Pacific Journal of Mathematics

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Vol. 63, No. 2 April 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 63, No. 2, 1976

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The purpose of this paper is to show that in certain uniform algebras all analytic maps (for the definition see § 2) of the open unit disk onto a nontrivial Gleason part are mutually closely related (Theorem 2), and that these maps are isometries of the open unit disk with pseudo-hyperbolic metric onto a nontrivial Gleason part with part metric (Theorem 3).

The results of this paper are contained in § 2. Some necessary preliminaries are given in § 1.

1. Preliminaries. A *uniform algebra* A on a compact Hausdorff space X is a uniformly closed subalgebra of the algebra $C(X)$ of complex valued continuous functions on X which contains the constants and separates the points of X. Let $\mathscr{M}(A)$ denote the maximal ideal space of A which has the Gelfand topology. Let $\hat{f}$ be the Gelfand transform of f in A and let $\hat{A} = \{\hat{f}: f \in A\}$.

For φ and θ in $\mathscr{M}(A)$ we define

$$(1.1) \qquad G: G(\varphi, \theta) = \sup\{|\varphi(f) - \theta(f)|: f \in A, \|f\| \leqq 1\},$$

$$(1.2) \qquad \sigma: \sigma(\varphi, \theta) = \sup\{|\varphi(f)|: f \in A, \|f\| \leqq 1, \theta(f) = 0\},$$

where $\|f\| = \sup\{|f(x)|: x \in X\}$, and we write $\varphi \sim \theta$ when $G(\varphi, \theta) < 2$ (or, equivalently, $\sigma(\varphi, \theta) < 1$). Then $\sim$ is an equivalence relation in $\mathscr{M}(A)$, and an equivalence class $P(m) = \{\varphi: \varphi \in \mathscr{M}(A), \varphi \sim m\}(\supsetneqq \{m\})$ is called the (*nontrivial*) *Gleason part* of $m \in \mathscr{M}(A)$ (cf. Gleason [2]). $G(\varphi, \theta)$ and $\sigma(\varphi, \theta)$ are metrics on $P(m)$ (cf. König [5]).

If $m \in \mathscr{M}(A)$ has a unique representing measure μ_m, i.e., if m has a unique probability measure μ_m on X such that $m(f) = \int f d\mu_m$ for every $f \in A$, then every φ in $P(m)$ also has a unique representing measure μ_φ. It is also known that φ in M_A belongs to $P(m)$ if and only if there exist mutually absolutely continuous representing measures μ_φ, μ_m for φ, m respectively; indeed, there exists a constant c $(0<c<1)$ such that $c\mu_\varphi \leqq \mu_m$ and $c\mu_m \leqq \mu_\varphi$.

For example, let $A(D)$ denote the disk algebra of all continuous functions on the closed unit disk $\bar{D} = \{s: |s| \leqq 1\}$ in the plane wich are analytic in the open unit disk D. Then $\mathscr{M}(A(D))$ can be identified with $\bar{D}$, and the open unit disk D is one part. For $t, s \in D$, we see that

$$(1.3)\qquad G(t, s) = \sup\{|f(t) - f(s)|: f \in A(D), \|f\| \leqq 1\},$$

$$(1.4)\qquad \sigma(t, s) = \sup\{|f(t)|: f \in A(D), \|f\| \leqq 1, f(s) = 0\} = \left|\frac{t-s}{1-\bar{s}t}\right| \quad \text{(pseudo-hyperbolic metric)}.$$

Throughout the rest of this paper, we do not distinguish in notations $\varphi \in \mathscr{M}(A)$ from its representing measure when φ has a unique representing measure, and we suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m. Let $A_m = \{f \in A; m(f) = 0\}$, the kernel of a complex homomorphism m. Let $H^\infty(m)$, H^∞_m be the weak-star closures of A, A_m in $L^\infty(m)$ respectively, and for $1 \leqq p < \infty$ let $H^p(m)$, H^p_m be the closures of A, A_m in $L^p(m)$ norm respectively. If φ belongs to $P(m)(\supsetneqq \{m\})$, then for $1 \leqq p < \infty$ the spaces $H^p(m)$ and $H^p(\varphi)$ are identical as sets of (equivalence classes of) functions; as Banach spaces, they have distinct but equivalent norms. Under the same hypothesis, the Banach algebras $H^\infty(m)$ and $H^\infty(\varphi)$ are identical.

For a Dirichlet algebra Wermer [7] showed the following theorem, and Hoffman [3] generalized Wermer's result to a logmodular algebra (cf. Browder [1], Chap. IV). Functions in $H^\infty(m)$ of unit modulus are called *inner functions*.

THEOREM 1 (WERMER'S EMBEDDING THEOREM). *Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Then we have the following.*

(1) *There is an inner function Z*(= Wermer's embedding function) *such that $ZH^2(m) = H^2_m$.*

(2) *If $\varphi \in P$, define $\hat{Z}(\varphi) = \int Z\, d\varphi$. Then $\hat{Z}$ is a one-one map of the part P onto the open unit disk D in the plane. The inverse map τ of $\hat{Z}$ is a one-one continuous map of D onto P* (with the Gelfand topology).

(3) *For every f in A the composed function $\hat{f}\circ\tau$ is analytic on D.*

Let φ be an element of the Gleason part $P(m)$ of m in $\mathscr{M}(A)$. Then there is a function h in $L^\infty(m)$ such that $\varphi(f) = \int f\, d\varphi = \int fh\, dm$ for all $f \in A$, so φ has a unique extension to a linear functional $\tilde{\varphi}$ on $H^\infty(m)$ which is both multiplicative and weak-star continuous. For any $f \in H^\infty(m)$ $\tilde{\varphi}$ has the form

$$\tilde{\varphi}(f) = \int f\, d\varphi = \int fh\, dm\,.$$

We call $\tilde{\varphi}$ the *measure extension* of φ in $P(m)$.

PROPOSITION. *Let $A, m, P(m)$ and Z be as in Wermer's embedding theorem. Let $\mathscr{P} = \mathscr{P}(m)$ be the set of measure extension $\tilde{\varphi}$ of φ in $P(m)$. Then we have the following.*

(1) *$\mathscr{P}$ is the nontrivial Gleason part of $\tilde{m}$ in $\mathscr{M}(H^\infty(m))$.*

(2) *The map $\hat{Z}|\mathscr{P}$ is a one-one continuous map of the part $\mathscr{P}$* (with the Gelfand topology) *onto the open unit disk D, and thus the inverse map $\tilde{\tau}$ of $\hat{Z}|\mathscr{P}$ is a homeomorphism of D onto $\mathscr{P}$.*

Proof. If the Gelfand transform $\hat{H}^\infty(m) = \hat{H}^\infty$ of $H^\infty(m)$ is restricted to the maximal ideal space Y of $L^\infty(m)$, then $\hat{H}^\infty$ is a logmodular algebra on Y (see Hoffman [3], Theorem 6.4, corollary), and therefore every complex homomorphism φ of $H^\infty(m)$ has a unique representing measure on Y (see [3], Theorem 4.2). In particular, $\tilde{m} \in \mathscr{M}(H^\infty(m))$ has a unique representing (normal) measure $\tilde{m}$ on the hyperstonean space Y. Then for every f in $L^\infty(m)$ we have

$$\int_X f\,dm = \int_Y \hat{f}\,d\tilde{m} .$$

And we can identify $L^\infty(\tilde{m})$ with $C(Y) = \hat{L}^\infty(m)$ (cf. Srinivasan-Wang [6], pp. 221-223).

Now let Σ be the Gleason part of $\tilde{m}$. For $\tilde{\varphi}$ in $\mathscr{P}$, we have

$$\tilde{\varphi}(f) = \int_X f\,d\varphi = \int_X fh\,dm = \int_Y \hat{f}\hat{h}\,d\tilde{m} \quad (f \in H^\infty(m)) , \tag{1.5}$$

where h is a function in $L^\infty(m)$ with $a < h < b$ for some positive constants a and b. From this we see that $\tilde{\varphi}$ is in Σ.

Conversely if λ is an element of Σ, then λ has a unique representing measure $\tilde{\lambda}$ on Y, and we have for every $f \in H^\infty(m)$

$$\lambda(f) = \int_Y \hat{f}\,d\tilde{\lambda} = \int_Y \hat{f}\frac{d\tilde{\lambda}}{d\tilde{m}}\,d\tilde{m} .$$

Since $d\tilde{\lambda}/d\tilde{m}$ is an element of $L^\infty(\tilde{m})$, there is a function h in $L^\infty(m)$ such that $d\tilde{\lambda}/d\tilde{m} = \hat{h}$ a.e. $(d\tilde{m})$. Hence we have

$$\lambda(f) = \int_Y \hat{f}\hat{h}\,d\tilde{m} = \int_X fh\,dm$$

and thus $\lambda \in \mathscr{P}$. So we get $\mathscr{P} = \Sigma$. Then the rest part (2) of the proposition follows from Theorem 1.

2. Results.

DEFINITION. Let $P(m)$ be the nontrivial Gleason part of m in the maximal ideal space $\mathscr{M}(A)$ of a uniform algebra A. A one-one continuous map $\rho(t)$ of the open unit disk D onto $P(m)$ (with the Gelfand topolgy) is called an *analytic map* if the composition $\hat{f}(\rho(t))$ is analytic on D, for every f in A.

Now we are in a position to prove the following theorem.

THEOREM 2. *Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Let $\tau(t)$ be an analytic map of the open unit disk D onto P which is obtained in Theorem 1. If $\rho(t)$ is an analytic map of D onto P such that $\rho(\alpha) = m$, then we have*

$$\rho(t) = \tau\left(\beta\frac{t-\alpha}{1-\bar{\alpha}t}\right), \tag{2.1}$$

where β is a constant of modulus 1. Furthermore, $\tau(t)$ is a homeomorphism if and only if $\rho(t)$ is a homeomorphism.

Proof. Let $\mathscr{P}$, Z, and $\tilde{\tau}$ be as in Theorem 1 and proposition. For any $t \in D$, $\rho(t)$ has a unique representing measure $h_t dm$, where h_t is an element of $L^\infty(m)$. Let $\tilde{\rho}(t)$ be the measure extension of $\rho(t)$ i.e., $\tilde{\rho}(t)(f) = \int f h_t dm$ for all $f \in H^\infty(m)$. For each $f \in H^\infty(m)$ there exists a sequence $\{f_n\}$ in A such that $\|f_n\| \leqq \|f\|$ for all n and $f_n \to f$ a.e. (dm) (Hoffman-Wermer theorem, see [1], Theorem 4.2.5). Then, by Lebesgue's dominant convergence theorem, $\rho(t)(f_n) = \int f_n h_t dm \to \tilde{\rho}(t)(f)$ for every t in D. Since $\rho(t)(f_n)(n = 1, 2, \cdots)$ are analytic in D and $|\rho(t)(f_n)| \leqq \|f_n\| \leqq \|f\|$, we see that, for every f in $H^\infty(m)$, $\tilde{\rho}(t)(f)$ is analytic in D (Vitali's theorem). Hence we see that $\tilde{\rho}(t)$ is an analytic map of D onto $\mathscr{P}$. If we set $g(t) = (\tilde{\tau}^{-1}\circ\tilde{\rho})(t) = \hat{Z}(\tilde{\rho}(t))$, then $f(t)$ is a one-one holomorphic map of D onto D, and $g(\alpha) = 0$. Hence we see

$$g(t) = \beta\frac{t-\alpha}{1-\bar{\alpha}t},$$

where β is a constant of modulus 1, and thus we have

$$\tilde{\tau}\left(\beta\frac{t-\alpha}{1-\bar{\alpha}t}\right) = \tilde{\rho}(t).$$

Since $\tilde{\tau}(t)|A = \tau(t)$ and $\tilde{\rho}(t)|A = \rho(t)$ we have

$$\tau\left(\beta\frac{t-\alpha}{1-\bar{\alpha}t}\right) = \rho(t).$$

Next we prove that $\tau(t)$ is a homeomorphism if and only if $\rho(t) = \tau(\beta(t-\alpha)/(1-\bar{\alpha}t))$ is a homeomorphism. We put $L_\alpha(t) = (t+\alpha)/(1+\bar{\alpha}t)$ and $\beta = e^{i\theta}$. Then $\tau(t)$ is a homeomorphism of D onto P if and only if $\hat{Z}(\varphi)\left(=\int Z d\varphi\right)$ is a continuous map of P onto D if and only if $L_\alpha \circ e^{-i\theta} \circ \hat{Z}$ is a continuous map of P onto D if and only if

$$(L_\alpha \circ e^{-i\theta} \circ \hat{Z})^{-1}(t) = \tau(e^{i\theta} L_{-\alpha}(t)) = \tau\left(e^{i\theta}\frac{t-\alpha}{1-\bar{\alpha}t}\right) = \rho(t)$$

is a homeomorphism, and the theorem is proved.

Next we shall prove the following theorem which generalizes a formula (6.12) in Hoffman [4], p. 105.

THEOREM 3. *Let A be a uniform algebra on X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. If $\rho(t)$ is an analytic map of the open unit disk D onto $P(m)$, then we have*

$$\sigma(\rho(t), \rho(s)) = \sigma(t, s)\ ,$$
$$G(\rho(t), \rho(s)) = G(t, s)\ .$$

For the definitions of σ, G see (1.1) $\sim$ (1.4).

Proof. Let Z, $\mathscr{P}$, τ and $\tilde{\tau}$ be as Theorem 1 and proposition. Let $\tilde{\tau}(t) = \tilde{\varphi}$, $\tilde{\tau}(s) = \tilde{\theta}$, $\tau(t) = \varphi$ and $\tau(s) = \theta$. From Lemma 4.4.4 in Browder [1], we see that

$$f \in H^\infty_{\tilde{\theta}} = \left\{f : f \in H^\infty(m) = H^\infty(\theta),\ \tilde{\theta}(f) = \int f\, d\theta = 0\right\}$$

if and only if $f \in (Z - s)H^\infty(m)$, and from this we easily get $H^\infty_{\tilde{\theta}} = \{(Z-s)/(1-\bar{s}Z)\}H^\infty(m)$. So we have

$$\begin{aligned}\sigma(\tilde{\varphi}, \tilde{\theta}) &= \sup\{|\tilde{\varphi}(f)|;\ f \in H^\infty(m),\ \|f\| \leqq 1,\ \tilde{\theta}(f) = 0\}\\ &= \sup\left\{|\tilde{\varphi}(f)| : f \in \frac{Z-s}{1-\bar{s}Z}H^\infty(m),\ \|f\| \leqq 1\right\}\\ &= \sup\left\{\left|\tilde{\varphi}\left(\frac{Z-s}{1-\bar{s}Z}\right)\tilde{\varphi}(g)\right| : g \in H^\infty(m),\ \|g\| \leqq 1\right\}\\ &= \left|\frac{t-s}{1-\bar{s}t}\right| = \sigma(t, s)\ .\end{aligned}$$

Since the closures of A_θ and $H^\infty_{\tilde{\theta}}$ in $L^2(m)$ are the same $H^2_\theta = \left\{f : f \in H^2(m) = H^2(\theta),\ \int f\, d\theta = 0\right\}$, we have the following equalities from the result which is stated as "the perhaps surprising equality" in

Browder [1], p. 134. (Note that $\tilde{\varphi}(f) = \int_Y \hat{f}\hat{h}d\tilde{m} = \int_Y \hat{f}d\tilde{\varphi}$ (see (1.5)) and $\int_X |f|^2 d\varphi = \int_Y |\hat{f}|^2 d\tilde{\varphi}$, for any $f \in H^\infty_{\tilde{\theta}}$.)

$$\begin{aligned}\sigma(\varphi, \theta) &= \sup\{|\varphi(f)|: f \in A_\theta, \|f\| \leqq 1\} \\ &= \sup\left\{|\varphi(f)|: f \in A_\theta, \int |f|^2 d\varphi \leqq 1\right\} \\ &= \sup\left\{|\varphi(f)|: f \in H^2_\theta, \int |f|^2 d\varphi \leqq 1\right\} \\ &= \sup\left\{|\tilde{\varphi}(f)|: f \in H^\infty_{\tilde{\theta}}, \int |f|^2 d\varphi \leqq 1\right\} \\ &= \sup\{|\tilde{\varphi}(f)|: f \in H^\infty_{\tilde{\theta}}, \|f\| \leqq 1\} \\ &= \sigma(\tilde{\varphi}, \tilde{\theta}) .\end{aligned}$$

Hence we have

$$\sigma(\tau(t), \tau(s)) = \sigma(\tilde{\tau}(t), \tilde{\tau}(s)) = \sigma(t, s) .$$

If $\rho(t)$ is an analytic map of D onto $P(m)$, then by Theorem 2 we have $\rho(t) = \tau(\beta(t - \alpha)/(1 - \bar{\alpha}t))$, where β is a constant of modulus 1. Therefore we have

$$\sigma(\rho(t), \rho(s)) = \sigma\left(\beta\frac{t - \alpha}{1 - \bar{\alpha}t}, \beta\frac{s - \alpha}{1 - \bar{\alpha}s}\right) = \sigma(t, s) .$$

The following equality is proved by König [5], which holds for φ, θ in the maximal ideal space $\mathscr{M}(A)$ of any uniform algebra A.

$$2 \log \frac{2 + G(\varphi, \theta)}{2 - G(\varphi, \theta)} = \log \frac{1 + \sigma(\varphi, \theta)}{1 - \sigma(\varphi, \theta)} .$$

Using this we get

$$G(\rho(t), \rho(s)) = G(t, s) .$$

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Received July 20, 1973.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 63, No. 2 April, 1976