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SOLVABILITY OF CONVOLUTION EQUATIONS IN \mathcal{K}'_p , $p > 1$

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SOLVABILITY OF CONVOLUTION EQUATIONS IN $\mathcal{H}'_p, p > 1$

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Let S be a convolution operator in the space $\mathcal{H}'_p, p > 1$, of distributions in R^n growing no faster than $\exp(k|x|^p)$ for some k . A condition on S introduced by I. Cioranescu is proved to be equivalent to $S*\mathcal{H}'_p = \mathcal{H}'_p$.

We denote by $\mathcal{H}'_p, p > 1$, the space introduced in [4] and consisting of distributions in R^n which "grow" no faster than $\exp(k|x|^p)$, for some k .

I. Cioranescu [1] characterized distributions with compact support, i.e. in the space \mathcal{E}' , having fundamental solutions in \mathcal{H}'_p . We recall that a distribution E is a fundamental solution for $S \in \mathcal{E}'$ if

$$S * E = \delta,$$

where δ is the Dirac measure and $*$ denotes the convolution. Cioranescu proved that, if S is a distribution in \mathcal{E}' and \hat{S} its Fourier transform, the following conditions are equivalent:

(a) There exist positive constants A, N, C such that

$$\sup_{x \in R^n, |x| \leq A[1 + \log(2 + |\xi|)]^{1/q}} |\hat{S}(\xi + x)| \geq \frac{C}{(1 + |\xi|)^N}, \quad \xi \in R^n,$$

where $1/p + 1/q = 1$.

(b) S has a fundamental solution in \mathcal{H}'_p .

In this paper we study the solvability of convolution equations in \mathcal{H}'_p . If $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ is the space of convolution operators in \mathcal{H}'_p , we ask the question: Under what condition on $S \in \mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ is $S*\mathcal{H}'_p = \mathcal{H}'_p$? The last equation means that the mapping $u \rightarrow S*u$ of \mathcal{H}'_p into \mathcal{H}'_p is surjective.

We prove the following theorem which extends the results of Cioranescu mentioned above.

THEOREM. *If S is a distribution in $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ then each of the conditions (a) and (b) is equivalent to each of the following ones:*

(a') *There exist positive constants A', N', C' such that*

$$\sup_{z \in O^n, |z| \leq A'[1 + \log(2 + |\xi|)]^{1/q}} |\hat{S}(\xi + z)| \geq \frac{C'}{(1 + |\xi|)^{N'}}; \quad \xi \in R^n,$$

where $1/p + 1/q = 1$.

(c) $S*\mathcal{H}'_p = \mathcal{H}'_p$.

REMARK. For $p = 1$ a similar theorem was proved in [5].

Before presenting the proof we state the basic facts about the spaces \mathcal{H}'_p and $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$; for the proofs we refer to [4].

We denote by \mathcal{H}_p the space of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$v_k(\varphi) = \sup_{x \in \mathbb{R}^n, |\alpha| \leq k} e^{k|x|^p} |D^\alpha \varphi(x)| < \infty, \quad k = 0, 1, \dots,$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ and

$$D^\alpha = \left(\frac{1}{i} \frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{1}{i} \frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{1}{i} \frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The topology in \mathcal{H}_p is defined by the family of semi-norms v_k . Then \mathcal{H}_p becomes a Frechet space.

The dual \mathcal{H}'_p of \mathcal{H}_p is a space of distributions. A distribution u is in \mathcal{H}'_p if and only if there exists a multi-index α , an integer $k \geq 0$ and a bounded, continuous function f on \mathbb{R}^n such that

$$u = D^\alpha [e^{k|x|^p} f(x)].$$

If $u \in \mathcal{H}'_p$ and $\varphi \in \mathcal{H}_p$, then the convolution $u * \varphi$ is a function in $C^\infty(\mathbb{R}^n)$ defined by

$$u * \varphi(x) = \langle u, \varphi(x - y) \rangle,$$

where $\langle u, \varphi \rangle = u(\varphi)$.

The space $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ of convolution operators in \mathcal{H}'_p consists of distributions $S \in \mathcal{H}'_p$ satisfying one of the equivalent conditions:

(i) The products $S_x \exp[k(1 + |x|^2)^{p/2}]$, $k = 0, 1, \dots$, are tempered distributions

(ii) For every $k \geq 0$ there exists an integer $m \geq 0$ such that

$$S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha,$$

where f_α , $|\alpha| \leq m$, are continuous functions in \mathbb{R}^n whose products with $\exp(k|x|^p)$ are bounded

(iii) For every $\varphi \in \mathcal{H}_p$, the convolution $S * \varphi$ is in \mathcal{H}_p ; moreover, the mapping $\varphi \rightarrow S * \varphi$ of \mathcal{H}_p into \mathcal{H}_p is continuous.

If $S \in \mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ and \check{S} is the distribution in \mathcal{H}'_p defined by $\langle \check{S}, \varphi \rangle = \langle S_x, \varphi(-x) \rangle$, $\varphi \in \mathcal{H}_p$, then \check{S} is also in $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$. The convolution of S with $u \in \mathcal{H}'_p$ is then defined by

$$(1) \quad \langle S * u, \varphi \rangle = \langle u * S, \varphi \rangle = \langle u, \check{S} * \varphi \rangle, \quad \varphi \in \mathcal{H}_p.$$

For a function $\varphi \in \mathcal{H}_p$, the Fourier transform

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} \varphi(x) dx$$

can be continued in C^n as an entire function such that

$$w_k(\hat{\varphi}) = \sup_{\zeta \in C^n} (1 + |\xi|)^k e^{-|\eta|^{q/k}} |\hat{\varphi}(\zeta)| < \infty, \quad k = 1, 2, \dots,$$

where $\zeta = \xi + i\eta$. We denote by K_p the space of Fourier transforms of functions in \mathcal{H}_p . If the topology in K_p is defined by the family of semi-norms w_k , then the Fourier transformation is an isomorphism of \mathcal{H}_p onto K_p .

The dual K'_p of K_p is the space of Fourier transforms of distributions in \mathcal{H}'_p . The Fourier transform \hat{u} of a distribution $u \in \mathcal{H}'_p$ is defined by the Parseval formula

$$\langle \hat{u}, \hat{\varphi} \rangle = (2\pi)^n \langle u_x, \varphi(-x) \rangle.$$

For $S \in \mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$, the Fourier transform \hat{S} is a function which can be continued in C^n as an entire function having the following property: For every $k > 0$ there exist constants C'' and N'' such that

$$(2) \quad |\hat{S}(\xi + i\eta)| \leq C''(1 + |\xi|)^{N''} e^{|\eta|^{q/k}}.$$

Furthermore, if $S \in \mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ and $u \in \mathcal{H}'_p$, we have the formula

$$(3) \quad \widehat{S^*u} = \hat{S}\hat{u},$$

where the product on the right-hand side is defined in K'_p by $\langle \hat{S}\hat{u}, \psi \rangle = \langle \hat{u}, \hat{S}\psi \rangle$, $\psi \in K_p$.

In the proof of our theorem we shall make use of the following lemma of L. Hörmander (see [3], Lemma 3.2):

If F, G and F/G are entire functions and ρ is an arbitrary positive number, then

$$|F(\zeta)/G(\zeta)| \leq \sup_{|\zeta-z| < 4\rho} |F(z)| \sup_{|\zeta-z| < 4\rho} |G(z)| \left/ \left(\sup_{|\zeta-z| < \rho} |G(z)| \right)^2 \right.$$

where $\zeta, z \in C^n$.

Proof of the theorem. It is obvious that (a) \Rightarrow (a') and (c) \Rightarrow (b). The implication (b) \Rightarrow (a) was proved in [1] for $S \in \mathcal{E}'$. If $S \in \mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ the proof is the same and therefore we omit it. Our only task is to prove that (a') \Rightarrow (c).

Let S be a distribution in $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$ whose Fourier transform satisfies condition (a'), and let $T = \hat{S}$. Then the Fourier transform of T also satisfies condition (a'). We consider the mapping $S^*: u \rightarrow S^*u$ of \mathcal{H}'_p into \mathcal{H}'_p . By (1), it is the transpose of the mapping $T^*: \varphi \rightarrow T^*\varphi$ of \mathcal{H}_p into \mathcal{H}_p . In order to prove (c) it suffices to show that T^* is an isomorphism of \mathcal{H}_p onto $T^*\mathcal{H}_p$ (see [2], Corollary

on p. 92).

Since T is in $\mathcal{O}'_c(\mathcal{H}'_p: \mathcal{H}'_p)$, the mapping T^* is continuous, by (iii). Also, using Fourier transforms and formula (3), it is easy to see that T^* is injective. We now prove that the inverse of T^* , i.e. the mapping $T^*\varphi \rightarrow \varphi$, is continuous. Since the Fourier transformation is an isomorphism from \mathcal{H}'_p onto K_p , it suffices to prove the equivalent statement that the mapping $\hat{T}\hat{\varphi} \rightarrow \hat{\varphi}$ is continuous.

Suppose that

$$\hat{T}\hat{\varphi} = \hat{\psi},$$

where $\hat{\varphi}, \hat{\psi} \in K_p$. We recall that \hat{T} is an entire function satisfying condition (a') and estimates of the form (2). Given an arbitrary integer $k > 0$, we pick an integer k' such that

$$(4) \quad k' > (10^q + 1)k.$$

In view of (2), for k' there exist constants $N'', C'' > 0$ such that

$$|\hat{T}(\zeta)| \leq C''(1 + |\xi|)^{N''} e^{|\eta|^{q/k'}}, \quad \zeta = \xi + i\eta \in C^n.$$

Hence, setting

$$(5) \quad \rho = |\eta| + A'[\log(2 + |\xi|)]^{1/q}$$

and making use of the inequality

$$(a + b)^q \leq 2^q(a^q + b^q), \quad a, b \geq 0,$$

we obtain

$$(6) \quad \begin{aligned} \sup_{|\zeta-z| < 4\rho} |\hat{T}(z)| &= \sup_{|z| < 4\rho} |\hat{T}(\zeta + z)| \\ &\leq C''(1 + |\xi| + 4\rho)^{N''} e^{(|\eta| + 4\rho)^{q/k'}} \\ &\leq C_1(1 + |\xi|)^{N''}(1 + |\eta|)^{N''} e^{[(10|\eta|)^q + (8A')^q \log(2 + |\xi|)]^{1/q}/k'} \\ &\leq C_1(1 + |\xi|)^{N'' + (8A')^q/k'} e^{(10^q + 1)|\eta|^{q/k'}} \end{aligned}$$

where $z \in C^n$ and C_1, C'_1 are constants.

On the other hand

$$(7) \quad \begin{aligned} \sup_{|\zeta-z| < \rho} |\hat{T}(z)| &= \sup_{|z| < \rho} |\hat{T}(\zeta + z)| \geq \sup_{|z| < A'[\log(2 + |\xi|)]^{1/q}} |\hat{T}(\xi + z)| \\ &\geq \frac{C'}{(1 + |\xi|)^{N''}}, \end{aligned}$$

by condition (a').

Applying now to the functions $\hat{\psi}, \hat{T}$ and $\hat{\psi}/\hat{T} = \hat{\varphi}$ Hörmander's lemma with ρ given by (5) and making use of the estimates (6) and (7), we obtain

$$(8) \quad \begin{aligned} |\hat{\varphi}(\zeta)| &\leq \sup_{|\xi-z| < 4\rho} |\hat{\psi}(z)| \sup_{|\zeta-z| < 4\rho} |T(z)| \left/ \left(\sup_{|\zeta-z| < \rho} |T(z)| \right)^2 \right. \\ &\leq C_2(1 + |\xi|)^{2N'+N''+(8A')^q/k'} e^{(10q+1)|\eta|^q/k'} \sup_{|z| < 4\rho} |\hat{\psi}(\zeta+z)|, \end{aligned}$$

where C_2 is another constant. But, for any integer $l > 0$ and all $z = x + iy \in C^n$ with $|z| < 4\rho$, we have

$$(9) \quad \begin{aligned} |\hat{\psi}(\zeta+z)| &\leq w_l(\hat{\psi})(1 + |\xi+x|)^{-l} e^{|\eta+y|^q/l} \\ &\leq w_l(\hat{\psi})(1 + |x|)^l (1 + |\xi|)^{-l} e^{(|\eta|+|y|)^q/l} \\ &\leq w_l(\hat{\psi})(1 + 4\rho)^l (1 + |\xi|)^{-l} e^{(|\eta|+4\rho)^q/l} \\ &\leq C_3 w_l(\hat{\psi})(1 + |\eta|)^l (1 + |\xi|)^{-l} e^{[(10|\eta|)^q + (8A')^q \log(2+|\xi|)]/l} \\ &\leq C'_3 w_l(\hat{\psi})(1 + |\xi|)^{1-l+(8A')^q/l} e^{(10q+1)|\eta|^q/l}, \end{aligned}$$

where C_3 and C'_3 depend only on l and q . We choose the integer l so that

$$l > \max \left\{ k + 1 + 2N' + N'' + 2(8A')^q, (10^q + 1) \left/ \left(\frac{1}{k} - \frac{10^q + 1}{k'} \right) \right. \right\},$$

which is possible because of (4). Then

$$k + 1 + 2N' + N'' + (8A')^q \left(\frac{1}{k'} + \frac{1}{l} \right) - l < 0$$

and

$$(10^q + 1) \left(\frac{1}{k'} + \frac{1}{l} \right) - \frac{1}{k} < 0.$$

Consequently from (8) and (9) it follows that

$$w_k(\hat{\varphi}) \leq C_4 w_l(\hat{\psi}) = C_4 w_l(\hat{T}\hat{\varphi}),$$

for some C_4 independent of $\hat{\varphi}$. This proves the continuity of the mapping $\hat{T}\hat{\varphi} \rightarrow \hat{\varphi}$ and thus completes the proof of the implication (a') \Rightarrow (c).

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