assume $Y = \langle y \rangle$,

\[
y = \begin{pmatrix} 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1
\end{pmatrix}.
\]

Take

\[
k(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & 0 & \alpha & \beta \\
0 & 1 & \beta & \gamma \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

a typical element of $0_3(K)$. A matrix calculation show that $y^{-1}k(\alpha, \beta, \gamma)y = k(\alpha - 2\beta + \gamma, \beta - \gamma, \gamma)$. To show $0_3(K)$ is a free $Y$-module, it suffices, since $0_3(K) \cong Z_3 \times Z_3 \times Z_3$, to find a triple $(\alpha, \beta, \gamma)$ such that the three elements $y^{-1}k(\alpha, \beta, \gamma)y^i, i = 0, 1, 2$ are linearly independent. Any $(\alpha, \beta, \gamma)$ with $\beta \neq 0$ does the trick.

We now prove statement (iv). First (iii) implies that $H^i(Y, 0_3(K)) = 0$ for $i \geq 1$. Secondly, if $X \leq K, X \cap 0_3(K) = 1$, then a Sylow 3-subgroup $X_3$ of $X$ is conjugate to a subgroup of $Y$, and so $H^i(X_3, 0_3(K)) = 0$ for $i \geq 1$. Finally, we quote the injectiveness of the restriction $H^i(X, 0_3(K)) \rightarrow H^i(X_3, 0_3(K))$. A consequence is that $X$ is conjugate in $0_3(K)X$ to $L \cap 0_3(K)X$, whence $X$ stabilizes a maximal totally isotropic subspace complementing $I$. 
THEOREM. The extension (*) is split. Consequently, $H^4(SL(3, 3), F_3^3) = 0$.

Proof. Let $S \cong GL(2, 3)$ complement $\langle v_\circ \rangle$ in $C_6(h^*)$ (use Lemma 4 and Gaschütz' theorem). Easily, we see that $S$ is faithful on $Q$ and the map $\text{Aut} (Q) \rightarrow \text{Sp}_6(4, 3)$ embeds $S$ as a subgroup $S_6$ of $K$, where, in the notation of Lemma 6, $M = \overline{Q}$, $I = \overline{V}$. Also, $0_6(S) = 1$ implies $0_6(K) \cap S_6 = 1$. Hence, by Lemma 6 (iv), $S_6$ stabilizes a complement $\overline{J}$ to $\overline{V}$ in $\overline{Q}$, where $\overline{J}$ is totally singular. Letting $J$ be the preimage of $\overline{J}$ in $Q$, $J$ is elementary abelian. Since $h^*$ inverts $\overline{J}$ and centralizes $v_\circ$, we have $J = \langle v_\circ \rangle \times \langle J, h^* \rangle$. Then $[J, h^*]S$ complements $V$ in $R$. Since $(|G: R|^, 3) = 1$, Gaschütz theorem implies that $G^*$ splits over $V$, as required.

An alternate proof was suggested by V. Landazuri in a conversation. We sketch the argument. Using Lemmas 2 and 3, we get

(i) every element of order 3 in $G$ is represented in $G^*$ by an element of order 3.

Let $y \in G^*$ represent $x_{\iota}(1)$, $|y| = 3$. Since $[V, y, y] = 1$, a simple calculation shows

(ii) every element of the coset $Vx_{\iota}(1)^n = Vy$ has order 3.

Now take $a, b \in U^*$, $a^\times = x_{12}(1)$, $b^\times = x_{14}(-1)x_{23}(1)$, $|b| = 3$ (using (i)). By (ii), $|a| = |ab| = |ba| = 3$. An elementary argument shows that, if $\xi_1, \xi_2$ are elements in any group such that $|\xi_1| = |\xi_2| = |\xi_1^2| = 3$, then $\langle \xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1} \rangle$ is a normal abelian subgroup of index 3 in $\langle \xi_1, \xi_2 \rangle$. Applying this to $\xi_1 = ab$, $\xi_2 = ba$ we see that $\langle a, b \rangle$ has a normal abelian subgroup $H = \langle a^{-1}, b^{-1}, [a, b] \rangle$ of index 3. By (ii), $H$ is elementary abelian. Therefore, $|\langle a, b \rangle| = 3^3$. It is easily seen that $\langle a^\times, b^\times \rangle = U$, and this means $\langle a, b \rangle \cap V = 1$. Our theorem now follows from Gaschütz' theorem.

REFERENCES


Received October 31, 1974.

UNIVERSITY OF MICHIGAN
Pacific Journal of Mathematics
Vol. 63, No. 2 April, 1976

Joseph Anthony Ball and Arthur R. Lubin, On a class of contractive perturbations of restricted shifts ........................................... 309
Joseph Becker and William C. Brown, On extending higher derivations generated by cup products to the integral closure ............................................................. 325
Andreas Blass, Exact functors and measurable cardinals .............................................. 335
Joseph Eugene Collison, A variance property for arithmetic functions ...................... 347
Craig McCormack Cordes, Quadratic forms over nonformally real fields with a finite number of quaternion algebras ................................................... 357
Freddy Delbaen, Weakly compact sets in $H^1$. ......................................................... 367
G. D. Dikshit, Absolute Nörlund summability factors for Fourier series .................. 371
Edward Richard Fadell, Nielsen numbers as a homotopy type invariant ............... 381
Josip Globevnik, Analytic extensions of vector-valued functions .............................. 389
Robert Gold, Genera in normal extensions ................................................................. 397
Solomon Wolf Golomb, Formulas for the next prime ............................................ 401
Robert L. Griess, Jr., The splitting of extensions of $SL(3, 3)$ by the vector space $F_3^3$ ................................................................. 405
Thomas Alan Keagy, Matrix transformations and absolute summability ................. 411
Kazuo Kishi, Analytic maps of the open unit disk onto a Gleason part ..................... 417
Kwangil Koh, Jiang Luh and Mohan S. Putcha, On the associativity and commutativity of algebras over commutative rings ........................................ 423
James C. Lillo, Asymptotic behavior of solutions of retarded differential difference equations ............................................................................................................. 431
John Alan MacBain, Local and global bifurcation from normal eigenvalues ............. 445
Anna Maria Mantero, Sets of uniqueness and multiplicity for $L^p$ .............................. 467
J. F. McClendon, Embedding metric families ............................................................. 481
L. Robbiano and Giuseppe Valla, Primary powers of a prime ideal ......................... 491
Wolfgang Ruess, Generalized inductive limit topologies and barreledness properties ................................................................. 499
Judith D. Sally, Bounds for numbers of generators of Cohen-Macaulay ideals ........ 517
Helga Schirmer, Mappings of polyhedra with prescribed fixed points and fixed point indices ............................................................................................................. 521
Cho Wei Sit, Quotients of complete multipartite graphs ............................................. 531
S. Sznajder and Zbigniew Zielezny, Solvability of convolution equations in $\mathcal{F}'_p$, $p > 1$ ..................................................................................................................................... 539
Mitchell Herbert Taibleson, The existence of natural field structures for finite dimensional vector spaces over local fields .................................................................................... 545
William Yslas Vélez, A characterization of completely regular fields ......................... 553
P. S. Venkatesan, On right unipotent semigroups ...................................................... 555
Kenneth S. Williams, A rational octic reciprocity law ................................................ 563
Robert Ross Wilson, Lattice orderings on the real field ............................................ 571
Harvey Eli Wolff, $V$-localizations and $V$-monads. II ............................................. 579