ON RIGHT UNIPOTENT SEMIGROUPS

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We investigate the implications of certain conditions on right unipotent semigroups. We describe the greatest idempotent-separating congruence $\beta$ on a right unipotent semigroup $S$. Necessary and sufficient conditions for (i) $S$ to be a union of groups, (ii) $S$ to be an inverse semigroup, (iii) the idempotents of $S$ to be in the centre of $S$ and (iv) the quotient semigroup $S/\beta$ to be isomorphic with the subsemigroup of idempotents of $S$ are also obtained.

It is known that any regular semigroup has the greatest idempotent-separating congruence [5, 6]. Such a congruence on an inverse semigroup was obtained by Howie [4]. For the general terminology and notation the reader is referred to [1, 2].

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1. Preliminary matters. An orthodox semigroup $S$ is a regular semigroup in which the idempotents form a subsemigroup. An inverse of an idempotent of $S$ is an idempotent, and if $a', b'$ are inverses of the elements $a, b$ in $S$ then $b'a'$ is an inverse of $ab$ [7].

A semigroup $S$ is called a right (left) unipotent semigroup if every principal right (left) ideal of $S$ has a unique idempotent generator. Such semigroups are called left (right) inverse by the author [9, 10]. Lemma 1 below is a part of the left-right dual of Theorem 1 in [10].

**Lemma 1.** Let $S$ be a regular semigroup. Then the following statements are equivalent.

(A) $ef = fe$ for any two idempotents $e, f$ in $S$.

(B) If $a'$ and $a''$ are inverses of the element $a$ in $S$ then $aa' = aa''$.

(C) $S$ is a right unipotent semigroup.

**Lemma 2.** Let $S$ be a right unipotent semigroup and $e$ be an idempotent of $S$. Let $x \in S$ and $x', x''$ be inverses of $x$. Then $xx'$ is an idempotent and $xx' = xx''$.

**Proof.** By Lemma 1 we have $xe = xx'xe = x(x'xx'x) = xex'x$. So $xx'$ is an idempotent. Also $xx' = (xex')xx' = (xex')xx'' = (xex'x)x'' = xex'$, using Lemma 1.
2. The statements $(Px)$, $(Qx)$ and $(Rx)$. Let $S$ be a right unipotent semigroup and $x \in S$. Through $E = E(S)$ denotes the subsemigroup of idempotents of $S$ and $V(x)$ denotes the set of inverses of the element $x$. The symbols $(Px)$, $(Qx)$ and $(Rx)$ stand for the statements indicated below.

$(Px)$ $exe = ex$ and $ex'e = ex'$ for all $e \in E$ and for at least one $x' \in V(x)$.

$(Qx)$ $xx' = xx' e$ for all $e \in E$ and $x' \in V(x)$.

$(Rx)$ $xx' = ex'$ for all $e \in E$ and $x' \in V(x)$.

REMARK. Let $S$ be a left unipotent semigroup. Then the left-right dual of $(Px)$, $(Qx)$ and $(Rx)$ are obtained by replacing respectively the equations in them by $xe = xe$ and $ex'e = x'e$, $x'exe = ex'x$ and $x'exe = x'xe$.

**Theorem 1.** Let $S$ be a right unipotent semigroup and $E = E(S)$. Then

1. $(Rx) \implies (Qx) \implies (Px)$ for any $x \in S$.
2. $E$ is contained in the center of $S$ if and only if $(Rx)$ is satisfied for all $x \in S$.

**Proof.** (1) Let $x \in S$ and $x' \in V(x)$.

Assume $(Rx)$. Then for any $e \in E$, we have $xx' = ex'$ and hence $e = (xx')x = xex'x = x(x'exe) = xe$ by Lemma 1. So $xx'e = xx'(ex)x' = xx'(xe)x' = xx'x = xe$ by Lemma 1. Therefore, $xx' = xx'e$ for any $e \in E$. Then $xx' = xx'e$ for any $e \in E$. Therefore, by Lemma 1, we get $exe = ex(x'xe) = e(xexe')x = e(xx'e)xe = ex'exe = ex'exe = ex'exe = exe = ex'exe = ex'exe = exe$.

(2) The only if part is trivial. If $xx'$ is for any $x \in S$ and $e \in E$, as shown above, $(Rx)$ implies $exe = xe$.

Let $S$ be a right unipotent semigroup. Then the statements (1) $S$ is union of groups, (2) each $H$-class of $S$ is a left group and (3) each $P$-class of $S$ is a group are equivalent [8]. An alternate characterization for $S$ to be a union of groups is obtained in the following

**Theorem 2.** Let $S$ be a right unipotent semigroup and $E = E(S)$. Then $S$ is a union of groups if and only if $(Px)$ is satisfied for all $x$ in $S$.

**Proof.** Let $S$ be a union of groups. Let $x \in S$ and $e \in E$. Let $x^{-1}$ be the inverse of $x$ in the group $H_x$. Then $x^{-1}x = xx^{-1}$. Let $a$ and $b$ respectively be the group inverses of $ex$ and $ex^{-1}$. As $x^{-1}e$ is
an inverse of \( ex \), and \( xe \) is an inverse of \( ex^{-1} \), by Lemma 1 we have 
\[
exa = exx^{-1}e \quad \text{and} \quad ex^{-1}b = ex^{-1}xe.
\]
But \( ex^{-1}e = ex^{-1} \) and \( ex^{-1}xe = exx^{-1}e = exx^{-1} \) by Lemma 1. So both \( ex \) and \( ex^{-1} \) and hence their product \( exex^{-1} \) belong to the group with identity element \( exx^{-1} \). As \( ex^{-1} \) is an idempotent we conclude that \( exex^{-1} = ex^{-1} \). Therefore 
\[
ex = ex(x^{-1}xe) = ex(x^{-1}xx^{-1}x) = (exex^{-1})x = exx^{-1}x = ex \quad \text{by Lemma 1.}
\]
Further since \( ex^{-1} \) belongs to the group with identity element \( exx^{-1} \), we have 
\[
ex^{-1} = ex^{-1}(exx^{-1}) = ex^{-1}(xx^{-1}xx^{-1}) = ex^{-1}(xx^{-1}e) = ex^{-1}e, \quad \text{by Lemma 1.}
\]
So we get \( (Px) \).

Conversely let \( (Px) \) be satisfied for all \( x \in S \). (This part of the proof holds for any regular semigroup \( S \)). Let \( x \in S \) and \( x' \in V(x) \). Taking \( e = xx' \) in \( ex = exe \) we have \( x = x^xx' \in x^3S \). So \( S \) is a right regular semigroup and hence a union of group \([1],[3] \).

Let \( S \) be a right unipotent semigroup. Then \( S \) is an inverse semigroup if and only if \( E \) satisfies the left-right dual of any of the statements of Lemma 1. We now obtain a necessary and sufficient condition in terms of \( (Px) \) and \( (Rx) \) for \( S \) to be an inverse semigroup.

**Theorem 3.** Let \( S \) be a right unipotent semigroup and \( E = E(S) \). Then \( S \) is an inverse semigroup if and only if \( (Px) \) implies \( (Rx) \) for all \( x \) in \( S \).

**Proof.** Let \( S \) be an inverse semigroup. Let \( x \in S \). Assume \( (Px) \). Then for any \( e \in E \) we have \( exe = ex \) and \( ex^{-1}e = ex^{-1} \). As the idempotents in \( S \) commute we get 
\[
exx^{-1} = (exe)x^{-1} = e(xex^{-1}) = (exe^{-1})e = x(ex^{-1}e) = xex^{-1}, \quad \text{giving} \ (Rx).
\]

Conversely let \( (Px) \) imply \( (Rx) \) for all \( x \in S \). Let \( g,h \in E \). Then for any \( e \in E \), by Lemma 1, we have \( e(gh)e = egh \). As \( gh \in V(gh) \), by hypothesis we conclude \( gheg = egh \). So, by Lemma 1, we get \( ghe = eg \). Taking \( e = h \), by Lemma 1, we have \( gh = hg \). Thus \( S \) is an inverse semigroup.

**Corollary.** Let \( S \) be a right unipotent semigroup and \( E = E(S) \). Then \( S \) is an inverse semigroup if and only if \( (Px) \), \( (Qx) \) and \( (Rx) \) are equivalent for all \( x \) in \( S \).

**Remark.** The left-right dual of Theorems 1, 2 and 3 hold for a left unipotent semigroup.

3. The congruences \( \alpha \) and \( \beta \). In this section we construct the greatest idempotent-separating congruence on a right (left) unipotent semigroup.

Theorems 4 and 7 below generalize known results for inverse semigroups [4]. In [6] Munn relates the greatest idempotent-
separating congruence on an inverse semigroup to a certain full inverse semigroup. We need the following

**Lemma 3.** Let $S$ be an orthodox semigroup and $\sigma$ be an idempotent-separating congruence on $S$. If $(x, y) \in \sigma$ then there exist $u \in V(x)$ and $v \in V(y)$ such that $(u, v) \in \sigma$.

**Proof.** Let $(x, y) \in \sigma$, $x' \in V(x)$ and $y' \in V(y)$. Since $\sigma$ is a congruence we get $(x'x, x'y) \in \sigma$ and hence $(x'xy'y, x'y) \in \sigma$. By transitivity of $\sigma$ we conclude $(x'xy'y, x'x) \in \sigma$. This, since $\sigma$ is idempotent-separating, implies $x'xy'y = x'x$. So $xy'y = x$. Similarly we get $xx'y = yy' \land xx'y = y$.

Set $u = y'yx'$ and $v = y'yx'$. Then $u \in V(x)$ and $v \in V(y)$. Now from $(x, y) \in \sigma$ we have $(y'xx', y'yx') \in \sigma$, that is $(v, u) \in \sigma$ and thus $(u, v) \in \sigma$. Hence the lemma.

Let $S$ be a regular semigroup and $E$ be the set of idempotents of $S$. Define the binary relations $\alpha$ and $\beta$ on $S$ thus:

$$\alpha = \{(x, y) \in S \times S : x'\alpha = y'\beta \text{ for all } e \in E, x' \in V(x) \text{ and } y' \in V(y)\}.$$

$$\beta = \{(x, y) \in S \times S : xex = yey \text{ for all } e \in E, x' \in V(x) \text{ and } y' \in V(y)\}.$$

**Theorem 4.** Let $S$ be a right (left) unipotent semigroup and $E = E(S)$. Then $\beta(\alpha)$ is an idempotent-separating congruence on $S$. Further, if $\sigma$ is any idempotent-separating congruence on $S$ then $\sigma \subseteq \beta(\sigma \subseteq \alpha)$.

**Proof.** We prove the theorem for the right unipotent semigroup $S$. Clearly $\beta$ is an equivalence relation on $S$. Let $(x, y) \in \beta$. Let $c \in S$ and $c' \in V(c)$ and $x' \in V(x)$. Then $x'c' \in V(cx)$ and $y'c' \in V(cy)$. As $c(xex')c' = c(yey')c'$, by Lemma 2, we get $(cx, cy) \in \beta$ and $\beta$ is a left congruence. Further, since $cex$ is an idempotent for any $e \in E$, $c'x' \in V(cx)$ and $c'y' \in V(cy)$ we have $x(cex)c' = y(cey)c'$. So by Lemma 2, $(xc, yc) \in \beta$. Therefore $\beta$ is a right congruence and hence a congruence relation on $S$.

Now let $g, h \in E$ and suppose that $(g, h) \in \beta$. Then by Lemma 2, for any $e \in E$ we have $g \text{eg} = h \text{eh}$. Taking $e = g$ and $e = h$ in turn we obtain $g = hgh = hg$ and $h = ghg = gh$ using Lemma 1. Therefore $g = h(hg) = hh = h$ proving that $\beta$ is idempotent-separating.

Now let $\sigma$ be any idempotent-separating congruence on $S$. Let $(x, y) \in \sigma$. Then by Lemma 3 there exist $x' \in V(x)$ and $y' \in V(y)$ such that $(x', y') \in \sigma$. As $\sigma$ is a congruence, for any $e \in E$ we have $(xe, ye) \in \sigma$ and hence $(xex', yey') \in \sigma$. But $xex'$ and $yey'$ are idempotents and $\sigma$ is idempotent-separating. Therefore $xex' = yey'$. This, by
Lemma 2, implies \((x, y) \in \beta\) and thus \(\sigma \subseteq \beta\). Hence the theorem.

**COROLLARY [4].** Let \(S\) be an inverse semigroup. Then \(\alpha(=\beta)\) is the greatest idempotent-separating congruence on \(S\).

**THEOREM 5.** Let \(S\) be a right unipotent semigroup and \(E = E(S)\). For each \(x \in S\) let \(\theta_x: E \to E\) be the mapping defined by \(\theta_x(e) = xex'\) where \(x' \in V(x)\). Then

1. \(\theta_x\) is an endomorphism, and
2. the following statements are equivalent.
   (A) \(\theta_x\) is an idempotent.
   (B) The \(\mathcal{H}\)-class \(H_x\) is a subgroup of \(S\) and \(xx^{-1}e = xx^{-1}x = x^{-1}ex\) for all \(e \in E\) where \(x^{-1}\) is the group inverse of \(x\) in \(H_x\).
   (C) \(\theta_x = \theta_y\) where \(g = xx'\).

**Proof.** Let \(x \in S\) and \(x' \in V(x)\).

1. For any \(e, f \in E\), by Lemma 1, we have \((xex')(xfx') = (xex')fx' = xefx',\) proving (1).

2. Assume (A). Then \(xex' = xxe'x'\) for all \(e \in E\). Taking \(x'x\) for \(e\) we have \(xx' = xxx'x'\) and therefore \(x'x = x'(xx')x = x'xxx'x'x = x'xxx'\) using Lemma 1. So \(x = xx'x = x^2x'\) and \(x \in \mathcal{H}x^2\).

Now taking \(x'x'xx\) for \(e\) in \(xex' = xxe'x'\), and using \(x = x'x'\) and Lemma 1, we get \(xx'xx = xx'x\) and hence \(x'x'x = x\). Therefore \(x = xx'x = xx'xx'x \) and \(x \in \mathcal{H}x^3\). Thus \(x \in \mathcal{H}x^3\) and \(H_x\) is a subgroup of \(S\).

By hypothesis and Lemma 2, for all \(e \in E\) we have \(xx^{-1}e = xx^{-1}x = x^{-1}ex\) and therefore \(x^{-1}x = x^{-1}xx^{-1} = xx^{1} = xx^{-1}x = xx^{-1}ex\) since \(x^{-1}x = xx^{-1}\).

Again taking \(x^{-1}e\) for \(e\) in \(xx^{-1} = xx^{-2}\) we get \(x^{-1}ex = xx^{-1}xx^{-1} = xx^{-1}e\) using Lemma 1. So we get (B).

Assume (B). Then by Lemma 1, for all \(e \in E\) we have \(xx^{-1} = xx^{-1}e = xx^{-1}xx^{-1}\), giving (C). Clearly (C) implies (A).

**THEOREM 6.** Let \(S\) be a right unipotent semigroup and \(E = E(S)\). Then

1. \(T = \{\theta_x: x \in S\}\) is a right unipotent semigroup.
2. The mapping \(\theta: S \to T\) defined by \(\theta(x) = \theta_x\) is an onto homomorphism and \(\theta \circ \theta^{-1} = \beta\).
3. Set \(\gamma = \theta(\beta)\) (\(\theta\) restricted to \(E\)). Then \(\gamma\) is an isomorphism of \(E\) upon \(\theta(E)\).

**Proof.** As \(\theta \circ \theta = \theta \circ \theta^{-1}\) it follows that \(T\) is a regular semigroup. We now show directly that \(T\) is right unipotent. Let \(\theta_x\) and \(\theta_y\) be idempotents of \(T\). Then, for all \(e \in E\), using (B) of Theorem 5 repeatedly we have \(x(yxex^{-1}y^{-1})x^{-1} = xx^{-1}y(xex^{-1})y^{-1} = xx^{-1}yy^{-1}(xx^{-1}) = xx^{-1}(yy^{-1})xx^{-1} = xx^{-1}xx^{-1}ex^{-1} = x(yy^{-1}e)x^{-1} = xyey^{-1}x^{-1}\), and hence
\( \theta_{\text{uv}} = \theta_{\text{vy}} \) by Lemma 2. So, by Lemma 1, \( T \) is a right unipotent semigroup, proving (1).

(2) follows directly. As for (3) we need only to show that \( \gamma \) is one-to-one. Let \( \gamma(g) = \gamma(h) \) for \( g, h \in E \). Then by Lemma 2, we have \( (g, h) \in \beta \). This, by Theorem 4, implies \( g = h \) and so \( \gamma \) is an isomorphism.

We now consider the quotient semigroup \( S/\beta \). The following theorem gives a necessary and sufficient condition for \( S/\beta \) to be an idempotent semigroup.

**Theorem 7.** Let \( S \) be a right unipotent semigroup and \( E = E(S) \). Then the quotient semigroup \( S/\beta \) is isomorphic with \( E \) if and only if the statement \( (Qx) \) is satisfied for all \( x \) in \( S \). (The left-right dual holds for a left unipotent semigroup).

**Proof.** Let \( S/\beta \) be isomorphic with \( E \). As \( S/\beta \) is a homomorphic image of \( S \), each idempotent of \( S/\beta \) is the image an idempotent of \( S \) [10]. So each \( \beta \)-class of \( S \) contains at least one and hence exactly one idempotent of \( S \). Let \( x \in S \). Then there exists \( h \in E \) such that \( (x, h) \in \beta \). So for any \( e \in E \) and \( x' \in V(x) \) we have \( xx'e = heh \). In particular taking \( e = x'x \) and \( e = h \) in turn we get \( xx' = hx'xh \) and \( xx'h = h \). The first equation gives \( xx'h = xx' \) and the second \( xx'h = h \). So \( xx' = h \). Hence for any \( e \in E \), by Lemma 1, we have \( xx'e = heh = xe = xx'e \) giving \( (Qx) \).

Conversely let \( (Qx) \) be satisfied for all \( x \in S \). Let \( x \in S \). Then for any \( e \in E \) and \( x' \in V(x) \), by Lemma 1, we have \( xx'e = xx'e = xx' \). Therefore \( (x, xx') \in \beta \) and hence each \( \beta \)-class contains a unique idempotent. Let \( \beta^* \) be the natural homomorphism of \( S \) upon \( S/\beta \). Then the mapping \( \beta^* \) restricted to \( E \) is an isomorphism of \( E \) upon \( S/\beta \). This completes the proof of the theorem.

**Remark.** One may appeal to Theorems 5 and 6 to prove Theorem 7. Clearly, for any \( x \in S \), the statements \( (Qx) \), and (C) of Theorem 5 are equivalent. Therefore \( (Qx) \) is satisfied for all \( x \in S \) if and only if \( T = \theta(E) \) and hence if and only if \( S/\beta \) is isomorphic with \( E \).

From Theorems 1, 2, 7 and the corollary of Theorem 3 we have the following.

**Corollary [4].** Let \( S \) be an inverse semigroup. Then the following statements are equivalent.

(A) The quotient semigroup \( S/\beta \) is isomorphic with \( E \).
(B) \( S \) is a union groups.
(C) The idempotents of \( S \) are contained in the centre of \( S \).
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