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EMBEDDING METRIC FAMILIES

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The embedding of a metric space in a Banach space plays an important role in metric space theory. In the present paper we consider the problem of embedding a metric family $X \to D$ in a Banach family. We obtain results under various hypotheses: (1) X a metric fiber bundle, (2) X an extended metric family, and (3) X has the coarse topology for a family of local cross-sections.

In §1 the basic definitions are given and a result is proved for metric fiber bundles. In §2 some general conditions are given which suffice for embedding. §3 studies family metrics which are restrictions of continuous pseudometrics. §4 describes the topology of a metric family in terms of a given family of local sections. In §5 a Banach family is associated with a given map and in §6 this is used to embed a locally sectioned family. In §7 an example is described relating to the question of embedding in a product family and also applying the techniques of §6 in a different way.

1. Definitions. In this section various definitions are given and the embedding question is posed. The question is answered in the case of metric fiber bundles.

Suppose $p: E \to D$ a function. Define $E_d = E(d) = p^{-1}(d)$, for $d \in D$, $E_S = E(S) = p^{-1}(S)$ for $S \subset D$, $E \times_D E = \{(e, e') \in E \times E \mid pe = pe'\}$. A continuous function will be called as map.

DEFINITION 1.1. A $\langle$continuous$\rangle$ [pseudo] metric family is a pair $(p: E \to D, m)$ where $p: E \to D$ is a map, $m: E \times_D E \to R$ is an upper semi-continuous $\langle$continuous$\rangle$ function, and $m \mid E(d) \times E(d)$ is [pseudo] metric.

Usually we speak of E as being a metric family rather than (p, m). Recall that a function $u: Z \to R$ from a topological space Z to the real numbers R is called upper semi-continuous provided that $u^{-1}(-\infty, b)$ is open for all $b \in R$. Note that the "metric family" of [2] is called a continuous metric family here.

Suppose $p: E \to D$ a map. A map $s: U \to E$ is called a local section of p if U is open in D and $ps =$ identity (on U). s gives

$$E(U) \xrightarrow[(1,\, sp)]{} E(U) \times_D E(U) \xrightarrow[m]{} R$$

if E is a metric family. So $B(s, r) = \{e \in E \mid m(e, spe) < r\}$ is open in

E since it is $(m\cdot(1, sp))^{-1}(-\infty, r)$. Let $\mathscr{S}$ be the family of all local sections of p and $\mathscr{B} = \mathscr{B}(\mathscr{S}) = \{B(s, r) \mid r > 0, s \in \mathscr{S}\}$. If

$$\bigcup \{s(d) \mid s \in \mathscr{S}\}$$

is dense in $E(d)$ then it follows as in [2, §2] (see, also, §4) that $\mathscr{B}$ is a basis of a topology on E. If this is the same as the given topology of E, we say that E is a coarse $\langle$continuous$\rangle$ [pseudo] metric family. The density condition is always assumed when the word "coarse" is used.

Let $p: E \to D$ be a map and suppose $E(d)$ a vector space and $a: E \times_D E \to E$, $a(e, e') = e + e'$, $b: R \times E \to E$, $b(c, e) = ce$, are both continuous. Then E is called a vector family. Suppose each $E(d)$ is a [pseudo] normed vector space with [pseudo] norm $n(d): E(d) \to R$. If $n: E \to R$, $n \mid E(d) = n(d)$, is an upper semi-continuous $\langle$continuous$\rangle$ function and the relative topology on $E(d)$ from E is the norm topology from $n(d)$, then E or $(p: E \to D, n)$ will be called a $\langle$continuous$\rangle$ [pseudo] normed vector family. Define $m: E \times_D E \to R$ by $m(e, e') = n(e - e')$. This makes E into a $\langle$continuous$\rangle$ [pseudo] metric family.

Let $p: A \to D$ and $q: B \to D$ be pseudo metric families. A map $f: A \to B$ with $qf = p$ will be called a D-map. It is an isometric embedding if it is a topological embedding (homeomorphism onto $f(A)$) and an isometry on each $A(d)$. We will consider the following question. When can a given metric family be embedded in a coarse normed vector family? There are some related questions, not all to be considered in the present paper. If such an embedding is possible, can the vector family be taken to be a product family? a vector bundle? Can a bound be put on the dimensions of the fibers? In the present paper we usually assume A is a coarse metric family. In a later paper we will consider cases where this is not true.

Suppose that M is a metric space with bounded metric m. Let $B(M)$ be the set of bounded maps $M \to R$ with norm

$$n(f) = \sup \{f(x) \mid x \in M\} .$$

Then $M \to B(M)$, $x \to m(x)$, $m(x)(y) = m(x, y)$, is an isometric embedding. If D is any space then $D \times M \to D \times B(M)$, $(d, x) \to (d, m(x))$, is an isometric embedding of continuous metric family $D \times M$ in the continuous normed vector family $D \times B(M)$.

More generally, let $X \to D$ be a metric fiber bundle (group action on fiber preserves metric) with fiber M and group G. Then $G \times B(M) \to B(M)$, $(g\cdot f)(x) = f(gx)$, gives an action of G on $B(M)$. Then we can form the associated bundle $B_D(X) \to D$, $B_D(X) = \bigcup B(X(d))$, there is a natural isometric embedding $X \to B_D(X)$ which in the fibers is

$M \to B(M)$ as above.

Note that for any metric family $X \to D$ it is possible to form the set $B_D(X)$ and get a $1-1$ function $X \to B_D(X)$. It would be interesting to know necessary and sufficient condtions, or even different sufficient conditions, under which $B_D(X)$ can be topologized in such a way as to make $X \to B_D(X)$ an embedding of the desired type.

2. Embedding conditions.

Assume that $(\hat{a}: A \to D, m)$ and $(\hat{b}: B \to D, m')$ are pseudo-metric families and $f: A \to B$ is a D-function. Here we consider some conditions on A, B, and f, and some easy consequences of them.

1. f is isometric, i.e., each $f(d)$ is isometric and f is $1-1$.
2. If s is a local section of A then fs is a local section of B.
3. $f(A)$ has a basis $\{B(fs, r) \mid r > 0,\ s$ a local section of $A\}$.
4. A has the coarse topology.
5. B has the coarse topology.

THEOREM 2.1. *If condidition* $1-4$ *are satisfied then* f *is an embedding.*

Proof. Note first that Condition 1 implies that f is injective and that the following two formulas are true

$$fB(s;\, r) = B(fs;\, r) \cap fA$$
$$f^{-1}B(fs;\, r) = B(s;\, r)$$

for any $s: S \to A$, $S \subset D$, $\hat{a}s = \mathrm{id}$. From 2, 3, and the second formula we see that f is continuous. Similarly from 2, 4, and the first formula we see that $f: A \to f(A)$ is open.

THEOREM 2.2. *If condition* 1, 2, 4, *and* 5, *are satisfied then* f *is an embedding.*

Proof. We need to show Condition 3. Let $b = f(a) \in B(t;\, r)$, $t: W \to B$ a local section. Let $\hat{a}a = d_0$, $m'(t(d_0), b) = r(1) < r(2) < r$. Select a local section $s: U \to A$, $d_0 \in U$, $m(a, sd_0) < c = \min\{r(2) - r(1), r - r(2)\}$. Then $m'(fsd_0, td_0) \leqq m'(fsd_0, fa) + m'(fa, td_0) = m(sd_0, a) + m'(b, td_0) < r(2) - r(1) + r(1) = r(2)$. So there is an open V, $d_0 \in V$, and $d \in V$ implies $m'(fsd, td) < r(2)$. I claim $b \in B(fs \mid V, c) \subset B(t;\, r)$ (which will prove 3). First $m'(fsd_0, b) = m'(fsd_0, fa) = m(sd_0, a) < r$. Now suppose $e \in B(fs \mid V, c)$, $\hat{b}e = d \in V$. Then $m'(e, t\hat{b}e) = m'(e, td) \leqq m'(e, fsd) + m'(fsd, td) < c + r(2) \leqq (r - r(2)) + r(2) = r$. Hence $e \in B(t;\, r)$.

3. Extended metric families.

In this section we will study

metric families which satisfy a strong extension condition and prove two embedding theorems for them.

DEFINITION 3.1. $(\hat{x}: X \to D, m)$ is an extended metric family if $\hat{x}$ is continuous and

(1) $m: X \times X \to R$ is a continuous pseudo metric

(2) $m | X \times_D X$ is a family metric.

Note that m is continuous if and only if it is upper semi-continuous. Also it is clear that an extended metric family is a continuous metric family.

If m is actually a bounded metric on X and X has the metric topology then $X \to B(X)$ is an embedding so $X \to D \times B(X)$ is an isometric embedding into a product family.

In general if we take m be bounded, as we can, then $v: X \to D \times B(X)$, $x \to (\hat{x}x, m(x))$, $m(x)(y) = m(x, y)$, is continuous and $1-1$ (since it is fiber preserving and $1-1$ on each fiber (since $m | X(d) \times X(d)$ is a metric)). This proves the following theorem.

THEOREM 3.2. *Suppose that $(X \to D, m)$ is an extended metric family. Suppose that X is compact and D is Hausdorff. Then v is an isometric embedding of X in the product family $D \times B(X)$.*

THEOREM 3.3. *Suppose that $(X \to D, m)$ is an extended metric family. Suppose also that it has the coarse topology. Then v is an isometric embedding of X in the product family $D \times B(X)$.*

Proof. This will follow from Theorem 2.2. We have that v is $1-1$, isometric, and continuous. This gives Conditions 1 and 2. Condition 4 is assumed and Condition 5 is true because $D \times B(X)$ is a product family.

4. Families of local sections.

Let X be a set, D a topological space, and $\hat{x}: X \to D$ a function. Suppose $m: X \times_D X \to R$ is a function such that the restriction is a [pseudo] metric on $X(d)$ for all $d \in D$. Let $\mathscr{S}$ be a family of local sections of x, i.e., functions $s: W \to X$ where $W = W(s)$ is open in D and $\hat{x}s =$ identity on W. Assume $\{s(d) | s \in \mathscr{S}\}$ is dense in $X(d)$ for all $d \in D$. For $s, s' \in \mathscr{S}$, define $u = u(s, s'): W(s) \cap W(s') \to R$ by $u(d) = m(s(d), s'(d))$. Also for $s \in \mathscr{S}$, W open in D, define $B(s; r) = B(s; r, m) = \{x \in X | m(x, s\hat{x}x) < r\}$. Write $B(s|W; r)$ for $B(t; r)$ where $t =$ restriction of s to $W \cap W(s)$, so $B(s|W; r) = B(s; r) \cap \hat{x}^{-1}(W)$. Let $\mathscr{B} = \mathscr{B}(\mathscr{S}, m) = \{B(s|W; r) | s \in \mathscr{S}, r > 0, W$ open in $D\}$. Theorems 4.1 and 4.2 below should be compared to [Fell, 1, Prop 1.6, p. 10].

THEOREM 4.1. *Assume* $u(s, s')$ *upper semi-continuous* $\langle$continuous$\rangle$ *for all* $s, s' \in \mathscr{S}$. *Then*

(1) $\mathscr{B}$ *is a basis for a topology* $\mathscr{T} = \mathscr{T}(\mathscr{S}, m)$ *on* X. $\hat{x}$ *is continuous and open.*

(2) *Each* s *is a continuous local section.*

(3) m *is upper semi-continuous* $\langle$continuous$\rangle$.

(4) $(X, \mathscr{S}, m)$ *is a coarse* [pseudo] $\langle$continuous$\rangle$ *metric family.*

(5) *Let* $t: U \to X$ *be a function,* $\hat{x}t = id, U$ *open in* D, *then*

(a) t *continuous implies* $u(s, t)$ *upper semi-continuous* $\langle$continuous$\rangle$ *all* $s \in \mathscr{S}$.

(b) $u(s, t)$ *upper semi-continuous all* $s \in \mathscr{S}$ *implies* t *is continuous.*

Proof of 1. Suppose $x \in B(s | W; r) \cap B(s' | W'; r')$, $\hat{x}x = d$, $m(sd, x) = r(1) < r(2) < r$, $m(s'd, x) = r(1)' < r(2)' < r'$. Let $a = \min(r(2) - r(1), r - r(2), r(2)' - r(1)', r' - r(2)')$. Select $t \in \mathscr{S}$ with $m(x, td) < a$. Let $u = u(s, t)$, $u' = u(s', t')$, $U = u^{-1}(-\infty, r(2)) \cap u'^{-1}(-\infty, r(2)') \cap W \cap W'$. It is not hard to show $x \in B(t | U; a) \subset B(s | W; r) \cap B(s'W'; r')$, (cf. proof of 2.2).

Proof of 2. $s^{-1}B(t | W; r) = \{d \,|\, m(s(d), t(d)) < r\} \cap W = u^{-1}(-\infty, r) \cap W$, $u = u(s, t)$. So $s \in \mathscr{S}$ implies s continuous.

Proof of 3. If m is upper semi-continuous use $a = -\infty$ in what follows. Let $(x', y') \in m^{-1}(a, b) = \{(x, y) \,|\, \hat{x}x = \hat{x}y,\ a < m(x, y) < b\}$. Let $m(x', y') = c$, $\hat{x}x = d' = \hat{x}y'$. Select r, a', and b', such that $a < a' - r < a' < c < b' < b' + r < b$, and set $z = \min((b' - c)/2, r/2, (c - a')/2)$. Select $t, t' \in \mathscr{S}$ with $m(x', td') < z$, $m(y', t'd') < z$. Let

$$W = u(t, t')^{-1}(a', b') .$$

It is not hard to check that $(x', y') \in B(tW; r/2) \times_D B(t' | W, r/2) \subset m^{-1}(a, b)$. So $m^{-1}(a, b)$ is open and m is upper semi-continuous [continuous].

Now 4 follows from 1, 2, and 3, and 5 is easy.

It is interesting to see from part 5 that if $u(s, s')$ is continuous for all $s, s' \in \mathscr{S}$ then if $t: U \to X$, $\hat{x}t = $ id, U open, we have that $u(s, t)$ is upper semi-continuous if and only if it is continuous. Theorem 4.1 permits us to complete a metric family and preserve whatever condit ions we had. Let $X(d)'$ be the completion of $X(d)$ and $i: X(d) \to X(d)'$ the embedding. Set $X' = \bigcup X(d)'$ giving

$$\begin{tikzcd} X \arrow[rr, "i"'] \arrow[dr, "\hat{x}"'] & & X' \arrow[dl, "\hat{x}'"] \\ & D & \end{tikzcd}$$

The [pseudo] metrics m'_d give $m'\colon X' \times_D X' \to R$. Let $\mathscr{S}' = \{is \mid s \in \mathscr{S}\}$. Then $\{s'(d) \mid s' \in \mathscr{S}'\}$ is dense in $X(d)'$. Also $u(s', t') = u(s, t)$ so the hypotheses on X carry over to X'. We give X' the topology from $\mathscr{S}'$. Conditions 1, 2, 4, 5, of §2 are clear so i is an isometric embedding.

Let E be a set, D a topological space, and $\hat{e}\colon E \to D$ a function. Assume $E(d)$ a [pseudo] normed real vector space. Let $\mathscr{S}$ be a given family of local sections of e. Let $s + s'$ and cs be defined by operating on values, $\operatorname{dom}(s + s') = \operatorname{dom}(s) \cap \operatorname{dom}(s')$. The function $\phi\colon \phi \to E$ is a local section. Suppose

(S1) $\{s(d) \mid s \in \mathscr{S}, d \in \operatorname{dom}(s)\}$ is dense in $E(d)$

(S2) $s, t \in \mathscr{S}$ imply $s + t \in \mathscr{S}$

(S3) $s \in \mathscr{S}$ implies $cs \in \mathscr{S}$ all $c \in R$

(S4) $s \in \mathscr{S}$ implies the function $d \to n(s(d))$ is upper semi-continuous ⟨continuous⟩. Define $m\colon E \times_D E \to R$ by $m(e, e') = n(e - e')$ and $\mathscr{B} = \mathscr{B}(\mathscr{S}) = \{B(s \mid W; r) \mid s \in \mathscr{S}, W$ open in $D, r > 0\}$. The following theorem should be compared to [Fell, 1, p. 10].

THEOREM 4.2 (1) *$\mathscr{B}$ is a basis for a topology on E. $\hat{e}$ is continuous and open.*

(2) *E is a* [pseudo] ⟨continuous⟩ *normed vector family.*

Proof. Let $s, t \in \mathscr{S}, u = u(s, t), u(d) = m(s(d), t(d)) = n((s - t)(d))$. By S2 and $S3$, $s - t \in \mathscr{S}$ and by S4, u is upper semi-continuous ⟨continuous⟩ so by 4.1 we see that $\mathscr{B}$ is a basis for a topology on E and $\hat{e}$ is continuous and open. Also each $s \in \mathscr{S}$ is a continuous cross section. It is clear that each $E(d)$ gets the [pseudo] norm topology. It remains only to show addition and scalor multiplication are continuous. This is not difficult (cf. proof of 2.2).

The completion process above gives for E as in Theorem 4.2 a D-map $E \to E'$ where $E' \to D$ is a [pseudo] ⟨continuous⟩ complete normed vector family = (definition) a [pseudo] ⟨continuous⟩ Banach family.

5. **A metric family associated with $X \to D$.** Recall that we can associate to any topological space Z the vector space $B_a(Z)$ of all bounded real valued functions with norm $M(b) = \sup\{|b(z)| \mid z \in Z\}$ and metric $d(b, b') = M(b - b')$. Let $B(z)$ be the subspace of continuous functions and $B_s(Z)$ the subspace generated by the upper semi-continuous functions. Below let $B(Z)$ stand for any of these.

Let $\hat{x}\colon X \to D$ be a map. Define $P(d) = \lim\{B(X(U)) \mid U \ni d, U$ open$\}$ as a set. So if $e \in P(d)$ then $e = [F](d)$ (equivalence class of F at d) where $F\colon X(U) \to R$ is a bounded function. Also $[F], [G] \in$

$P(d)$, $[F] = [G]$ iff $F | X(W) = G | X(W)$, some open W, $d \in W$. Now define $P(\hat{x}) = P(X) = \bigcup \{P(d) | d \in D\}$ as a set (a disjoint union). Let $F \in e \in P$ and define $n(e) = \inf \{M(F | X(U)) | d \in U,\ U \text{ open}\}$. For $F \in B(X(W))$ define $s = s(F): W \to E$ by $s(F)(d) = [F](d)$ and let $\mathscr{S} = \{s(F) | F \in B(X(W)),\ W \text{ open in } D\}$. Let $\mathscr{B} = \{B(s | W; r) | s \in \mathscr{S},\ r > 0,\ W \text{ open in } D\}$.

THEOREM 5.1. *$\mathscr{B}$ is a basis for a topology on $P = P(X) \cdot P \to D$ is a pseudo normed vector family.*

Proof. If $e \in P(d)$ then $e = [F](d) = s(F)(d)$, proving S1 of §4. $s + s' = s(F) + s(F') = s(F + F')$, $cs = cs(F) = s(cF)$ prove S2 and S3. Suppose $s = s(F)$ given and $u: W \to R$, $u(d) = n(s(d)) = n([F](d)) = \inf \{M(F | X(U)) | d \in U,\ U \text{ open}\}$. Suppose $u(d') < r$. Select V open, $d' \in V$, with $M(F | X(V)) < r$. Then for any

$$d \in V,\ n(s(d)) \leqq M(F | X(V)) < r .$$

Thus u is upper semi-continuous.

Now we can complete $P(X)$ to $P'(X)$ and form $N(X) = P'(X)/\mathscr{R}$ where $e\mathscr{R}0$ iff $n(e) = 0$. This gives $N(X) \to D$ a Banach family.

Note that in general n is only upper semi-continuous no matter which B is used (it may be continuous if $X \to D$ is nice).

The above process generalizes to treat $F_D(X, Y) \to D$ where $Y \to D$ is any normed vector family ($Y = D \times R$ above) or even any metric family.

6. **Embedding coarse metric families.** In this section we assume that $(X \to D, m)$ is a metric family with a local section through each point, X has the coarse topology, and m is bounded.

Let P, P', N, be the families constructed in §5. If $X \to D$ is a continuous metric family assume $B(X(U))$ was used. If $X \to D$ is only a metric family assume $B_s(X(U))$ was used. Write B in both cases. Let n denote the norm and K the metric for any of these families. For $F \in B(X(U))$ let $[F](d)$ be the equivalence class in $P(d)$ or $P'(d)$ and $\langle F \rangle(d)$ the class in $N(d)$. For a local section s: $U \to X$, define $m(s)$: $X(U) \to R$ by $m(s)(y) = m(y, s\hat{x}y)$ so $m(s) \in B(X(U))$. For $x \in X$ select a local section s through x and define $u(x) = \langle m(s) \rangle(d) \in N(d)$. The fact that u is single valued follows from the lemma below.

THEOREM 6.1. *$u: X \to N$ is an isometric embedding.*

LEMMA 6.2. *Let s and t be local sections and d in* dom (s) *and* dom (t). *Then*

$$K([m(s)](d), [m(t)](d)) = m(s(d), t(d)) .$$

Proof of 6.2.

$$K([m(s)](d), [m(t)](d)) = \inf_{U \ni d} \sup_{x \in X(U)} |(m(s) - m(t))(x)|$$
$$= \inf_{U \ni d} \sup_{d' \in U} \left(\sup_{x \in X(d')} |m(s)(x) - m(t)(x)| \right) .$$

But

$$\sup_{x \in X(d')} |m(s)(x) - m(t)(x)| = \sup_{x \in X(d')} |m(s\hat{x}x, x) - m(t\hat{x}x, x)| = m(sd', td')$$

and $d \longrightarrow m(sd, td)$ is upper semi-continuous, proving the result.

Proof of 6.1. Condition 1 of §2 follows from 6.2. The other conditions are also met so 6.1 follows from 2.2.

I'll give a result on embedding X in P'. The proof given seems to require an extra assumption on X. Let $a: P' \longrightarrow N$ be the natural projection. Define a multifunction $F: X \longrightarrow P'$ by $F = a^{-1}u$. Then $\hat{p}F = \hat{x}: X \longrightarrow D$ and F is lower semi-continuous since a is continuous. Furthermore, it is easily seen that $F(x)$ is convex and closed in $P'(d)$ for each x, $d = \hat{x}x$. Thus 5.1 of [2] applies and gives a continuous selection $u: X \longrightarrow P'$, provided that X is paracompact. u is isometric because v is. Thus the theorem below follows from 2.1.

THEOREM 6.3. *X paracompact. Then $u: X \longrightarrow P'$ is an isometric embedding.*

7. An example. Here an example will be described which illustrates the problem of embedding in a product family. It also shows that the embedding method of 6.3 applies in other situations.

Let $T = I \times R$ as a set where I is the unit interval. Define

$$N(x, y, r) = \begin{cases} ((x - r, x + r) \times (y - r, y + r)) \cap T & x \neq 0 \\ & \text{(usual nbhd.)} \\ T & x = 0,\ r > 1 \\ (0, y) \cup \{(x, y') \mid 0 < x < r, |y - y'| < xr & x = 0,\ r \leqq 1 . \end{cases}$$

These form a basis for a topology in T which is finer than the Euclidean topology. In fact, T is a well-known example of a space which is completely regular but not normal. $p: T \longrightarrow I$, $p(x, y) = x$, is continuous. Define $s(y): I \longrightarrow T$ by $s(y)(x) = (x, y)$. Then $s(y)$ is a continuous section. Now define $m: T \times_I T \longrightarrow R$ by

$$m(x, y, x, y') = \begin{cases} \min\{|y - y'|/x, 1\} & x \neq 0 \\ 1 & x = 0, y \neq y' \\ 0 & x = 0, y = y' . \end{cases}$$

It is not hard to see that m is continuous. In fact, $T \rightarrow I$ is a coarse continuous metric family with a global section through each point.

Now define $F(y): T \rightarrow R$ by $F(y)(x', y') = m((x', y'), s(y)(x')) = m(x', y', x', y)$. So, in the notation of §6, $F(y) = m(s(y))$. Define $u: T \rightarrow P'$ by $u(x, y) = [F(y)](x)$. Condition 1 of §2 follows just as in §6. Conditions 4 and 5 are clear. In the notation of §5, $us(y) = s(F(y))$ so us is a continuous section of P' establishing Condition 2. Now Theorem 2.2 shows that u is an isometric embedding. Note that T is not paracompact.

Finally we note that there is no embedding $T \rightarrow I \times M$ into a product family, M metric, since this would force T to be normal.

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