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**SOLVABILITY OF CONVOLUTION EQUATIONS IN  $\mathcal{K}'_p$ ,  $p > 1$**

S. SZNAJDER AND ZBIGNIEW ZIELEZNY

# SOLVABILITY OF CONVOLUTION EQUATIONS IN $\mathcal{H}'_p$ , $p > 1$

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Let  $S$  be a convolution operator in the space  $\mathcal{H}'_p$ ,  $p > 1$ , of distributions in  $R^n$  growing no faster than  $\exp(k|x|^p)$  for some  $k$ . A condition on  $S$  introduced by I. Cioranescu is proved to be equivalent to  $S*\mathcal{H}'_p = \mathcal{H}'_p$ .

We denote by  $\mathcal{H}'_p$ ,  $p > 1$ , the space introduced in [4] and consisting of distributions in  $R^n$  which "grow" no faster than  $\exp(k|x|^p)$ , for some  $k$ .

I. Cioranescu [1] characterized distributions with compact support, i.e. in the space  $\mathcal{E}'$ , having fundamental solutions in  $\mathcal{H}'_p$ . We recall that a distribution  $E$  is a fundamental solution for  $S \in \mathcal{E}'$  if

$$S * E = \delta,$$

where  $\delta$  is the Dirac measure and  $*$  denotes the convolution. Cioranescu proved that, if  $S$  is a distribution in  $\mathcal{E}'$  and  $\hat{S}$  its Fourier transform, the following conditions are equivalent:

(a) There exist positive constants  $A, N, C$  such that

$$\sup_{x \in R^n, |x| \leq A[\log(2+|\xi|)]^{1/q}} |\hat{S}(\xi + x)| \geq \frac{C}{(1 + |\xi|)^N}, \quad \xi \in R^n,$$

where  $1/p + 1/q = 1$ .

(b)  $S$  has a fundamental solution in  $\mathcal{H}'_p$ .

In this paper we study the solvability of convolution equations in  $\mathcal{H}'_p$ . If  $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$  is the space of convolution operators in  $\mathcal{H}'_p$ , we ask the question: Under what condition on  $S \in \mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$  is  $S*\mathcal{H}'_p = \mathcal{H}'_p$ ? The last equation means that the mapping  $u \rightarrow S*u$  of  $\mathcal{H}'_p$  into  $\mathcal{H}'_p$  is surjective.

We prove the following theorem which extends the results of Cioranescu mentioned above.

**THEOREM.** *If  $S$  is a distribution in  $\mathcal{O}'_c(\mathcal{H}'_p; \mathcal{H}'_p)$  then each of the conditions (a) and (b) is equivalent to each of the following ones:*

(a') There exist positive constants  $A', N', C'$  such that

$$\sup_{z \in C^n, |z| \leq A'[\log(2+|\xi|)]^{1/q}} |\hat{S}(\xi + z)| \geq \frac{C'}{(1 + |\xi|)^{N'}}; \quad \xi \in R^n,$$

where  $1/p + 1/q = 1$ .

(c)  $S*\mathcal{H}'_p = \mathcal{H}'_p$ .

REMARK. For  $p = 1$  a similar theorem was proved in [5].

Before presenting the proof we state the basic facts about the spaces  $\mathcal{H}'_p$  and  $\mathcal{O}'_C(\mathcal{H}'_p: \mathcal{H}'_p)$ ; for the proofs we refer to [4].

We denote by  $\mathcal{H}_p$  the space of all functions  $\varphi \in C^\infty(R^n)$  such that

$$v_k(\varphi) = \sup_{x \in R^n, |\alpha| \leq k} e^{k|x|^p} |D^\alpha \varphi(x)| < \infty, \quad k = 0, 1, \dots,$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$  and

$$D^\alpha = \left(\frac{1}{i} \frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{1}{i} \frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{1}{i} \frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The topology in  $\mathcal{H}_p$  is defined by the family of semi-norms  $v_k$ . Then  $\mathcal{H}_p$  becomes a Frechet space.

The dual  $\mathcal{H}'_p$  of  $\mathcal{H}_p$  is a space of distributions. A distribution  $u$  is in  $\mathcal{H}'_p$  if and only if there exists a multi-index  $\alpha$ , an integer  $k \geq 0$  and a bounded, continuous function  $f$  on  $R^n$  such that

$$u = D^\alpha [e^{k|x|^p} f(x)].$$

If  $u \in \mathcal{H}'_p$  and  $\varphi \in \mathcal{H}_p$ , then the convolution  $u * \varphi$  is a function in  $C^\infty(R^n)$  defined by

$$u * \varphi(x) = \langle u_y, \varphi(x - y) \rangle,$$

where  $\langle u, \varphi \rangle = u(\varphi)$ .

The space  $\mathcal{O}'_C(\mathcal{H}'_p: \mathcal{H}'_p)$  of convolution operators in  $\mathcal{H}'_p$  consists of distributions  $S \in \mathcal{H}'_p$  satisfying one of the equivalent conditions:

(i) The products  $S_x \exp[k(1 + |x|^2)^{p/2}]$ ,  $k = 0, 1, \dots$ , are tempered distributions

(ii) For every  $k \geq 0$  there exists an integer  $m \geq 0$  such that

$$S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha,$$

where  $f_\alpha$ ,  $|\alpha| \leq m$ , are continuous functions in  $R^n$  whose products with  $\exp(k|x|^p)$  are bounded

(iii) For every  $\varphi \in \mathcal{H}_p$ , the convolution  $S * \varphi$  is in  $\mathcal{H}_p$ ; moreover, the mapping  $\varphi \rightarrow S * \varphi$  of  $\mathcal{H}_p$  into  $\mathcal{H}_p$  is continuous.

If  $S \in \mathcal{O}'_C(\mathcal{H}'_p: \mathcal{H}'_p)$  and  $\check{S}$  is the distribution in  $\mathcal{H}'_p$  defined by  $\langle \check{S}, \varphi \rangle = \langle S_x, \varphi(-x) \rangle$ ,  $\varphi \in \mathcal{H}_p$ , then  $\check{S}$  is also in  $\mathcal{O}'_C(\mathcal{H}'_p: \mathcal{H}'_p)$ . The convolution of  $S$  with  $u \in \mathcal{H}'_p$  is then defined by

$$(1) \quad \langle S * u, \varphi \rangle = \langle u * S, \varphi \rangle = \langle u, \check{S} * \varphi \rangle, \quad \varphi \in \mathcal{H}_p.$$

For a function  $\varphi \in \mathcal{H}_p$ , the Fourier transform

$$\hat{\varphi}(\xi) = \int_{R^n} e^{-i\langle \xi, x \rangle} \varphi(x) dx$$

can be continued in  $C^n$  as an entire function such that

$$w_k(\hat{\varphi}) = \sup_{\zeta \in C^n} (1 + |\xi|)^k e^{-|\gamma|^{q/k} |\hat{\varphi}(\zeta)|} < \infty, \quad k = 1, 2, \dots,$$

where  $\zeta = \xi + i\eta$ . We denote by  $K_p$  the space of Fourier transforms of functions in  $\mathcal{H}_p$ . If the topology in  $K_p$  is defined by the family of semi-norms  $w_k$ , then the Fourier transformation is an isomorphism of  $\mathcal{H}_p$  onto  $K_p$ .

The dual  $K'_p$  of  $K_p$  is the space of Fourier transforms of distributions in  $\mathcal{H}'_p$ . The Fourier transform  $\hat{u}$  of a distribution  $u \in \mathcal{H}'_p$  is defined by the Parseval formula

$$\langle \hat{u}, \hat{\varphi} \rangle = (2\pi)^n \langle u_x, \varphi(-x) \rangle.$$

For  $S \in \mathcal{O}'_c(\mathcal{H}'_p: \mathcal{H}'_p)$ , the Fourier transform  $\hat{S}$  is a function which can be continued in  $C^n$  as an entire function having the following property: For every  $k > 0$  there exist constants  $C''$  and  $N''$  such that

$$(2) \quad |\hat{S}(\xi + i\eta)| \leq C'' (1 + |\xi|)^{N''} e^{|\gamma|^{q/k}}.$$

Furthermore, if  $S \in \mathcal{O}'_c(\mathcal{H}'_p: \mathcal{H}'_p)$  and  $u \in \mathcal{H}'_p$ , we have the formula

$$(3) \quad \widehat{S^*u} = \hat{S}\hat{u},$$

where the product on the right-hand side is defined in  $K'_p$  by  $\langle \hat{S}\hat{u}, \psi \rangle = \langle \hat{u}, \hat{S}\psi \rangle$ ,  $\psi \in K_p$ .

In the proof of our theorem we shall make use of the following lemma of L. Hörmander (see [3], Lemma 3.2):

If  $F, G$  and  $F/G$  are entire functions and  $\rho$  is an arbitrary positive number, then

$$|F(\zeta)/G(\zeta)| \leq \sup_{|\zeta-z| < 4\rho} |F(z)| \sup_{|\zeta-z| < 4\rho} |G(z)| \left/ \left( \sup_{|\zeta-z| < \rho} |G(z)| \right)^2 \right.$$

where  $\zeta, z \in C^n$ .

*Proof of the theorem.* It is obvious that (a)  $\Rightarrow$  (a') and (c)  $\Rightarrow$  (b). The implication (b)  $\Rightarrow$  (a) was proved in [1] for  $S \in \mathcal{E}'$ . If  $S \in \mathcal{O}'_c(\mathcal{H}'_p: \mathcal{H}'_p)$  the proof is the same and therefore we omit it. Our only task is to prove that (a')  $\Rightarrow$  (c).

Let  $S$  be a distribution in  $\mathcal{O}'_c(\mathcal{H}'_p: \mathcal{H}'_p)$  whose Fourier transform satisfies condition (a'), and let  $T = \hat{S}$ . Then the Fourier transform of  $T$  also satisfies condition (a'). We consider the mapping  $S^*: u \rightarrow S^*u$  of  $\mathcal{H}'_p$  into  $\mathcal{H}'_p$ . By (1), it is the transpose of the mapping  $T^*: \varphi \rightarrow T^*\varphi$  of  $\mathcal{H}_p$  into  $\mathcal{H}_p$ . In order to prove (c) it suffices to show that  $T^*$  is an isomorphism of  $\mathcal{H}_p$  onto  $T^*\mathcal{H}_p$  (see [2], Corollary

on p. 92).

Since  $T$  is in  $\mathcal{O}'_c(\mathcal{K}_p': \mathcal{K}_p')$ , the mapping  $T^*$  is continuous, by (iii). Also, using Fourier transforms and formula (3), it is easy to see that  $T^*$  is injective. We now prove that the inverse of  $T^*$ , i.e. the mapping  $T^*\varphi \rightarrow \varphi$ , is continuous. Since the Fourier transformation is an isomorphism from  $\mathcal{K}_p$  onto  $K_p$ , it suffices to prove the equivalent statement that the mapping  $\hat{T}\hat{\varphi} \rightarrow \hat{\varphi}$  is continuous.

Suppose that

$$\hat{T}\hat{\varphi} = \hat{\psi},$$

where  $\hat{\varphi}, \hat{\psi} \in K_p$ . We recall that  $\hat{T}$  is an entire function satisfying condition (a') and estimates of the form (2). Given an arbitrary integer  $k > 0$ , we pick an integer  $k'$  such that

$$(4) \quad k' > (10^q + 1)k.$$

In view of (2), for  $k'$  there exist constants  $N'', C'' > 0$  such that

$$|\hat{T}(\zeta)| \leq C''(1 + |\xi|)^{N''} e^{|\eta|^{q/k'}}, \quad \zeta = \xi + i\eta \in C^n.$$

Hence, setting

$$(5) \quad \rho = |\eta| + A'[\log(2 + |\xi|)]^{1/q}$$

and making use of the inequality

$$(a + b)^q \leq 2^q(a^q + b^q), \quad a, b \geq 0,$$

we obtain

$$(6) \quad \begin{aligned} \sup_{|\zeta - z| < 4\rho} |\hat{T}(z)| &= \sup_{|z| < 4\rho} |\hat{T}(\zeta + z)| \\ &\leq C''(1 + |\xi| + 4\rho)^{N''} e^{(|\eta| + 4\rho)^{q/k'}} \\ &\leq C_1(1 + |\xi|)^{N''}(1 + |\eta|)^{N''} e^{[(10|\eta|)^q + (8A')^q \log(2 + |\xi|)]^{1/q}/k'} \\ &\leq C'_1(1 + |\xi|)^{N'' + (8A')^q/k'} e^{(10^q + 1)|\eta|^{q/k'}} \end{aligned}$$

where  $z \in C^n$  and  $C_1, C'_1$  are constants.

On the other hand

$$(7) \quad \begin{aligned} \sup_{|\zeta - z| < \rho} |\hat{T}(z)| &= \sup_{|z| < \rho} |\hat{T}(\zeta + z)| \geq \sup_{|z| < A'[\log(2 + |\xi|)]^{1/q}} |\hat{T}(\xi + z)| \\ &\geq \frac{C'}{(1 + |\xi|)^{N''}}, \end{aligned}$$

by condition (a').

Applying now to the functions  $\hat{\psi}, \hat{T}$  and  $\hat{\psi}/\hat{T} = \hat{\varphi}$  Hörmander's lemma with  $\rho$  given by (5) and making use of the estimates (6) and (7), we obtain

$$(8) \quad \begin{aligned} |\hat{\varphi}(\zeta)| &\leq \sup_{|\xi-z|<4\rho} |\hat{\psi}(z)| \sup_{|\zeta-z|<4\rho} |T(z)| \left/ \left( \sup_{|\zeta-z|<\rho} |T(z)| \right)^2 \right. \\ &\leq C_2(1 + |\xi|)^{2N'+N''+(8A')^q/k'} e^{(10q+1)|\eta|q/k'} \sup_{|z|<4\rho} |\hat{\psi}(\zeta+z)|, \end{aligned}$$

where  $C_2$  is another constant. But, for any integer  $l > 0$  and all  $z = x + iy \in C^n$  with  $|z| < 4\rho$ , we have

$$(9) \quad \begin{aligned} |\hat{\psi}(\zeta+z)| &\leq w_i(\hat{\psi})(1 + |\xi+x|)^{-l} e^{|\eta+y|q/l} \\ &\leq w_i(\hat{\psi})(1 + |x|)^l (1 + |\xi|)^{-l} e^{(|\eta|+|y|)q/l} \\ &\leq w_i(\hat{\psi})(1 + 4\rho)^l (1 + |\xi|)^{-l} e^{(|\eta|+4\rho)q/l} \\ &\leq C_3 w_i(\hat{\psi})(1 + |\eta|)^l (1 + |\xi|)^{-l} e^{[(10|\eta|)^q + (8A')^q \log(2+|\xi|)]/l} \\ &\leq C'_3 w_i(\hat{\psi})(1 + |\xi|)^{1-l+(8A')^q/l} e^{(10q+1)|\eta|q/l}, \end{aligned}$$

where  $C_3$  and  $C'_3$  depend only on  $l$  and  $q$ . We choose the integer  $l$  so that

$$l > \max \left\{ k + 1 + 2N' + N'' + 2(8A')^q, (10^q + 1) \left/ \left( \frac{1}{k} - \frac{10^q + 1}{k'} \right) \right. \right\},$$

which is possible because of (4). Then

$$k + 1 + 2N' + N'' + (8A')^q \left( \frac{1}{k'} + \frac{1}{l} \right) - l < 0$$

and

$$(10^q + 1) \left( \frac{1}{k'} + \frac{1}{l} \right) - \frac{1}{k} < 0.$$

Consequently from (8) and (9) it follows that

$$w_k(\hat{\varphi}) \leq C_4 w_i(\hat{\psi}) = C_4 w_i(\hat{T}\hat{\varphi}),$$

for some  $C_4$  independent of  $\hat{\varphi}$ . This proves the continuity of the mapping  $\hat{T}\hat{\varphi} \rightarrow \hat{\varphi}$  and thus completes the proof of the implication (a')  $\Rightarrow$  (c).

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