

Pacific Journal of Mathematics

**MATRIX TRANSFORMATIONS AND ABSOLUTE
SUMMABILITY**

THOMAS ALAN KEAGY

MATRIX TRANSFORMATIONS AND ABSOLUTE SUMMABILITY

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The main results of this paper are two theorems which give necessary conditions for a matrix to map into \mathcal{C} the set of all subsequences (rearrangements) of a null sequence not in \mathcal{C} . These results provide affirmative answers to the following questions proposed by J. A. Fridy. Is a null sequence x necessarily in \mathcal{C} if there exists a sum-preserving $\mathcal{C}-\mathcal{C}$ matrix A that maps all subsequences (rearrangements) of x into \mathcal{C} ?

1. Introduction. Let s, m, c, c_0 and cs denote, respectively, the set of all complex sequences, the set of all bounded sequences in s , the set of all convergent sequences in s , the set of all null sequences in c , and the set of all sequences in s with sequence of partial sums in c . Let

$$\mathcal{C} = \{x \in s: \Sigma |x_p| < \infty\} \text{ and } \mathcal{C}^2 = \{x \in s: \Sigma |x_p|^2 < \infty\}.$$

A matrix A which maps each element of \mathcal{C} into \mathcal{C} is called an $\mathcal{C}-\mathcal{C}$ matrix and may be characterized [3] and [6] by the property: $\{\sum_{p=1}^{\infty} |a_{pq}| \}_{q=1}^{\infty} \in m$. If, in addition, $\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{pq}x_q = \sum_{q=1}^{\infty} x_q$, whenever $x \in \mathcal{C}$, then A is a sum-preserving $\mathcal{C}-\mathcal{C}$ matrix; this is characterized by $\sum_{p=1}^{\infty} a_{pq} = 1$, for each q .

In 1943, R. C. Buck [1] showed that a sequence x is convergent if some regular matrix sums every subsequence of x . J. A. Fridy [5] has obtained an analog to Buck's theorem in which "subsequence" is replaced by "rearrangement." In addition, he has characterized \mathcal{C} by showing that $x \in \mathcal{C}$ if there is a sum-preserving $\mathcal{C}-\mathcal{C}$ matrix that transforms every rearrangement of x into \mathcal{C} . In §2 of the present paper, necessary conditions are obtained for a matrix to map into \mathcal{C} the set of all subsequences of a null sequence not in \mathcal{C} . This result yields as a corollary the affirmative answer to the following question proposed by J. A. Fridy [5]. Is a null sequence x necessarily in \mathcal{C} if there exists a sum-preserving $\mathcal{C}-\mathcal{C}$ matrix that maps all subsequences of x into \mathcal{C} ? In §3, necessary conditions are obtained for a matrix to map into \mathcal{C} all rearrangements of a null sequence not in \mathcal{C} . This yields as a corollary Fridy's characterization of \mathcal{C} mentioned above. Finally, §4 contains examples of matrix mappings involving both subsequences and rearrangements.

2. Subsequences. The following two lemmas will be instru-

mental in the proof of Theorem 1.

LEMMA 1. Suppose x and a are sequences such that $\sum_{q=1}^{\infty} a_q y_q$ converges for every subsequence y of x . If $\varepsilon > 0$, then there exist $M > 0$ and a strictly increasing function $\delta: I^+ \rightarrow I^+$ such that if $t > M$, then $|\sum_{q=t}^{\infty} a_q y_q| \leq \varepsilon$ for every subsequence $(y_q)_{q=t}^{\infty}$ of $(x_q)_{q=\delta(t)}^{\infty}$.

LEMMA 2. If x is a null sequence not in \mathcal{L} and a is a nonnull convergent sequence, then there exists a subsequence y of x such that $\lim_t |\sum_{q=1}^t y_q| = \infty$ and $(\sum_{q=1}^n a_q y_q)_{n=1}^{\infty}$ is not bounded.

THEOREM 1. Let x be a null sequence not in \mathcal{L} , and suppose A is a matrix such that $Ay \in \mathcal{L}$ for every subsequence y of x . Then

- (i) $\sum_{p=1}^{\infty} |a_{pq}| < \infty$ for $q = 1, 2, 3, \dots$; and
- (ii) if $\lim_q \sum_{p=1}^{\infty} a_{pq} = L$, then $L = 0$.

Proof. To show (i), let k be fixed and $j > i > k$ such that $x_i \neq x_j$. Let y be the subsequence of x such that $y_q = x_q$ for $q = 1, 2, \dots, k-1$; $y_k = x_i$; and $y_{k+t} = x_{j+t}$ for $t = 1, 2, 3, \dots$. Let z be the subsequence of x such that $z_k = x_j$ and $z_q = y_q$ otherwise. Then

$$\infty > \sum_{p=1}^{\infty} \left| \sum_{q=1}^{\infty} a_{pq} y_q - \sum_{q=1}^{\infty} a_{pq} z_q \right| = |x_i - x_j| \sum_{p=1}^{\infty} |a_{pk}|.$$

Therefore $\sum_{p=1}^{\infty} |a_{pk}| < \infty$.

Suppose $\lim_q \sum_{p=1}^{\infty} a_{pq} = L$ and $L \neq 0$. Let (y_1, \dots, y_{M-1}) be a subsequence of x with $y_{M-1} = x_p$. Since $x \notin \mathcal{L}$ there exists a subsequence $(w_q)_{q=M}^{\infty}$ of $(x_q)_{q=r+1}^{\infty}$ such that $\lim_t |\sum_{q=M}^t w_q| = \infty$. By Lemma 2 there exists a subsequence $(z_q)_{q=M}^{\infty}$ of $(w_q)_{q=M}^{\infty}$ such that $\lim_t |\sum_{q=M}^t z_q| = \infty$ and $\limsup_t |\sum_{q=M}^t z_q \sum_{p=1}^{\infty} a_{pq}| = \infty$. Choose $k > M$ such that

$$\left| \sum_{q=M}^k z_q \sum_{p=1}^{\infty} a_{pq} \right| > M + \sum_{q=1}^{M-1} |y_q| \sum_{p=1}^{\infty} |a_{pq}| + 3.$$

Let $K > 0$ such that $|\sum_{p=K+1}^{\infty} a_{pq}| < 1/(k(|z_q| + 1))$ for $q = M, \dots, k$. By Lemma 1, letting $\varepsilon = 1/K$, there exist N'_p and δ'_p for $1 \leq p \leq K$, such that if $N = \max\{N'_1, \dots, N'_K, k + 2\}$ and $\delta(i) = \max\{\delta'_p(i): p = 1, \dots, K\}$, then $\sum_{p=1}^K |\sum_{q=N}^{\infty} a_{pq} v_q| < 1$ for every subsequence $(v_q)_{q=N}^{\infty}$ of $(x_q)_{q=\delta(N)}^{\infty}$. Let $y_q = z_q$ for $M \leq q \leq k$, and choose $(y_{k+1}, \dots, y_{N-1})$ a subsequence of $(w_q)_{q=\delta(N)}^{\infty}$ such that $\sum_{q=k+1}^{N-1} |y_q| \sum_{p=1}^{\infty} |a_{pq}| < 1$. Note that the first $N-1$ terms of a fixed sequence y have now been determined. If y^* is any subsequence of x that agrees with y for these first $N-1$ terms, then $\sum_{p=1}^K |\sum_{q=1}^{\infty} a_{pq} y_q^*| > M$.

This process for defining terms of y may be continued so that if $T > 0$, then there exist $M \geq T$ and $K > 0$ such that

$$\sum_{p=1}^K \left| \sum_{q=1}^{\infty} a_{pq} y_q \right| > M.$$

Thus a subsequence y of x can be constructed such that $Ay \notin \mathcal{I}$, a contradiction.

COROLLARY 1. *A null sequence x is in \mathcal{I} if and only if there exists a sum-preserving $\mathcal{I} - \mathcal{I}$ matrix A such that $Ay \in \mathcal{I}$ for every subsequence y of x .*

3. Rearrangements. Following J. A. Fridy [5], the sequence y is called a rearrangement of the sequence x provided that there is a 1-1 function π from the positive integers onto themselves such that for each k , $x_k = y_{\pi(k)}$. The word "permutation" will be reserved to indicate the reordering of a finite sequence.

THEOREM. *If x is a null sequence not in \mathcal{I} and A is a matrix such that $Ay \in \mathcal{I}$ for every rearrangement y of x , then $\lim_q \sum_{p=1}^{\infty} |a_{pq}| = 0$.*

Proof. Let $x_i \neq x_j$ be nonzero elements of x . Suppose the k th column of A is not in \mathcal{I} . Let $q \neq k$ and y be a rearrangement of x with $y_k = x_i$ and $y_q = x_j$. Let z be the rearrangement of x such that $z_k = x_j$, $z_q = x_i$, and $z_i = y_i$ otherwise. Then

$$|x_i - x_j| \sum_{p=1}^{\infty} |a_{pk} - a_{pq}| = \sum_{p=1}^{\infty} \left| \sum_{q=1}^{\infty} a_{pq} y_q - \sum_{q=1}^{\infty} a_{pq} z_q \right| < \infty.$$

Therefore $\sum_{p=1}^{\infty} |a_{pk} - a_{pq}| < \infty$ for every $q \neq k$. Since $\sum_{p=1}^{\infty} |a_{pk}| = \infty$, it now follows that $\sum_{p=1}^{\infty} |a_{pq}| = \infty$ for $q \geq 1$. Suppose $N > 0$ and a permutation (r_1, \dots, r_M) of M terms of x has been chosen such that $\sum_{q=1}^M r_q \neq 0$. If $\lambda = \sum_{p=1}^{\infty} |\sum_{q=1}^M a_{pq} r_q| < \infty$, then

$$\infty > \lambda + \sum_{q=2}^M |r_q| \sum_{p=1}^{\infty} |a_{p1} - a_{pq}| \geq \left| \sum_{q=1}^M r_q \right| \sum_{p=1}^{\infty} |a_{p1}|,$$

a contradiction. Therefore $\lambda = \infty$ and there exists $K > N$ such that $\sum_{p=N}^K |\sum_{q=1}^M a_{pq} r_q| > 2$. Let $i = \min \{q: x_q \in x \setminus (r_1, \dots, r_M)\}$. J. A. Fridy [5] has shown that each row of A is null. Therefore there exists $T > M + 1$ such that $|x_i| \sum_{p=1}^K |a_{pT}| < 2^{-(M+1)}$. Let $r_T = x_i$ and $(r_{M+1}, \dots, r_{T-1})$ be a subsequence of $x \setminus (r_1, \dots, r_M, r_T)$ such that $\sum_{p=1}^K \sum_{q=M+1}^{T-1} |a_{pq}| |r_q| < 2^{-(M+2)}$ and $\sum_{q=1}^T r_q \neq 0$. Then

$$\begin{aligned} \sum_{p=N}^K \left| \sum_{q=1}^T a_{pq} r_q \right| &\geq \sum_{p=N}^K \left| \sum_{q=1}^M a_{pq} r_q \right| - \sum_{p=N}^K \sum_{q=M+1}^{T-1} |a_{pq} r_q| \\ &\quad - |r_T| \sum_{p=N}^K |a_{pT}| > 1. \end{aligned}$$

But this process may be continued. Therefore there exists a rearrangement r of x such that if $L > 0$, then there exist $K > N \geq L$ such that $\sum_{p=N}^K |\sum_{q=1}^{\infty} a_{pq}r_q| > 1$, a contradiction. Hence each column of A is in \mathcal{L} .

Now suppose there exists $\varepsilon > 0$ such that if $N > 0$, then there exists $q > N$ such that $\sum_{p=1}^{\infty} |a_{pq}| > \varepsilon$. Let $z \in \mathcal{L}$ be a subsequence of x that includes all zero terms of x . Let $j_1 = \min\{q: x_q \neq 0\}$. Let $N_1 > 0$ such that $\sum_{p=1}^{\infty} |a_{pN_1}| > \varepsilon$. Let $r_{N_1} = x_{j_1}$. Also let (r_1, \dots, r_{N_1-1}) be a subsequence of z such that $\sum_{q=1}^{N_1-1} |r_q| \sum_{p=1}^{\infty} |a_{pq}| < 1/2$ and $z_t = r_a$ only if for each $s < t$ such that $z_s = 0$ there exists $b < a$ such that $z_s = r_b$. Let $M_1 > 0$ such that

$$\sum_{p=1}^{M_1} |a_{pN_1}| > \frac{\varepsilon}{2} \quad \text{and} \quad |r_{N_1}| \sum_{p=M_1+1}^{\infty} |a_{pN_1}| < \frac{1}{4}.$$

Let $j_2 = \min\{q: x_q \in x \setminus (r_1, \dots, r_{N_1}), \text{ and } x_q \neq 0\}$. Since each row of A is null, there exists $N_2 > N_1 + 1$ such that $\sum_{p=M_1+1}^{\infty} |a_{pN_2}| > \varepsilon/2$ and $|x_{j_2}| \sum_{p=1}^{M_1} |a_{pN_2}| < 1/8$. Let $r_{N_2} = x_{j_2}$. Also let $(r_{N_1+1}, \dots, r_{N_2-1})$ be a subsequence of $z \setminus (r_1, \dots, r_{N_1}, r_{N_2})$ such that $\sum_{q=N_1+1}^{N_2-1} |r_q| \sum_{p=1}^{\infty} |a_{pq}| < 1/16$ and $z_t = r_a$ only if for each $s < t$ such that $z_s = 0$ there exists $b < a$ such that $z_s = r_b$. Let $M_2 > M_1$ such that $\sum_{p=M_2+1}^{M_2} |a_{pN_2}| > \varepsilon/2$ and $|r_{N_2}| \sum_{p=M_2+1}^{\infty} |a_{pN_2}| < 1/32$. This selection process may be continued so that if k is fixed, then

$$\begin{aligned} \sum_{p=1}^{M_k} \left| \sum_{q=1}^{\infty} a_{pq}r_q \right| &\geq \left(\sum_{p=1}^{M_1} |a_{pN_1}r_{N_1}| + \dots + \sum_{p=M_{k-1}+1}^{M_k} |a_{pN_k}r_{N_k}| \right) \\ &\quad - \left(\sum_{q=1}^{N_1-1} |r_q| \sum_{p=1}^{M_k} |a_{pq}| + \sum_{p=M_1+1}^{M_k} |a_{pN_1}r_{N_1}| \right) \\ &\quad + \sum_{q=N_1+1}^{N_2-1} |r_q| \sum_{p=1}^{M_k} |a_{pq}| + |r_{N_2}| \sum_{p=1}^{M_1} |a_{pN_2}| \\ &\quad + \sum_{p=M_2+1}^{M_k} |a_{pN_2}r_{N_2}| + \dots \Big) \geq \frac{\varepsilon}{2} \sum_{i=1}^k |r_{N_i}| - 1. \end{aligned}$$

But r has been selected so that $\lim_k \sum_{i=1}^k |r_{N_i}| = \infty$. Therefore $Ar \notin \mathcal{L}$, a contradiction. Hence $\lim_q \sum_{p=1}^{\infty} |a_{pq}| = 0$.

The proof of Theorem 2 is now complete, and Corollary 2, which was first proved by J. A. Fridy [5], follows directly.

COROLLARY 2. *The null sequence x is in \mathcal{L} if and only if there exists a sum-preserving $\mathcal{L} - \mathcal{L}$ matrix A such that $Ay \in \mathcal{L}$ for every rearrangement y of x .*

4. Examples. By Theorem 2 a matrix A that maps all rearrangements of a sequence $x \in c_0 \setminus \mathcal{L}$ into \mathcal{L} must be an $\mathcal{L} - \mathcal{L}$ matrix.

But Theorem 1 gives little insight into the question of whether A must be $\mathcal{C}-\mathcal{C}$ if it maps all subsequences of x into \mathcal{C} . The following example shows that A need not be $\mathcal{C}-\mathcal{C}$ in this case. Let $x_n = 1/n$ for $n = 1, 2, 3, \dots$; $a_{qq} = q^{1/3}$ for $q = 1, 8, 27, 64, \dots$; and $a_{pq} = 0$ otherwise. If y is a subsequence of x and $Ay = z$, then $|z_q| \leq q^{-2/3}$ for $q = 1, 8, 27, \dots$ and $z_q = 0$ otherwise. Thus $z \in \mathcal{C}$, but clearly $x \in c_0 \setminus \mathcal{C}$ and A is not $\mathcal{C}-\mathcal{C}$.

I. J. Maddox [7] showed that a matrix A is Schur if it maps all subsequences of some divergent sequence x into c . This might cause one to suspect that if A maps all subsequences (rearrangements) of a sequence $x \in c_0 \setminus \mathcal{C}$ into \mathcal{C} , then $Az \in \mathcal{C}$ for every $z \in cs$. The following example shows that this is not true. (The author wishes to thank the referee for his comments which aided in the simplification of this example.) Let $x_n = 1/n$ for $n = 1, 2, 3, \dots$; $a_{1q} = (-1)^q/q$ for $q \geq 1$ and $a_{pq} = 0$ otherwise. Since $(a_{1q})_{q=1}^\infty$ and x are both in \mathcal{C}^2 , each subsequence (rearrangement) y of x is also in \mathcal{C}^2 ; hence, $Ay \in \mathcal{C}$. But if z is defined by $z_q = (-1)^q/(\log(q+1))$ for each q , then $z \in cs$ and $(a_{1q}z_q)_{q=1}^\infty \notin cs$; thus, $Az \notin \mathcal{C}$.

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