ANALYTIC MAPS OF THE OPEN UNIT DISK ONTO A GLEASON PART

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The purpose of this paper is to show that in certain uniform algebras all analytic maps (for the definition see §2) of the open unit disk onto a nontrivial Gleason part are mutually closely related (Theorem 2), and that these maps are isometries of the open unit disk with pseudo-hyperbolic metric onto a nontrivial Gleason part with part metric (Theorem 3).

The results of this paper are contained in §2. Some necessary preliminaries are given in §1.

1. Preliminaries. A uniform algebra $A$ on a compact Hausdorff space $X$ is a uniformly closed subalgebra of the algebra $C(X)$ of complex valued continuous functions on $X$ which contains the constants and separates the points of $X$. Let $\mathcal{M}(A)$ denote the maximal ideal space of $A$ which has the Gelfand topology. Let $\hat{f}$ be the Gelfand transform of $f$ in $A$ and let $\hat{A} = \{\hat{f}: f \in A\}$.

For $\varphi$ and $\theta$ in $\mathcal{M}(A)$ we define

(1.1) $G: G(\varphi, \theta) = \sup \{||\varphi(f) - \theta(f)||: f \in A, ||f|| \leq 1\},$

(1.2) $\sigma: \sigma(\varphi, \theta) = \sup \{||\varphi(f)||: f \in A, ||f|| \leq 1, \theta(f) = 0\},$

where $||f|| = \sup \{||f(x)||: x \in X\}$, and we write $\varphi \sim \theta$ when $G(\varphi, \theta) < 2$ (or, equivalently, $\sigma(\varphi, \theta) < 1$). Then $\sim$ is an equivalence relation in $\mathcal{M}(A)$, and an equivalence class $P(m) = \{\varphi: \varphi \in \mathcal{M}(A), \varphi \sim m\}(\cong \{m\})$ is called the (nontrivial) Gleason part of $m \in \mathcal{M}(A)$ (cf. Gleason [2]). $G(\varphi, \theta)$ and $\sigma(\varphi, \theta)$ are metrics on $P(m)$ (cf. König [5]).

If $m \in \mathcal{M}(A)$ has a unique representing measure $\mu_m$, i.e., if $m$ has a unique probability measure $\mu_m$ on $X$ such that $m(f) = \int fd\mu_m$ for every $f \in A$, then every $\varphi$ in $P(m)$ also has a unique representing measure $\mu_\varphi$. It is also known that $\varphi$ in $M_A$ belongs to $P(m)$ if and only if there exist mutually absolutely continuous representing measures $\mu_\varphi, \mu_m$ for $\varphi, m$ respectively; indeed, there exists a constant $c$ $(0 < c < 1)$ such that $c\mu_\varphi \leq \mu_m$ and $c\mu_m \leq \mu_\varphi$.

For example, let $A(D)$ denote the disk algebra of all continuous functions on the closed unit disk $\bar{D} = \{s: |s| \leq 1\}$ in the plane which are analytic in the open unit disk $D$. Then $\mathcal{M}(A(D))$ can be identified with $\bar{D}$, and the open unit disk $D$ is one part. For $t, s \in D$, we see that
Throughout the rest of this paper, we do not distinguish in notations $\varphi \in \mathcal{M}(A)$ from its representing measure when $\varphi$ has a unique representing measure, and we suppose that $m \in \mathcal{M}(A)$ has a unique representing measure $m$. Let $A_m = \{f \in A; m(f) = 0\}$, the kernel of a complex homomorphism $m$. Let $H^\infty(m)$, $H^p_m$ be the weak-star closures of $A$, $A_m$ in $L^\infty(m)$ respectively, and for $1 \leq p < \infty$ let $H^p(m)$, $H^p_m$ be the closures of $A$, $A_m$ in $L^p(m)$ norm respectively. If $\varphi$ belongs to $P(m)\left(\mathbb{R} \{m\}\right)$, then for $1 \leq p < \infty$ the spaces $H^p(m)$ and $H^p(\varphi)$ are identical as sets of (equivalence classes of) functions; as Banach spaces, they have distinct but equivalent norms. Under the same hypothesis, the Banach algebras $H^\infty(m)$ and $H^\infty(\varphi)$ are identical.

For a Dirichlet algebra Wermer [7] showed the following theorem, and Hoffman [3] generalized Wermer's result to a logmodular algebra (cf. Browder [1], Chap. IV). Functions in $H^\infty(m)$ of unit modulus are called inner functions.

**THEOREM 1 (WERMER’S EMBEDDING THEOREM).** Let $A$ be a uniform algebra on a compact space $X$. Suppose that $m \in \mathcal{M}(A)$ has a unique representing measure $m$ on $X$, and that the part $P$ of $m$ consists of more than one point. Then we have the following.

1. There is an inner function $Z(=\text{Wermer's embedding function})$ such that $ZH^\infty(m) = H^\infty_m$.

2. If $\varphi \in P$, define $\tilde{Z}(\varphi) = \{Z \, d\varphi\}$. Then $\tilde{Z}$ is a one-one map of the part $P$ onto the open unit disk $D$ in the plane. The inverse map $\tau$ of $\tilde{Z}$ is a one-one continuous map of $D$ onto $P$ (with the Gelfand topology).

3. For every $f$ in $A$ the composed function $\tilde{f} \circ \tau$ is analytic on $D$.

Let $\varphi$ be an element of the Gleason part $P(m)$ of $m$ in $\mathcal{M}(A)$. Then there is a function $h$ in $L^p(m)$ such that $\varphi(f) = \int f \, d\varphi = \int f \, h \, dm$ for all $f \in A$, so $\varphi$ has a unique extension to a linear functional $\tilde{\varphi}$ on $H^\infty(m)$ which is both multiplicative and weak-star continuous. For any $f \in H^\infty(m)$ $\tilde{\varphi}$ has the form

$$\tilde{\varphi}(f) = \int f \, d\varphi = \int f \, h \, dm.$$
We call $\bar{\varphi}$ the measure extension of $\varphi$ in $P(m)$.

**Proposition.** Let $A, m, P(m)$ and $Z$ be as in Wermer's embedding theorem. Let $\mathcal{P} = \bar{P}(m)$ be the set of measure extension $\bar{\varphi}$ of $\varphi$ in $P(m)$. Then we have the following.

1. $\mathcal{P}$ is the nontrivial Gleason part of $\bar{m}$ in $\mathcal{M}(H^\infty(m))$.
2. The map $Z|\mathcal{P}$ is a one-one continuous map of the part $\mathcal{P}$ (with the Gelfand topology) onto the open unit disk $D$, and thus the inverse map $\tilde{\tau}$ of $Z|\mathcal{P}$ is a homeomorphism of $D$ onto $\mathcal{P}$.

**Proof.** If the Gelfand transform $\hat{H^\infty(m)} = \hat{H^\infty}$ of $H^\infty(m)$ is restricted to the maximal ideal space $Y$ of $L^\infty(m)$, then $\hat{H^\infty}$ is a logmodular algebra on $Y$ (see Hoffman [3], Theorem 6.4, corollary), and therefore every complex homomorphism $\varphi$ of $H^\infty(m)$ has a unique representing measure on $Y$ (see [3], Theorem 4.2). In particular, $\bar{m} \in \mathcal{M}(H^\infty(m))$ has a unique representing (normal) measure $\bar{m}$ on the hyperstonean space $Y$. Then for every $f$ in $L^\infty(m)$ we have

$$\int_X f \, dm = \int_Y \hat{f} \, d\bar{m}.$$

And we can identify $L^\infty(\bar{m})$ with $C(Y) = \hat{L^\infty(m)}$ (cf. Srinivasan-Wang [6], pp. 221–223).

Now let $\Sigma$ be the Gleason part of $\bar{m}$. For $\bar{\varphi}$ in $\mathcal{P}$, we have

$$\bar{\varphi}(f) = \int_X f \, d\varphi = \int_Y f \, h \, d\bar{m} = \int_Y \hat{f} \, d\bar{m} \quad (f \in H^\infty(m)),$$

where $h$ is a function in $L^\infty(m)$ with $a < h < b$ for some positive constants $a$ and $b$. From this we see that $\bar{\varphi}$ is in $\Sigma$.

Conversely if $\lambda$ is an element of $\Sigma$, then $\lambda$ has a unique representing measure $\hat{\lambda}$ on $Y$, and we have for every $f \in H^\infty(m)$

$$\lambda(f) = \int_Y \hat{f} \, d\hat{\lambda} = \int_Y \hat{f} \, \frac{d\hat{\lambda}}{d\bar{m}} \, d\bar{m}.$$

Since $d\hat{\lambda}/d\bar{m}$ is an element of $L^\infty(\bar{m})$, there is a function $\bar{h}$ in $L^\infty(m)$ such that $d\hat{\lambda}/d\bar{m} = \bar{h}$ a.e. $(d\bar{m})$. Hence we have

$$\lambda(f) = \int_Y \hat{f} \, d\bar{h} \bar{m} = \int_X f \, \bar{h} \, dm$$

and thus $\lambda \in \mathcal{P}$. So we get $\mathcal{P} = \Sigma$. Then the rest part (2) of the proposition follows from Theorem 1.

2. **Results.**

**Definition.** Let $P(m)$ be the nontrivial Gleason part of $m$ in
the maximal ideal space $\mathcal{M}(A)$ of a uniform algebra $A$. A one-one continuous map $\rho(t)$ of the open unit disk $D$ onto $P(m)$ (with the Gelfand topology) is called an analytic map if the composition $\hat{f}(\rho(t))$ is analytic on $D$, for every $f$ in $A$.

Now we are in a position to prove the following theorem.

**Theorem 2.** Let $A$ be a uniform algebra on a compact space $X$. Suppose that $m \in \mathcal{M}(A)$ has a unique representing measure $m$ on $X$, and that the part $P$ of $m$ consists of more than one point. Let $\tau(t)$ be an analytic map of the open unit disk $D$ onto $P$ which is obtained in Theorem 1. If $\rho(t)$ is an analytic map of $D$ onto $P$ such that $\rho(\alpha) = m$, then we have

$$\rho(t) = \tau(\beta \frac{t - \alpha}{1 - \alpha t}),$$

where $\beta$ is a constant of modulus 1. Furthermore, $\tau(t)$ is a homeomorphism if and only if $\rho(t)$ is a homeomorphism.

**Proof.** Let $\mathcal{P}$, $Z$, and $\bar{\tau}$ be as in Theorem 1 and proposition. For any $t \in D$, $\rho(t)$ has a unique representing measure $h_t dm$, where $h_t$ is an element of $L^\infty(m)$. Let $\rho(t)$ be the measure extension of $\rho(t)$ i.e., $\rho(t)(f) = \int f h_t dm$ for all $f \in H^\infty(m)$. For each $f \in H^\infty(m)$ there exists a sequence $\{f_n\}$ in $A$ such that $||f_n|| \leq ||f||$ for all $n$ and $f_n \rightarrow f$ a.e. (dm) (Hoffman-Wermer theorem, see [1], Theorem 4.2.5). Then, by Lebesgue's dominant convergence theorem, $\rho(t)(f_n) = \int f_n h_t dm \rightarrow \rho(t)(f)$ for every $t$ in $D$. Since $\rho(t)(f_n)(n = 1, 2, \cdots)$ are analytic in $D$ and $|\rho(t)(f_n)| \leq ||f_n|| \leq ||f||$, we see that, for every $f$ in $H^\infty(m)$, $\rho(t)(f)$ is analytic in $D$ (Vitali's theorem). Hence we see that $\rho(t)$ is an analytic map of $D$ onto $\mathcal{P}$. If we set $g(t) = (\bar{\tau}^{-1} \circ \rho)(t) = \Lambda(\rho(t))$, then $f(t)$ is a one-one holomorphic map of $D$ onto $D$, and $g(\alpha) = 0$. Hence we see

$$g(t) = \beta \frac{t - \alpha}{1 - \alpha t},$$

where $\beta$ is a constant of modulus 1, and thus we have

$$\bar{\tau}(\beta \frac{t - \alpha}{1 - \alpha t}) = \rho(t).$$

Since $\bar{\tau}(t) | A = \tau(t)$ and $\rho(t) | A = \rho(t)$ we have

$$\tau(\beta \frac{t - \alpha}{1 - \alpha t}) = \rho(t).$$
Next we prove that $\tau(t)$ is a homeomorphism if and only if $\rho(t) = \tau(\beta(t - \alpha)/(1 - \alpha t))$ is a homeomorphism. We put $L_\alpha(t) = (t + \alpha)/(1 + \alpha t)$ and $\beta = e^{i\theta}$. Then $\tau(t)$ is a homeomorphism of $D$ onto $P$ if and only if $Z(\phi) = \left[ Z \phi \right]$ is a continuous map of $P$ onto $D$ if and only if $L_\alpha e^{-i\theta} \tilde{Z}$ is a continuous map of $P$ onto $D$ if and only if

$$(L_\alpha e^{-i\theta} \tilde{Z})^{-1}(t) = \tau(e^{i\theta} L_\alpha(t)) = \tau\left(e^{i\theta} \frac{t - \alpha}{1 - \alpha t}\right) = \rho(t)$$

is a homeomorphism, and the theorem is proved.

Next we shall prove the following theorem which generalizes a formula (6.12) in Hoffman [4], p. 105.

**Theorem 3.** Let $A$ be a uniform algebra on $X$. Suppose that $m \in \mathcal{M}(A)$ has a unique representing measure $m$ on $X$, and that the part $P$ of $m$ consists of more than one point. If $\rho(t)$ is an analytic map of the open unit disk $D$ onto $P(m)$, then we have

$$\sigma(\rho(t), \rho(s)) = \sigma(t, s),$$
$$G(\rho(t), \rho(s)) = G(t, s).$$

For the definitions of $\sigma, G$ see (1.1) ~ (1.4).

**Proof.** Let $Z, \mathcal{P}, \tau$ and $\tau$ be as Theorem 1 and proposition. Let $\tau(t) = \phi, \tau(s) = \theta, \tau(t) = \psi$ and $\tau(s) = \theta$. From Lemma 4.4.4 in Browder [1], we see that

$$f \in H^\infty_\phi = \left\{ f : f \in H^\infty(m) = H^\infty(\theta), \tilde{\partial}(f) = f d\theta = 0 \right\}$$

if and only if $f \in (Z - s)H^\infty(m)$, and from this we easily get $H^\infty_\phi = \{(Z - s)/(1 - \bar{s}Z)\}H^\infty(m)$. So we have

$$\sigma(\phi, \theta) = \sup \{|\tilde{\phi}(f)|; f \in H^\infty(m), ||f|| \leq 1, \tilde{\partial}(f) = 0\}$$
$$= \sup \left\{ |\tilde{\phi}(f)|; f \in \frac{Z - s}{1 - \bar{s}Z}H^\infty(m), ||f|| \leq 1 \right\}$$
$$= \sup \left\{ |\tilde{\phi}(\frac{Z - s}{1 - \bar{s}Z})\psi(g)|; g \in H^\infty(m), ||g|| \leq 1 \right\}$$
$$= \left| \frac{t - s}{1 - \bar{s}t} \right| = \sigma(t, s).$$

Since the closures of $A_\phi$ and $H^\infty_\phi$ in $L^\infty(m)$ are the same $H^\infty_\phi = \left\{ f : f \in H^\infty(m) = H^\infty(\theta), \int f d\theta = 0 \right\}$, we have the following equalities from the result which is stated as “the perhaps surprising equality” in
Browder [1], p. 134. (Note that $\bar{\varphi}(f) = \int f \bar{\mu} = \int f \tilde{\varphi}$ (see (1.5)) and $\int_{x} \| f \|^2 d\varphi = \int f \| \tilde{\varphi}^2$, for any $f \in H^s$).

$$\sigma(\varphi, \theta) = \sup \{ \| \varphi(f) \| : f \in A, \| f \| \leq 1 \}$$
$$= \sup \{ \| \varphi(f) \| : f \in A, \int f \| ^2 d\varphi \leq 1 \}$$
$$= \sup \{ \| \varphi(f) \| : f \in H^s, \int f \| ^2 d\varphi \leq 1 \}$$
$$= \sup \{ \| \varphi(f) \| : f \in H^s, \| f \| \leq 1 \}$$
$$= \sigma(\tilde{\varphi}, \tilde{\theta}).$$

Hence we have

$$\sigma(\tau(t), \tau(s)) = \sigma(\tilde{\tau}(t), \tilde{\tau}(s)) = \sigma(t, s).$$

If $\rho(t)$ is an analytic map of $D$ onto $P(m)$, then by Theorem 2 we have $\rho(t) = \tau(\beta(t - \alpha)/(1 - \alpha t))$, where $\beta$ is a constant of modulus 1. Therefore we have

$$\sigma(\rho(t), \rho(s)) = \sigma(\tilde{\beta}(t - \alpha)/(1 - \alpha t), \tilde{\beta}(s - \alpha)/(1 - \alpha s)) = \sigma(t, s).$$

The following equality is proved by König [5], which holds for $\varphi, \theta$ in the maximal ideal space $\mathcal{M}(A)$ of any uniform algebra $A$.

$$2 \log \frac{2 + G(\varphi, \theta)}{2 - G(\varphi, \theta)} = \log \frac{1 + \sigma(\varphi, \theta)}{1 - \sigma(\varphi, \theta)}.$$

Using this we get

$$G(\rho(t), \rho(s)) = G(t, s).$$

REFERENCES


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Joseph Anthony Ball and Arthur R. Lubin, *On a class of contractive perturbations of restricted shifts* ................................................................. 309
Joseph Becker and William C. Brown, *On extending higher derivations generated by cup products to the integral closure* ........................................ 325
Andreas Blass, *Exact functors and measurable cardinals* ............................ 335
Joseph Eugene Collison, *A variance property for arithmetic functions* .......... 347
Craig McCormack Cordes, *Quadratic forms over nonformally real fields with a finite number of quaternion algebras* ........................................ 357
Freddy Delbaen, *Weakly compact sets in $H^1$* ........................................ 367
G. D. Dikshit, *Absolute Nörlund summability factors for Fourier series* ........ 371
Edward Richard Fadell, *Nielsen numbers as a homotopy type invariant* ....... 381
Josip Globevnik, *Analytic extensions of vector-valued functions* .................. 389
Robert Gold, *Genera in normal extensions* ............................................. 397
Solomon Wolf Golomb, *Formulas for the next prime* .................................. 401
Robert L. Griess, Jr., *The splitting of extensions of $SL(3,3)$ by the vector space $F_3^3$* ................................................................. 405
Thomas Alan Keagy, *Matrix transformations and absolute summability* ........ 411
Kazuo Kishi, *Analytic maps of the open unit disk onto a Gleason part* ......... 417
Kwangil Koh, Jiang Luh and Mohan S. Putcha, *On the associativity and commutativity of algebras over commutative rings* ............................... 423
James C. Lillo, *Asymptotic behavior of solutions of retarded differential difference equations* ................................................................. 431
John Alan MacBain, *Local and global bifurcation from normal eigenvalues* .... 445
Anna Maria Mantero, *Sets of uniqueness and multiplicity for $L^p$* ............... 467
J. F. McClendon, *Embedding metric families* ........................................... 481
L. Robbiano and Giuseppe Valla, *Primary powers of a prime ideal* ............. 491
Wolfgang Ruess, *Generalized inductive limit topologies and barrelledness properties* ................................................................. 499
Judith D. Sally, *Bounds for numbers of generators of Cohen-Macaulay ideals* .... 517
Helga Schirmer, *Mappings of polyhedra with prescribed fixed points and fixed point indices* ................................................................. 521
Cho Wei Sit, *Quotients of complete multipartite graphs* ............................... 531
S. Sznajder and Zbigniew Zielezny, *Solvability of convolution equations in $H^1_p$, $p > 1$* ................................................................. 539
Mitchell Herbert Taibleson, *The existence of natural field structures for finite dimensional vector spaces over local fields* ............................... 545
William Yslas Vélez, *A characterization of completely regular fields* ............ 553
P. S. Venkatesan, *On right unipotent semigroups* ..................................... 555
Kenneth S. Williams, *A rational octic reciprocity law* ............................... 563
Robert Ross Wilson, *Lattice orderings on the real field* ............................... 571
Harvey Eli Wolff, *V-localizations and V-monads. II* .................................. 579