

Pacific Journal of Mathematics

A CHARACTERIZATION OF COMPLETELY REGULAR FIELDS

WILLIAM YSLAS VÉLEZ

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Dedicated to H. B. Mann on the occasion of his Seventieth Birthday

**We prove a theorem on tamely ramified extensions and
 apply this theorem to obtain a characterization of completely
 regular fields.**

Let Q_p denote the p -adic completion of Q , F a finite extension of Q_p , $e(F|Q_p) = a$, the ramification degree of F over Q_p , and ζ_p a primitive p^{th} root of unity.

We say that F is regular if $\zeta_p \notin F$.

When first studying the extension $F(\zeta_p)$ over F , it is a common error to assume that the extension $F(\zeta_p)$ over F is ramified. Or, to put it another way, that if K is an unramified extension of F , then K must also be regular.

On this question, Borevič, [1], has made the following definition:

Let $K \supset F$, $e(K|F) = 1$. If $\zeta_p \notin K$, for all such K , then we say that F is completely regular.

Using class field theoretic techniques, Borevič, [1], has given a characterization of completely regular fields. This characterization is a corollary to the following theorem.

THEOREM. *Let $L \cap F = L'$, where $e(L'|Q_p) = l$, degree $(L|L') = e(L|L') = f$, $(f, p) = 1$ and $d = (a/l, f)$. Then $L = L'(\pi^{1/f})$, where π is some prime element in L' and $e(F(\pi^{1/f})|F) = f/d$. Furthermore, if K is an unramified extension of F , then $e(K(\pi^{1/f})|K) = f/d$.*

Proof. Given $(a/l, f) = d$, we find an x such that $(x, f) = 1$ and $(a/l)x - yf = d$. To find x , set $d = d_1 d_2$, where d_2 is the largest divisor of d such that $(f_1, d_2) = 1$, where $f_1 = f/d$. Then $(f_1 d_1, d_2) = 1$. Let $(a/l d) \cdot x_1 \equiv 1 \pmod{f_1}$. Since $(f_1 d_1, d_2) = 1$, we can solve the system of congruences $x \equiv x_1 \pmod{f_1 d_1}$, $x \equiv 1 \pmod{d_2}$.

Since $(f, p) = 1$, we have that $L = L'(\pi^{1/f})$ (Weiss, page 89), where π is some prime element in L' . Then $\pi = \alpha \pi_1^{a/l}$, where π_1 is some prime element in F and α is a unit. Since $(f, x) = 1$, we have that

$$F(\pi^{1/f}) = F((\pi^x)^{1/f}) = F((\alpha^x \pi_1^d)^{1/f}) = F((\pi_1(\alpha^x)^{1/d})^{1/f_1}).$$

But $F((\alpha^x)^{1/d})$ is unramified over F and $F(\pi^{1/f})$ over $F((\alpha^x)^{1/d})$ is defined by the polynomial $x^{f_1} - (\alpha^x)^{1/d} \pi_1$, which is an Eisenstein polynomial.

Hence $e(F(\pi^{1/f})|F) = f/d$.

Let $e(K|F) = 1$. Then $e(K(\pi^{1/f})|F(\pi^{1/f})) = 1$. So we have that $e(K(\pi^{1/f})|F) = e(K(\pi^{1/f})|K) \cdot e(K|F) = e(K(\pi^{1/f})|K)$. But also $e(K(\pi^{1/f})|F) = e(K(\pi^{1/f})|F(\pi^{1/f})) \cdot e(F(\pi^{1/f})|F) = e(F(\pi^{1/f})|F)$. Hence $e(K(\pi^{1/f})|F) = e(K(\pi^{1/f})|K) = e(F(\pi^{1/f})|F) = f/d$.

COROLLARY. *The field F is completely regular iff $p - 1 \nmid e(F|Q_p)$.*

Proof. Let $\text{degree}(F \cap Q_p(\zeta_p)|Q_p) = l < p - 1$, with $fl = p - 1$, and

$$\text{degree}(Q_p(\zeta_p)|F \cap Q_p(\zeta_p)) = e(Q_p(\zeta_p)|F \cap Q_p(\zeta_p)) = f.$$

Then $(f, p) = 1$. Let $d = (a/l, f)$. Then $d = f$ iff $p - 1 | a$. If K is an unramified extension of F , then $e(K(\zeta_p)|K) > 1$ iff $p - 1 \nmid a$.

COROLLARY. *Let K be an unramified extension of F and $p - 1 | a$, then $\zeta_p \in K$ iff $(p - 1)/l | \text{degree}(K|F)$, where $l = \text{degree}(F \cap Q_p(\zeta_p)|Q_p)$.*

Proof. If $p - 1 | a$, then $d = f = (a/l, f)$. So $e(F(\zeta_p)|F) = 1$ and $\text{degree}(F(\zeta_p)|F) = (p - 1)/l$.

If $\zeta_p \in K$, then $K \supset F(\zeta_p) \supset F$. But $\text{degree}(F(\zeta_p)|F) = (p - 1)/l$, so $(p - 1)/l | \text{degree}(K|F)$.

Conversely, if $(p - 1)/l | \text{degree}(K|F)$, then since K over F has cyclic galois group, we have a field F_1 , $K \supset F_1 \supset F$, where $\text{degree}(F_1|F) = (p - 1)/l$ and $e(F_1|F) = 1$. But we have shown that $e(F(\zeta_p)|F) = 1$, and $\text{degree}(F(\zeta_p)|F) = (p - 1)/l$. Hence $F_1 = F(\zeta_p)$, since there is exactly one unramified extension for each degree. So $\zeta_p \in K$.

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Received February 3, 1976. The author was supported in part by a Fellowship from the Ford Foundation and by the U. S. Energy Research and Development Administration (ERDA). The author is presently at Sandia Laboratories.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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Manufactured and first issued in Japan

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