ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF RETARDED DIFFERENTIAL DIFFERENCE EQUATIONS

JAMES C. LILLO
ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF RETARDED DIFFERENTIAL DIFFERENCE EQUATIONS

J. C. Lillo

The asymptotic behavior of the solutions of nonautonomous \( n^{th} \) order linear retarded differential difference equations is studied in this paper. It is shown that if the coefficients satisfy certain restrictions, then for any real \( K \) there exists a finite dimensional subspace \( F(K) \) of the solution space having the following property. For any solution \( x \) of the equation one has for all \( t > 0 \) that \( x(t) = x_k(t) + x_r(t) \) where \( x_k \) belongs to \( F(K) \) and \( x_r(t) = 0 (\exp(-Kt)) \) as \( t \to \infty \). As in the author's earlier papers, considering the periodic and almost periodic cases, the spaces \( F(K) \) are obtained by treating the nonautonomous equation as a perturbation of an \( n^{th} \) order autonomous equation.

1. Introduction and notation. We consider perturbations of the autonomous \( n^{th} \) order equation

\begin{equation}
L_0(x(t)) = x^{(n)}(t) + \sum_{k=0}^{m} \sum_{l=0}^{l(k)} c_{lk}x^{(l)}(t - \Delta_k) = 0
\end{equation}

where \( 0 = \Delta_0 < \Delta_1 < \Delta_m \) and the \( c_{lk} \), for all pairs \( (l, k) \) occurring in (1.1), are real numbers. We assume \( m \geq 1 \). We also assume that \( l(0) < n \) and that \( l(k) < n \), \( c_{l(k)k} \neq 0 \) for \( k = 1, \ldots, m \). The perturbed equations will be of the form

\begin{equation}
L_0(x(t)) = D(x(t))
\end{equation}

where

\begin{equation}
D(x(t)) = \sum_{g=0}^{2n} \sum_{h=0}^{g(h)} q_{g,h}(t)x^{(g)}(t - \sigma_h).
\end{equation}

Here it is assumed that the \( q_{g,h} \) belong to \( C^{2n}(-\infty, \infty) \) and there exists an \( M_1 > 0 \) such that

\begin{equation}
|q_{g,h}^{(j)}(t)| \leq M_1
\end{equation}

for \( t \in (-\infty, \infty) \), \( j \leq 2n \), and \((g, h) \in B\), where \( B \) denotes the set of all pairs \((g, h)\) in (1.3) for which \( q_{g,h}(t) \neq 0 \).

In earlier papers, the author has established, in the cases where the coefficients \( q_{g,h} \) are periodic [6] or almost periodic [7], that for \( K > 0 \) and sufficiently large there exist finite dimensional solution spaces \( F(K) \) of (1.2) possessing the following properties. Any so-
olution of (1.2) has a representation of the form

\[ x(t) = x_κ(t) + x_r(t). \]

Here \( x_κ \in F(K) \) and there exists an \( M(x) \) such that

\[ |x_r(t)| < M(x) \exp(-Kt) \quad \text{for all} \quad t \geq 0. \]

If \( H(K) \) denotes the corresponding subspace for (1.1) then there exists a sequence \( a_r, \lim a_r = -\infty \) such that

\[ n(a_r) = \dim H(-a_r) = \dim F(-a_r). \]

In §3 we shall extend these results to systems of the form (1.2) under certain additional restrictions on the perturbations (1.3). In order to state these restrictions we next introduce the notion of a distribution diagram for (1.1) and (1.3).

Associated with (1.1) we have the characteristic polynomials

\[ P(z) = \sum \sum c_{lk}z^l \exp(-A_kz) + z^n \]

and \( G(z) = P(z) \exp(A_mz) \) where \( \sum \sum \) denotes the double sum occurring in (1.1). For \( G(z) \) we define its distribution diagram [6] as follows. Let \( S(G) \) denote the set of points, in a Cartesian plane, consisting of the point \((A_m, n)\) and the points \((A_m - A_k, l)\) where \((l, k)\) is any pair of subscripts in (1.1) for which \( c_{lk} \neq 0\). We denote by \( p_j = (\beta_j, m_j), \ j = 0, \cdots, k, \) those points in \( S(G) \) which possess the following properties:

1. \( p_0 = (0, l(m)), \ p_k = (A_m, n) \)
2. \( \beta_j < \beta_{j+1} \) and \( m_j < m_{j+1} \) for \( j = 0, 1, \cdots, k - 1 \)
3. if \( l_j \) denotes the line segment connecting \( p_{j-1} \) and \( p_j \) then every point of \( S(G) \) lies on or below at least one of the \( l_j, \ j = 1, \cdots, k, \)
4. if \( u_j \) denotes the slope of \( l_j \) then \( u_1 > u_2, \cdots, > u_k. \)

The graph consisting of the line segments \( l_1, \cdots, l_k \) is referred to as the distribution diagram of \( G. \) For \( j = 1, \cdots, k \) we denote by \( G_j(z) \) those terms in \( G(z) \) which correspond to points in \( S(G) \) on the line segment \( l_j. \) Each of the polynomials \( G_j(z) \) may be factored

\[ G_j(z) = b_jz^{m_{j-1}} \exp(\beta_{j-1}z) \prod_{k=1}^{\sigma(j)} (z \exp(z/u_j) - \gamma^\beta(j,k)) \]

where \( \gamma^\beta_{jh} \neq \gamma^\beta_{ij} \) for \( i \neq h \) and \( b_j \) denotes the coefficient in \( G(z) \) of the term \( z^{m_j} \exp(\beta_j z). \) Since our estimates for \( |G(z)| \) depend directly on the numbers \( \beta(j, h) \) in (1.9) we associate with \( P(z) \) the following constant.
(1.10) \( \sigma(P) = \max_{j = 1, \ldots, k} \beta(j, h) \) for \( j = 1, \ldots, k \) and \( h \in [1, \ldots, a(j)] \).

**Definition.** Equation (1.1) is said to satisfy condition \( I \alpha \) if \( \sigma(P) = \alpha \) for its associated polynomial \( P(z) \) as defined in (2.1).

As mentioned earlier it is necessary to impose a restriction, condition \( \Pi \alpha \), on the perturbation term (1.3). In order to define this condition we associate with (1.3) a distribution diagram. Let

\[
F(z) = \exp(\Delta_m z) \sum_{h} r_{gh} z^h \exp(-\sigma_h z)
\]

where \( r_{gh} = 1 \) if \((g, h) \in B\), \( r_{gh} = 0 \) if \((g, h) \notin B\), and \( \sum\sum \) denotes the double sum in (1.3). As in the case of \( G(z) \) one now defines the set of points \( S(G) \) and then the distribution diagram of \( F \) which is also referred to as the distribution diagram associated with (1.3).

Let \( R(\sigma) \) denote the closed region bounded by the lines \( x = \sigma/u_k \), \( y = 0 \), \( x = \Delta_m \), and the line segments \( l_i(\sigma), \ldots, l_k(\sigma) \). Here \( l_j(\sigma) \) is the line segment joining the points \( p_j(\sigma) = (\beta_j + \sigma/u_k, m_j) \) and \( p_{j-1}(\sigma) = (\beta_{j-1} + \sigma/u_k, m_{j-1}) \) where \( p_j \) and \( p_{j-1} \) are the endpoints of \( l_j \) in the distribution diagram of \( G(z) \).

**Definition.** We say that (1.3) satisfies conditions \( \Pi \alpha \) for any real \( \alpha \) if its distribution diagram is contained in \( R(\sigma) \) for some \( \sigma > \alpha \).

For an intuitive discussion of condition \( \Pi \alpha \) the reader is referred to the author's earlier paper [6]. Before stating the estimates that we shall need in § 2 we introduce the following notation.

For any complex number \( z_i \) we have

\[
k(z_i) = | \exp(-z_i/u_k) | \quad \text{and} \quad l(z_i) = \{ z : \Re(z) = \Re(z_i) \} .
\]

For any pair \((g, h) \in B\) we define

\[
H(g, h, w, z) = \exp[ -\Delta_h(z - iw)](z - iw)^{i/p}(z - iw) .
\]

Let \( Z \) denote the set of zeros of \( P(z) \). In an earlier work [6] the author has shown that there exists a decreasing sequence of real numbers \( \{ a_r \} \), \( \lim a_r = -\infty \), \( a_0 = 2 \) lies to the right of \( Z \) and 1, and an \( M_r \) such that the strips \( S_r = \{ z : \Re(z - a_r) < M_r k^{-r}(a_r) \} \) do not contain any points of \( Z \). Using the estimates established in [6] one has the following results. If (1.1) satisfies \( I \alpha \) and (1.3) satisfies \( \Pi \gamma + r \), \( \gamma \geq 0 \), then there exists \( \beta > 0 \), \( H_r > 0 \) such that for any pair \((g, h) \in B, r \geq H_i \) and \( \delta_r = M_r k^{-r}(a_r)/4 \) one has the estimates

\[
H(g, h, 0, z) \leq k^{-\gamma}(a_r) \quad \text{for} \quad \Re z \in (a_r - \delta_r, a_r + \delta_r) = \alpha_r
\]
where the integrals in (1.16) and (1.17) are along any line \( l(z) \) for \( \Re z \in \alpha_r \). We also have that there exists an \( M \geq 0 \) such that for any \( \sigma \geq 0 \)

\[
|H(g, h, 0, z)| \leq M |z|^2 \quad \text{for } z \in l(a_0 + \sigma)
\]

(1.19)

\[
\left| \int H(g, h, 0, z)\,dz \right| \leq M/(a_0 + \sigma)
\]

(1.20)

where the integrals in (1.19) and (1.20) are along the line \( l(a_0 + \sigma) \).

2. The Greens function. In this section we shall obtain in Theorem 2.1 a representation result for the Greens function associated with equation (1.2). The proof of Theorem 2.1 consists of treating equation (1.2) as a perturbation of (1.1) and solving by successive approximations. To facilitate this discussion we introduce the following notation. Let \( G(t, s) \), called the Greens function, denote the function which vanishes for \( t < s \), satisfies (1.2) as a function of \( t \) for \( t > s \), \( G^{(n-1)}(s^+, s) - G^{(n-1)}(s^-, s) = 1 \) and \( G^{(j)}(t, s) \) is continuous in \( t \) for all \( j \leq n-2 \). Here \( f^{(j)}(t, s) \) denotes the \( j \)th partial derivative of \( f(t, s) \) with respect to \( t \). The \( j \)th partial derivative of \( f(t, s) \) with respect to \( s \) will be denoted by \( f^{(j)}(t, s) \). We shall denote by \( G(0, t, s) \) the Greens function for equation (1.1). Associated with \( G(0, t, s) \) we have

\[
G(0, t, s, r, 1) = \int_r \exp [z(t - s)] \mathcal{A}(z)dz \quad \text{for all values of } t ,
\]

(2.1)

\[
G(0, t, s, r, 2) = \int_{a_r} \exp [z(t - s)] \mathcal{A}(z)dz \quad \text{for } t > s ,
\]

(2.2)

where \( \mathcal{A}(z) = 1/P(z) \) and the integrals are defined as follows. The symbol \( \int_r \) denotes the line integral \( 1/(2\pi i) \int_{a_r + \infty i} \) and \( \int_{a_r} \) denotes the negative of this integral. The symbol \( \int_{a_r} \) denotes the sum of the two integrals \( \int_{a_r} \) and \( \int_{a_r} \) where the \( a_r \) are defined as in § 1. Since \( Z \) contains only a finite set of points to the right of \( l(a_r) \) it follows easily [1] that

\[
G(0, t, s) = G(0, t, s, r, 1) + G(0, t, s, r, 2)
\]

(2.3)
for all values of $t$. Here $G(0, t, s, r, 1)$ satisfies equation (1.1) for all values of $t$ and $G(0, t, s, r, 2)$ satisfies equation (1.1) for all $t > s$. It also follows [1] that for any $k \geq 0$ and $r$ sufficiently large that there exist constants $M(k, r)$ such that for $j \leq n - 1$

\[
|G^{(j)}(0, t, s, r, 1)| \leq M(k, r) \exp [b(t - s)]
\]
for $t \geq s$ and $b \in [a + k, a + k + 1]$

(2.4)

\[
|G^{(j)}(0, t, s, r, 1)| \leq M(k, r) \exp [\gamma(t - s)]
\]
for $t \leq s$ and $\gamma \in \alpha_r$

\[
|G^{(j)}(0, t, s, r, 2)| \leq M(k, r) \exp [\alpha(t - s)]
\]
for $t \geq s$ and $\alpha \in \alpha_r$.

Since $G(0, t, s, r, 1)$ and $G(0, t, s, r, 2)$ satisfy equation (1.1) for $t$ and $s$ in appropriate domains it follows that except for a finite set of points in $[0, (n + 1)\Delta u]$ the derivatives of order $j \leq 2n$ will exist and the constants $M(k, r)$ may be chosen so that one has for these higher derivatives the estimates (2.4) for the indicated values of $t$ and $s$.

We now establish, in Theorem 2.1, a similar representation result for $G(t, 0)$. For $i = 1, 2$ we define $G(0, t, r, i) \equiv G(0, t, 0, r, i)$. For $j \geq 1$ we define for all values of $t$

\[
G(j, t, r, 1) = \int_{-\infty}^{t} D(G(j - 1, w, r, 1))G(0, t - w, r, 2)dw
\]

(2.5)

\[
+ \int_{0}^{t} D(G(j - 1, w, r, 1))G(0, t - w, r, 1)dw
\]

\[
+ \int_{0}^{\infty} D(G(j - 1, w, r, 2))G(0, t - w, r, 1)dw
\]

and for $t > 0$

\[
G(j, t, r, 2) = \int_{\infty}^{t} D(G(j - 1, w, r, 2))G(0, t - w, r, 1)dw
\]

(2.6)

\[
+ \int_{0}^{-\infty} D(G(j - 1, w, r, 1))G(0, t - w, r, 2)dw
\]

\[
+ \int_{0}^{t} D(G(j - 1, w, r, 2))G(0, t - w, r, 2)dw.
\]

For $t \leq 0$, $j \geq 1$ we set $G(j, t, r, 2) = -G(j, t, r, 1)$. We then set

\[
G(t, r, 1) = \sum_{j=1}^{\infty} G(j, t, r, 1)
\]

(2.7)

\[
G(t, r, 2) = \sum_{j=0}^{\infty} G(j, t, r, 2).
\]

**Theorem 2.1.** If equation (1.1) satisfies $I_{a}$ and Equation (1.3)
satisfies $II_{a+2}$ then there exists an $H_i > 0$ such that for any $r \geq H_i$ the function $G(t, 0)$ can be written for all values of $t$

$$G(t, 0) = G(t, r, 1) + G(t, r, 2).$$

(2.8)

The function $G(t, r, 1)$ satisfies equation (1.2) for all values of $t$ and $G(t, r, 2)$ satisfies (1.2) for all $t > 0$. Furthermore there exists a constant $k$ and constants $N(k, r)$ for $r \geq H_i$ such that

$$|G^{(j)}(t, r, 1)| \leq N(k, r) \exp (bt)$$

for $t \geq 0$, $b \in [a_0 + k, a_0 + k + 1]$

(2.9)

$$|G^{(j)}(t, r, 1)| \leq N(k, r) \exp (\gamma t)$$

for $t \leq 0$, $\gamma \in [a_* - \delta_*/2, a_* + \delta_*/2] = \beta_r$

$$|G^{(j)}(t, r, 2)| \leq N(k, r) \exp (at)$$

for $t \geq 0$, $\alpha \in \beta_r$.

Proof. For any function $f$ we denote by $[f]^+$ the function which equals $f$ for $t \geq 0$ and is zero for $t < 0$. We denote by $[f]^-$ the function $f - [f]^+$. We denote by $||f||_1$ and $||f||_2$ the $L^1(-\infty, \infty)$ $L^2(-\infty, \infty)$ norms of $f$. We also introduce the functions

$$f(\gamma, m, h, g, r, j, t) \equiv \exp (-\gamma t)G^{(\varphi)}(m, t - \sigma_h, r, j).$$

Then using (2.4), (2.9), and (1.17) it follows for $\gamma \in \beta_r$, $r$ sufficiently large, and any pair $(g, h) \in B$

$$||[f(\gamma, 0, h, g, r, 1, -)]^-||_1 < k^{-\beta/2}(a_*)$$

(2.10)

$$||[f(\gamma, 0, h, g, r, 2, -)]^+||_1 < k^{-\beta/2}(a_*)$$

and the $L^i[-\infty, \infty]$ norms of the above functions are bounded by $k^{-\beta_2 - \delta_2/(a_*)}$. Similarly using (1.19) in place of (1.17) for any $k > 0$, $(g, h) \in B$ and $b \in [a_0 + k, a_0 + k + 1]$ it follows that

$$||[f(b, 0, h, g, 1, -)]^+||_1 < M/[a_0 + k]$$

(2.11)

and its $L^i[-\infty, \infty]$ norm is bounded by $M^{\nu_2}/(a_0 + k)^{\nu_2}$. Next we note that if $f \in L^i[-\infty, \infty]$, $g \in L^i[-\infty, \infty]$ then

$$h(y) = \int_{-\infty}^{\infty} f(y + x)g(x)dx$$

belongs to $L^i[-\infty, \infty]$ and

$$||h(y)|| \leq ||f||_2 ||g||_2$$

(2.12)

$$||h||_2 \leq ||f||_1 ||g||_2.$$ (2.13)

For $k = 1, 2$ let $C(k, m, \gamma, b)$ denote max ($||[f(\gamma, m, h, g, r, 2, \cdot)]^+||_k$,

$||[f(\gamma, m, h, g, r, 1, \cdot)]^-||_k$, $||[f(b, m, h, g, r, 1, \cdot)]^+||_k$ where the max is over all pairs $(g, h) \in B$, and $\gamma \in \beta_r.$
Then from (2.5), (2.6), (2.10), (2.13) we have for all \((g, h) \in B, m \geq 1\) that

\[
\| f(b, m, h, g, r, 1, \cdot) \|_2 \leq 2M_v(n - 3)C(2, m - 1, \gamma, b) [k^{-\beta/2}(a_r) + M/[a_0 + k]]
\]

\[
\| f(\gamma, m, h, g, r, 1, \cdot) \|_2 \leq 3M_v(n - 3)C(2, m - 1, r, b)k^{-\beta/2}(a_r)
\]

(2.14)

Thus for \(m \geq 1\) one has

\[
C(2, m, \gamma, b) \leq M_v(n - 3)[3k^{-\beta/2}(a_r) + 2M/[a_0 + k]C(2, m - 1, \gamma, b)
\]

(2.15)

Similarly from (2.12) we obtain the estimates for all values of \(t\)

\[
| f(b, m, h, g, r, 1, t) | \leq M_v(n - 3)C(2, m - 1, \gamma, b) \times [3k^{-\beta/2-3/2}(a_r) + 2M^{1/2}/[a_0 + b]^{3/2}]
\]

(2.16)

Thus it follows that if \(k\) and \(r\) are chosen so that

\[
M_v(n - 3)[3k^{-\beta/2}(a_r) + 2M/[a_0 + k]] < \frac{1}{2}
\]

(2.17)

then the series in (3.7) will converge uniformly and absolutely in every finite interval and for \(j = 0\) and \(\gamma = \alpha\) one has the estimates (2.9). For \(m = 0, j = 1, 2\) and \(r > H\), we note that the functions \(G(m, t, r, j)\) are independent of the choice of \(\gamma \in \beta_r\). By induction this then holds for \(m \geq 1\). Thus we have the estimates in (2.9) for the case \(j = 0\). It also follows that \(G(t, r, 2)\) and \(G(t, r, 1)\) satisfy the equations

\[
G(t, r, 1) = \int_{-\infty}^{t} D(G(w, r, 1))G(0, t - w, r, 2)dw
+ \int_{0}^{t} D(G(w, r, 1))G(0, t - w, r, 1)dw + \int_{0}^{\infty} D(G(w, r, 2))
\times G(0, t - w, r, 1)dw
\]

(2.18)

\[
G(t, r, 2) = \int_{-\infty}^{t} D(G(w, r, 2))G(0, t - w, r, 2)dw
+ \int_{0}^{-\infty} D(G(w, r, 1))G(0, t - w, r, 2)dw
\]

for \(t \geq 0\).
Now using the estimates (2.4) it follows from (2.18) that $G(t, r, 1)$ satisfies equation (1.2) for all values of $t$ and $G(t, r, 2)$ satisfies equation (1.2) for $t > 0$. It also follows from (2.18), the estimates for the terms $D(G(w, r, i))$ above, and (2.17) that the estimates given in (2.9) hold for $j \leq n - 1$ with $N(k, r) = 2M(k, r)$. This completes the proof of Theorem 2.1.

Associated with equations (1.1) and (1.2) we have the adjoint equations (2.19) and (2.20) given below. These equations are adjoint relative to the inner product given by (3.1) in the next section.

\[(2.19) \quad L_0^+(y(s)) = y^{(n)}(s) + \sum_{k} \sum_{i} (-1)^{i+n} c_{ikh} y^{(k)}(s + \Delta h)\]
\[(2.20) \quad L_0^+(y(s)) = D^+(y(s)) = \sum_{k} \sum_{i} (-1)^{k+n} [q_{sh}(s + \sigma h)y(s + \sigma h)]^{(s)} .\]

Now in place of the equations (3.5) and (3.6) we have for $j \geq 1$ and all value of $t$

\[
G(j, t, s, r, 1) = \int_{-\infty}^{t} D(G(j - 1, w, s, r, 1))G(0, t, w, r, 2)dw \\
+ \int_{s}^{t} D(G(j - 1, w, s, r, 1))G(0, t, w, r, 1)dw \\
+ \int_{s}^{\infty} D(G(j - 1, w, s, r, 2))G(0, t, w, r, 1)dw,
\]

and for $t \geq s$

\[
G(j, t, s, r, 2) = \int_{\infty}^{t} D(G(j - 1, w, s, r, 2))G(0, t, w, r, 1)dw \\
+ \int_{-\infty}^{s} D(G(j - 1, w, s, r, 1))G(0, t, w, r, 2)dw \\
+ \int_{s}^{\infty} D(G(j - 1, w, s, r, 2))G(0, t, w, r, 2)dw .
\]

For $t < s$ and $j \geq 1$ we set $G(j, t, s, r, 2) = -G(j, t, s, r, 1)$. Then defining

\[
G(t, s, r, 1) = \sum_{j=0}^{\infty} G(j, t, s, r, 1)
\]
\[
G(t, s, r, 2) = \sum_{j=0}^{\infty} (G(j, t, s, r, 2)
\]

we have the following theorem.

**Theorem 2.2.** If equation (1.1) satisfies $I_a$ and equation (1.3) satisfies $II_{a+2}$ for some $a$ then there exists $H_2 > 0$ such that for any $r \geq H_2$ the function $G(t, s)$ can be written for all value of $t, s$ in the form
\[ G(t, s) = G(t, s, r, 1) + G(t, s, r, 2) . \]

The function \( G(t, s, r, 1) \) satisfies equation (1.2) as a function of \( t \) and equation (2.20) as a function of \( s \) for all values of \( s \) and \( t \). The function \( G(t, s, r, 2) \) satisfies (1.2) as a function of \( t \) and (2.20) as a function of \( s \) for all \( t > s \). The mixed partials \( \partial^{i+j} G(t, s, r, 1) / \partial t^i \partial s^j \) are continuous and independent of order for all \( i \leq n - 1, j \leq n - 1 \), and all values of \( t \) and \( s \). Furthermore these partials satisfy estimates of the form (2.9) with appropriate constants \( N^*(k, r) \) and \( t \) replaced by \( t - s \) on the right side of the inequalities.

**Proof.** We first note that the change of variable \( t_1 = t - s \) converts the equations (2.21), (2.22) into equations of the form (2.18) where the coefficients used in defining \( D \) have been shifted. Since the estimates on the coefficients were uniform in \( t \) we have from Theorem 2.1 the assertions that \( G(t, s, r, 1) \) and \( G(t, s, r, 2) \) are solutions as functions of \( t \), of equation (1.2) and satisfy equations (2.25), (2.26) for the appropriate values of \( t \).

\[
G(t, s, r, 1) = \int_{-\infty}^t D(G(w, s, r, 1))G(0, t, w, r, 2)dw \\
+ \int_s^t D(G(w, s, r, 1))G(0, t, w, r, 1)dw \\
+ \int_s^\infty D(G(w, s, r, 2))G(0, t, w, r, 1)dw \\
(2.25)
\]

\[
G(t, s, r, 2) = \int_{-\infty}^t D(G(w, s, r, 2))G(0, t, w, r, 1)dw \\
+ \int_s^\infty D(G(w, s, r, 2))G(0, t, w, r, 1)dw \\
+ \int_s^t D(G(w, s, r, 2))G(0, t, w, r, 2)dw . \\
(2.26)
\]

The results of Theorem 2.1 also justify integrating the equations (2.21) and (2.22) by parts, obtaining

\[
G(j, t, s, r, 1) \\
= \int_{-\infty}^t \sum_{\sigma_h} G(j - 1, w - \sigma_h, s, r, 1)[q_{sh}(w)G(0, t, w, r, 2)]^{(p)}dw \\
+ \int_s^t \sum_{\sigma_h} G(j - 1, w - \sigma_h, s, r, 1)[q_{sh}(w)G(0, t, w, r, 1)]^{(p)}dw \\
+ \int_s^\infty \sum_{\sigma_h} G(j - 1, w - \sigma_h, s, r, 2)[q_{sh}(w)G(0, t, w, r, 1)]^{(p)}dw \\
(2.27)
\]

and
\[
G(j, t, s, r, 2) = \int_{-\infty}^{\infty} \sum_{\phi k} G(j - 1, w - \sigma_h, s, r, 2)[q_{\phi h}(w)G(0, t, w, r, 1)]^{(q)}dw
\]

(2.28)
\[
+ \int_{-\infty}^{t} \sum_{\phi k} G(j - 1, w - \sigma_h, s, r, 1)[q_{\phi h}(w)G(0, t, w, r, 2)]^{(q)}dw
\]
\[
+ \int_{t}^{\infty} \sum_{\phi k} G(j - 1, w - \sigma_h, s, r, 2)[q_{\phi h}(w)G(0, t, w, r, 2)]^{(q)}dw
\]

where \([ \ ]^{(q)}\) denotes the \(q\)th derivative with respect to \(w\). But then it follows from (2.27) and (2.28) for \(i = 1\) that

(2.29)
\[
L_0^+ G(i, t, s, r, 1) = D^+(G(i - 1, t, s, r, 1))
\]

for all values of \(t\) and \(s\), and for \(t \geq s\) that

(2.30)
\[
L_0^+ G(i, t, s, r, 2) = D^+(G(i - 1, t, s, r, 2))
\]

Then assuming the relations (2.29) and (2.30) hold for \(i = n\) one extends them to \(i = n + 1\). This is done by noting that due to the smoothness properties of \(G(n, t, s, r, k), k = 1, 2, \) and its derivatives for \(n \geq 1\) and the identity \(G(n, t, s, r, 2) = -G(n, t, s, r, 1)\) for \(t < s\) one may commute the operator \(L_0^+\) with the integral and summation signs in equations (2.27) and (2.28). Then by the induction hypothesis one obtains \(D^+ G(n - 2, w - \sigma_h, s, r, k)\) inside the integral as \(k = 1\) or 2 in the given integral. But then, for the reasons mentioned above, one is able to commute \(D^+\) with the integral and summation signs to obtain \(D^+ G(n - 1, w - \sigma_h, s, r, 1)\) in (2.27) and \(D^+ G(n - 1, w - \sigma_h, s, r, 2)\) in (2.28). This completes the desired induction. The uniform convergence results of Theorem 2.1 now establish the assertions that \(G(t, s, r, 1)\) and \(G(t, s, r, 2)\) satisfy equation (2.20) for \(t\) and \(s\) in the appropriate domains. The assertions concerning the mixed partial derivatives of \(G(t, s, r, 1)\) are obtained from the identity

\[
G(t, s, r, 1)
\]

(2.31)
\[
= \int_{-\infty}^{t} \sum_{\phi k} G(w - \sigma_h, s, r, 1)[q_{\phi h}(w)G(0, t, w, r, 2)]^{(q)}dw
\]
\[
+ \int_{t}^{\infty} \sum_{\phi k} G(w - \sigma_h, s, r, 1)[q_{\phi h}(w)G(0, t, w, r, 1)]^{(q)}dw
\]
\[
+ \int_{t}^{\infty} \sum_{\phi k} G(w - \sigma_h, s, r, 2)[q_{\phi h}(w)G(t, w, r, 1)]^{(q)}dw
\]

In light of the existence of the partials of \(G(t, s, r, k)\) and \(G(0, t, s, r, k)\) with respect to \(s\) and \(t\) and the fact that they satisfy bounds of the form (2.4), (2.9), the mixed partials \(\frac{\partial^{i+j}}{\partial t^i \partial s^j} \) are obtained by taking these partials inside the integrals along with the additional terms due to the simple discontinuities of \(G^{(n-1+k)}(0, t, w, r, 2)\) at the finite set.
of isolated points $t_{kj}$ in the interval $[0, (n + 1)\Delta m]$. These terms are
of the form $G^{(l)}(t_{kj} - \sigma_h, s, r, 1)$ multiplied by appropriate constants
where $l = g + i + j - (n - 1 + k)$. Thus the mixed partials possess
the stated properties. This completes the proof of Theorem 2.2.

We now denote by $K$ the max $g(h)$ for $h = 0, \cdots, v$. We then
have the following result.

**Theorem 2.3.** The conclusions of Theorem 2.2 remain valid if
the hypothesis that (1.3) satisfies $II_{a+2}$ is replaced by the hypothesis
that (1.3) satisfies $II_{a+1/2}$ and the assumption that for all $j \leq k$ and
pairs $(g, h) \in B$ one has that $q_{gh}^{(j)}$ are in $L[\infty, \infty]$.

**Proof.** Referring to the proof of Theorem 2.1 we note that in
obtaining the estimates (2.14) we used (2.13). The restriction $II_{a+2}$
was used in obtaining a bound for the $L^1$ norm of the terms playing
the role of $f$. But now using the fact that the $q_{gh} \in L[\infty, \infty]$ one
can obtain for the terms playing the role of $g$, in Theorem 2.1, an
estimate on their $L^1$ norm. Thus one needs only an estimate on the
$L^2$ norm of the terms playing the role of $f$ in the proof of Theorem
2.1. But for this it is sufficient to have (1.3) satisfy the condition
$II_{a+1/2}$. Thus the results of Theorem 2.1 remain valid if the $q_{gh} \in L[\infty, \infty]$ for all pairs $(g, h) \in B$. The additional hypothesis that
the $q_{gh}^{(j)} \in L[-\infty, \infty]$ for $j \leq k$ are needed in the proof of Theorem 2.2
where the adjoint equation is considered. In particular, they are
required for the representation given by equation (2.30). This com-
pletes the proof of Theorem 2.3.

3. Representation result. In this section we shall establish
the representation result given by equation (1.5) and described in
§1. This is done by showing that the function $G(t, s, r, 1)$, occurring
in Theorem 2.2 defines the finite dimensional subspace $F(-a_r)$ described
in §1. This result is then used to establish Theorem 3.2.

Let $C$ denote the space of functions having $n - 1$ continuous
derivatives on $[-A_0, 0]$ with the uniform norm $|| \cdot ||$ and $C^+$ denote
the space of functions having $n - 1$ continuous derivatives on $[0, A_0]$ along with the uniform norm. Then associated with equation (1.2)
and its adjoint equation (2.20) we have for every real number $\sigma$ and
$f \in C$, $d \in C^+$ the following inner product

$$[f, d, \sigma] = \sum_{i=1}^n (-1)^{i-1} f^{(n-i)}(0)d^{(i-1)}(0)$$

$$- \sum_{k} \sum_{il} \left\{ \int_0^{A_{k_j}} f^{(l)}(\xi)c_{ik}d(\xi + A_{k}) \right\}$$
(3.1)\[+ \sum_{i=1}^{l} (-1)^i f^{(i-l)}(0)c_{1i}d^{i-1}(\Delta h)\]
\[+ \sum_{h} \sum_{j} \left\{ \int_{-\sigma_h}^{\sigma_h} f^{(j)}(\zeta)q_{\sigma_h}(\sigma + \zeta + \sigma_h)d(\zeta + \sigma_h)d\zeta \right\} .
\]

Here we have written
\[\left[ q_{\sigma_h}(\sigma + \sigma_h)d(\sigma_h) \right]^{(i)} \text{ for } \frac{d^i[q_{\sigma_h}(\sigma + \sigma_h + \zeta)d(\zeta + \sigma_h)]}{d\zeta^i}\]
evaluated at \( \zeta = 0 \). If \( y \) denotes a solution of the adjoint equation (2.20) for \( s \leq s_0 \), then for any \( \sigma \leq s_0 \) we define \( y_\sigma \in C \) by the equation \( y_\sigma(\xi) = y(\sigma + \xi) \) for \( \xi \in [0, \Delta_m] \). Similarly if \( x \) is a solution of (1.2) for \( t \geq t_0 \) we denote by \( x_\sigma \), \( \sigma \geq t_0 \), the function \( x_\sigma \in C \) defined by \( x_\sigma(\xi) = x(\sigma + \xi) \) for \( \xi \in [-\Delta_m, 0] \). For any \( r, r \geq H_2 \) as given in Theorem 2.2, we define the linear operator \( E(r) \) on \( C \) by setting
\[(3.2)\]
\[E(r, f)(t) = [f, G_0(t, \cdot, r, 1), 0] \text{ for } t \in [-\Delta_m, 0].\]

We then have the following result.

**Theorem 3.1.** For any \( r \geq H_2 \) the linear operator \( E(r) \) defines a projection of \( C \) into \( C \). The range of \( E(r) \) has the dimension \( n(a_r) \) as defined in § 1.

**Proof.** We first establish the identity
\[(3.3)\]
\[G(t, s, r, 1) = [G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 1), 0]\]
for \( r \geq H_2 \) and all values of \( t \) and \( s \). Since the mixed partials of \( G(t, s, r, 1) \) of order up to \( 2n \) are continuous and since \( G(t, s, r, 1) \) is a solution of (1.2) and (2.20) it follows that the right side of (3.3) is a solution of (1.2), as a function of \( t \), and a solution of (2.20), as a function of \( s \) for all values of \( s \) and \( t \). We consider (3.3) for the case in which \( t > \Delta_m \). For these values of \( s \) and \( t \) [4] one has the identity
\[(3.4)\]
\[G(t, s, r, 1) = [G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 1), 0]\]
\[= [G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 1), 0] + [G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 2), 0].\]

Now it is known [4] that for all values of \( \sigma \leq 0 \)
\[(3.5)\]
\[[G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 2), \sigma]\]
is a constant which is independent of \( \sigma \). From Theorem 2.2 it follows
that there exists an $M > 0$ such that (3.5) is bounded by

$$M \exp \left[ (a_r - \delta_{r,2})(t - \sigma) \right] \exp \left[ (a_r + \delta_{r,2})(\sigma - s) \right].$$

Letting $\sigma \to -\infty$ we have that the expression in (3.5) is zero and so we obtain (3.3) for all $t > \Delta_m$. Using the fact that solutions of (1.2) have unique backward continuations [5], [4], the identity (3.3) follows for all values of $t$. Using the continuity of the mixed partials of $G(t, s, r, 1)$ as given in Theorem 2.2 and Fubini's theorem one may carry out the interchanges in the orders of integrations and differentiations [6] to obtain the identity

$$[[f, G_0(t, \cdot, r, 1), 0], G_0(\alpha, \cdot, r, 1), 0] = [f, [G_0(\cdot, s, r, 1), G_0(\alpha, \cdot, r, 1), 0], 0]$$

where $t$ varies in $[-\Delta_m, 0]$ and $s$ in $[0, \Delta_m]$. Thus we have that $E(r, E(r, f)) = E(r, f)$ and it follows that $E(r)$ defines a projection on $C$. In order to prove that the dimension of the range of $E(r)$ is $n(a_r)$ one introduces the one parameter family of equations

$$L_0(x(t)) = uD(x(t)) \quad \text{for} \quad 0 \leq u \leq 1.$$ 

Then from Theorem 2.2 one again obtains a function $G(t, s, r, 1, u)$ which for $u = 1$ reduces to $G(t, s, r, 1)$. As in [6], the inner product (3.1) is modified to $[f, d, \sigma, u]$ by replacing the functions $q_{gh}$ by the functions $uq_{gh}$. One again shows that

$$E(r, f, u)(t) = [f, G_0(t, \cdot, r, 1, u), 0, u]$$

defines a projection. It then follows from the proof of Theorems 2.1 and 2.2 that the mixed partials $G^j(t, s, r, 1, u)$ are continuous functions of $u$ for $|t| \leq 2\Delta_m$, $|s| \leq 2\Delta_m$, and $j \leq 2n$. Thus it follows that the norm of $E(r, u)$ is a continuous function of $u$ for $0 \leq u \leq 1$. Since the dimension of the range of $E(r, 0)$ is $n(a_r)$ it then follows by the usual arguments [6] that the dimension of the range of $E(r, 1)$ is also $n(a_r)$. This completes the proof of Theorem 3.1.

Let $C(r)$ denote the range of $E(r)$. Then since $C(r)$ has the dimension $n(a_r)$ we may select a basis $\xi_{r,j}$, $j = 1, \cdots, n(a_r)$ for $C(r)$. Thus for every $f \in C(r)$ we have a unique representation of the form

$$E(r, f) = \sum_{j=1}^{n(a_r)} a(f, j)\xi_{r,j}.$$ 

Now for $t > 0$ we have [4] for any solution $x$ of (1.2)

$$x(t, f, 0) = [f, G_0(t, \cdot), 0] = [f, G_0(t, \cdot, r, 1), 0]$$

$$+ [f, G_0(t, \cdot, r, 2), 0].$$
Since $G(t, s, r, 1)$ satisfies equation (1.2) for all values of $t$ it follows that $[f, G_0(t, \cdot, r, 1), 0]$ has a unique backward extension to $[-\Delta_m, 0]$. Thus it follows from (3.9) and (3.10) that for all $t > 0$

$$x(t, f, 0) = \sum_{j=1}^{n(a_r)} a(f, j)x(t, \zeta_{r_j}, 0) + x_r(t).$$

From the results of Theorem 2.2 we have that there exists a constant $k(r)$, depending only on $r$, such that for all $t > 0$

$$\|x_r\| \leq k(r) \|f\| \exp(a_r t).$$

Summarizing these results we have the following theorem.

**Theorem 3.2.** Under the hypothesis of Theorem 2.2 or Theorem 2.3 one has for every $r > H_2$ and every solution $x(t, f, 0)$ of (1.2) a unique representation of the form (3.11) where the remainder term $x_r$ satisfies the estimate (3.12).

We note finally that the hypothesis of Theorem 2.3 are more restrictive than those of Hale [3] and Cooke [2]. Thus in this case their results are applicable. These results assure us that each of the solutions $x(t, \zeta_{r_j}, 0)$ is asymptotic as $t \to \infty$ to a solution of (1.1).

**References**


Received May 28, 1975. This research was supported by National Science Grant GP-28392.

PURDUE UNIVERSITY