ON RIGHT UNIPOTENT SEMIGROUPS

P. S. Venkatesan
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We investigate the implications of certain conditions on right unipotent semigroups. We describe the greatest idempotent-separating congruence $\beta$ on a right unipotent semigroup $S$. Necessary and sufficient conditions for (i) $S$ to be a union of groups, (ii) $S$ to be an inverse semigroup, (iii) the idempotents of $S$ to be in the centre of $S$ and (iv) the quotient semigroup $S/\beta$ to be isomorphic with the subsemigroup of idempotents of $S$ are also obtained.

It is known that any regular semigroup has the greatest idempotent-separating congruence [5], [6]. Such a congruence on an inverse semigroup was obtained by Howie [4]. For the general terminology and notation the reader is referred to [1], [2].

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1. Preliminary matters. An orthodox semigroup $S$ is a regular semigroup in which the idempotents form a subsemigroup. An inverse of an idempotent of $S$ is an idempotent, and if $a', b'$ are inverses of the elements $a, b$ in $S$ then $b'a'$ is an inverse of $ab$ [7].

A semigroup $S$ is called a right (left) unipotent semigroup if every principal right (left) ideal of $S$ has a unique idempotent generator. Such semigroups are called left (right) inverse by the author [9], [10]. Lemma 1 below is a part of the left-right dual of Theorem 1 in [10].

**Lemma 1.** Let $S$ be a regular semigroup. Then the following statements are equivalent.

(A) $eef = fe$ for any two idempotents $e, f$ in $S$.

(B) If $a'$ and $a''$ are inverses of the element $a$ in $S$ then $aa' = aa''$.

(C) $S$ is a right unipotent semigroup.

**Lemma 2.** Let $S$ be a right unipotent semigroup and $e$ be an idempotent of $S$. Let $x \in S$ and $x', x''$ be inverses of $x$. Then $xex'$ is an idempotent and $xex' = xex''$.

**Proof.** By Lemma 1 we have $xe = xx'xe = x(x'xex'x) = xex'x$. So $xex'$ is an idempotent. Also $xex' = (xex')xx' = (xex')xx'' = (xex')x'' = xex''$, using Lemma 1.
2. The statements \((Px), (Qx)\) and \((Rx)\). Let \(S\) be a right unipotent semigroup and \(x \in S\). Throught \(E = E(S)\) denotes the subsemigroup of idempotents of \(S\) and \(V(x)\) denotes the set of inverses of the element \(x\). The symbols \((Px), (Qx)\) and \((Rx)\) stand for the statements indicated below.

\((Px)\): \(exe = ex\) and \(ex'e = ex'\) for all \(e \in E\) and for at least one \(x' \in V(x)\).

\((Qx)\): \(xex' = xx'e\) for all \(e \in E\) and \(x' \in V(x)\).

\((Rx)\): \(xex' = xx'x\) for all \(e \in E\) and \(x' \in V(x)\).

**Remark.** Let \(S\) be a left unipotent semigroup. Then the left-right dual of \((Px), (Qx)\) and \((Rx)\) are obtained by replacing respectively the equations in them by \(xx = xe\) and \(ex' = x'e, x'x = ex'x\) and \(xex = x'xe\).

**Theorem 1.** Let \(S\) be a right unipotent semigroup and \(E = E(S)\). Then

1. \((Rx) \implies (Qx) \implies (Px)\) for any \(x \in S\).
2. \(E\) is contained in the center of \(S\) if and only if \((Rx)\) is satisfied for all \(x \in S\).

**Proof.** (1) Let \(x \in S\) and \(x' \in V(x)\).

Assume \((Rx)\). Then for any \(e \in E\) we have \(xex' = xx'\) and hence \(ex = (exx')x = xex'x = x(x'xex') = x(x'xe) = xe\) by Lemma 1. So \(xx'e = xx'(ex)x' = xx'(xe)x' = xx'x = ex\) and \(ex'e = xx'x) = ex'(xx'e) = xx'xe = ex', giving \((Qx)\).

Assume \((Qx)\). Then \(xex' = xx'\) for any \(e \in E\). Therfore, by Lemma 1, we get \(exe = ex(x'xe) = ex(x'xex') = e(xex')x = e(xx'e)x = exx'x = ex\) and \(ex'e = ex'(xx'e) = ex'(xex') = ex'xx' = ex', giving \((Px)\).

(2) The only if part is trivial. The if part follows since, for any \(x \in S\) and \(e \in E\), as shown above, \((Rx)\) implies \(ex = xe\).

Let \(S\) be a right unipotent semigroup. Then the statements (1) \(S\) is union of groups, (2) each \(L\)-class of \(S\) is a left group and (3) each \(R\)-class of \(S\) is a group are equivalent [8]. An alternate characterization for \(S\) to be a union of groups is obtained in the following.

**Theorem 2.** Let \(S\) be a right unipotent semigroup and \(E = E(S)\). Then \(S\) is a union of groups if and only if \((Px)\) is satisfied for all \(x\) in \(S\).

**Proof.** Let \(S\) be a union of groups. Let \(x \in S\) and \(e \in E\). Let \(x^{-1}\) be the inverse of \(x\) in the group \(H_x\). Then \(x^{-1}x = xx^{-1}\). Let \(a\) and \(b\) respectively be the group inverses of \(ex\) and \(ex^{-1}\). As \(x^{-1}e\) is
an inverse of $ex$, and $xe$ is an inverse of $ex^{-1}$, by Lemma 1 we have $exa = ex^{-1}e$ and $ex^{-1}b = ex^{-1}xe$. But $exx^{-1}e = exx^{-1}$ and $ex^{-1}xe = exx^{-1}e = exx^{-1}$ by Lemma 1. So both $ex$ and $ex^{-1}$ and hence their product $exx^{-1}$ belong to the group with identity element $exx^{-1}$. As $exx^{-1}$ is an idempotent we conclude that $exx^{-1} = exx^{-1}$. Therefore $exe = ex(x^{-1}xe) = ex(x^{-1}xex^{-1}x) = (exx^{-1})x = exx^{-1}x = ex$ by Lemma 1. Further since $exx^{-1}$ belongs to the group with identity element $exx^{-1}$, we have $ex^{-1} = ex^{-1}(exx^{-1}) = ex^{-1}(xx^{-1}exx^{-1}) = ex^{-1}(xx^{-1}e) = ex^{-1}e$, by Lemma 1. So we get $(Px)$. 

Conversely let $(Px)$ be satisfied for all $x \in S$. (This part of the proof holds for any regular semigroup $S$). Let $x \in S$ and $x' \in V(x)$. Taking $e = xx'$ in $ex = exe$ we have $x = x'a' \in x^2S$. So $S$ is a right regular semigroup and hence a union of group $[1], [3]$. 

Let $S$ be a right unipotent semigroup. Then $S$ is an inverse semigroup if and only if $S$ satisfies the left-right dual of any of the statements of Lemma 1. We now obtain a necessary and sufficient condition in terms of $(Px)$ and $(Rx)$ for $S$ to be an inverse semigroup.

**Theorem 3.** Let $S$ be a right unipotent semigroup and $E = E(S)$. Then $S$ is an inverse semigroup if and only if $(Px)$ implies $(Rx)$ for all $x$ in $S$.

**Proof.** Let $S$ be an inverse semigroup. Let $x \in S$. Assume $(Px)$. Then for any $e \in E$ we have $exe = ex$ and $ex^{-1}e = ex^{-1}$. As the idempotents in $S$ commute we get $exx^{-1} = (exe)x^{-1} = e(exx^{-1}) = (ex^{-1})e = x(ex^{-1}e) = xex^{-1}$, giving $(Rx)$. 

Conversely let $(Px)$ imply $(Rx)$ for all $x \in S$. Let $g, h \in E$. Then for any $e \in E$, by Lemma 1, we have $e(gh)e = egh$. As $gh \in V(gh)$, by hypothesis we conclude $gh = egh$. So, by Lemma 1, we get $gh = egh$. Taking $e = h$, by Lemma 1, we have $gh = hg$. Thus $S$ is an inverse semigroup.

**Corollary.** Let $S$ be a right unipotent semigroup and $E = E(S)$. Then $S$ is an inverse semigroup if and only if $(Px), (Qx)$ and $(Rx)$ are equivalent for all $x$ in $S$.

**Remark.** The left-right dual of Theorems 1, 2 and 3 hold for a left unipotent semigroup.

3. The congruences $\alpha$ and $\beta$. In this section we construct the greatest idempotent-separating congruence on a right (left) unipotent semigroup. 

Theorems 4 and 7 below generalize known results for inverse semigroups [4]. In [6] Munn relates the greatest idempotent-
separating congruence on an inverse semigroup to a certain full inverse semigroup. We need the following

**Lemma 3.** Let $S$ be an orthodox semigroup and $\sigma$ be an idempotent-separating congruence on $S$. If $(x, y) \in \sigma$ then there exist $u \in V(x)$ and $v \in V(y)$ such that $(u, v) \in \sigma$.

**Proof.** Let $(x, y) \in \sigma$, $x' \in V(x)$ and $y' \in V(y)$. Since $\sigma$ is a congruence we get $(x'x, x'y) \in \sigma$ and hence $(x'xy'y, x'y) \in \sigma$. By transitivity of $\sigma$ we conclude $(x'xy'y, x'x) \in \sigma$. This, since $\sigma$ is idempotent-separating, implies $x'xy'y = x'x$. So $xy' = x$. Similarly we get $xx'yy' = yy'$ and $xx'y = y$.

Set $u = y'yx'$ and $v = y'xx'$. Then $u \in V(x)$ and $v \in V(y)$. Now from $(x, y) \in \sigma$ we have $(y'xx', y'yy') \in \sigma$, that is $(v, u) \in \sigma$ and thus $(u, v) \in \sigma$. Hence the lemma.

Let $S$ be a regular semigroup and $E$ be the set of idempotents of $S$. Define the binary relations $\alpha$ and $\beta$ on $S$ thus:

\[ \alpha = \{(x, y) \in S \times S : x'x = y'y \text{ for all } e \in E, x' \in V(x) \text{ and } y' \in V(y)\} \]

\[ \beta = \{(x, y) \in S \times S : xex' = yey' \text{ for all } e \in E, x' \in V(x) \text{ and } y' \in V(y)\} \]

**Theorem 4.** Let $S$ be a right (left) unipotent semigroup and $E = E(S)$. Then $\beta(\alpha)$ is an idempotent-separating congruence on $S$. Further, if $\sigma$ is any idempotent-separating congruence on $S$ then $\sigma \subseteq \beta(\sigma) \subseteq \alpha$.

**Proof.** We prove the theorem for the right unipotent semigroup $S$. Clearly $\beta$ is an equivalence relation on $S$. Let $(x, y) \in \beta$. Let $e \in S$ and $'e' \in V(e)$ and $x' \in V(x)$. Then $x'c' \in V(cx)$ and $y'c' \in V(cy)$. As $c(xex')'c' = c(yey')'c'$, by Lemma 2, we get $(cx, cy) \in \beta$ and $\beta$ is a left congruence. Further, since $ce'c'$ is an idempotent for any $e \in E$, $c'x'c' \in V(cx)$ and $c'y'c' \in V(cy)$ we have $x(cec')x' = y(cec')y'$. So by Lemma 2, $(xc, yc) \in \beta$. Therefore $\beta$ is a right congruence and hence a congruence relation on $S$.

Now let $g, h \in E$ and suppose that $(g, h) \in \beta$. Then by Lemma 2, for any $e \in E$ we have $geg = heh$. Taking $e = g$ and $e = h$ in turn we obtain $g = hgh = hg$ and $h = ghg = gh$ using Lemma 1. Therefore $g = h(gh) = hh = h$ proving that $\beta$ is idempotent-separating.

Now let $\sigma$ be any idempotent-separating congruence on $S$. Let $(x, y) \in \sigma$. Then by Lemma 3 there exist $x' \in V(x)$ and $y' \in V(y)$ such that $(x', y') \in \sigma$. As $\sigma$ is a congruence, for any $e \in E$ we have $(xe, ye) \in \sigma$ and hence $(xex', yey') \in \sigma$. But $xex'$ and $yey'$ are idempotents and $\sigma$ is idempotent-separating. Therefore $xex' = yey'$. This, by
Lemma 2, implies \((x, y) \in \beta\) and thus \(\sigma \subseteq \beta\). Hence the theorem.

COROLLARY [4]. Let \(S\) be an inverse semigroup. Then \(\alpha(=\beta)\) is the greatest idempotent-separating congruence on \(S\).

THEOREM 5. Let \(S\) be a right unipotent semigroup and \(E = E(S)\). For each \(x \in S\) let \(\theta_x: E \to E\) be the mapping defined by \(\theta_x(e) = xex'\) where \(x' \in V(x)\). Then

1. \(\theta_x\) is an endomorphism, and
2. the following statements are equivalent.
   - \(A\) \(\theta_x\) is an idempotent.
   - \(B\) The \(H\)-class \(H_x\) is a subgroup of \(S\) and \(xx^{-1}e = xex^{-1} = x^{-1}ex\) for all \(e \in E\) where \(x^{-1}\) is the group inverse of \(x\) in \(H_x\).
   - \(C\) \(\theta_x = \theta_g\) where \(g = xx'\).

Proof. Let \(x \in S\) and \(x' \in V(x)\).
1. For any \(e, f \in E\), by Lemma 1, we have \((xex')(xfx') = (xex')fx' = xefx',\) proving (1).
2. Assume (A). Then \(xex' = xxex'x'\) for all \(e \in E\). Taking \(x'x\) for \(e\) we have \(xx' = xxex'x'\) and therefore \(x'x = x'(xx')x = x'xx'x\) using Lemma 1. So \(x = xx'x = x^2\) and \(xHx^2\).

Now taking \(x'x'xx\) for \(e \in xx' = xxex'x'\), and using \(x = x^2\) and Lemma 1, we get \(xx'x'x = xx'\) and hence \(x'x'x = x\). Therefore \(x = xx' = xx'x^2\) and \(xHx^2\). Thus \(xHx^2\).

By hypothesis and Lemma 2, for all \(e \in E\) we have \(xex^{-1} = x^2ex^{-2}\) and therefore \(x^{-1}xe = x^{-1}xxex^{-1}x = x^{-1}(x^2ex^{-2})x = xex^{-1}\) since \(x^{-1}x = xx^{-1}\). Again taking \(x^2ex^2\) for \(e\) in \(xex^{-1} = x^2ex^{-2}\) we get \(x^{-1}ex = xx^{-1}exx^{-1} = xx^{-1}e\) using Lemma 1. So we get (B).

Assume (B). Then by Lemma 1, for all \(e \in E\) we have \(xex^{-1} = xx^{-1}e = xx^{-1}exx^{-1}\), giving (C). Clearly (C) implies (A).

THEOREM 6. Let \(S\) be a right unipotent semigroup and \(E = E(S)\). Then

1. \(T = \{\theta_x: x \in S\}\) is a right unipotent semigroup.
2. The mapping \(\theta: S \to T\) defined by \(\theta(x) = \theta_x\) is an onto homomorphism and \(\theta \cdot \theta^{-1} = \beta\).
3. Set \(\gamma = \theta | E\) (\(\theta\) restricted to \(E\)). Then \(\gamma\) is an isomorphism of \(E\) upon \(\theta(E)\).

Proof. As \(\theta_x \theta_y = \theta_{xy}\) it follows that \(T\) is a regular semigroup. We now show directly that \(T\) is right unipotent. Let \(\theta_x\) and \(\theta_y\) be idempotents of \(T\). Then, for all \(e \in E\), using (B) of Theorem 5 repeatedly we have \(x(yex^{-1}y^{-1})x^{-1} = xx^{-1}y(xex^{-1})y^{-1} = xx^{-1}yy^{-1}(xex^{-1}) = xx^{-1}(yy^{-1})xex^{-1} = xx^{-1}(yy^{-1})ex^{-1} = x(yy^{-1})x^{-1} = xyey^{-1}x^{-1}\), and hence
\( \theta_{sv} = \theta_{sv} \) by Lemma 2. So, by Lemma 1, \( T \) is a right unipotent semigroup, proving (1).

(2) follows directly. As for (3) we need only to show that \( \gamma \) is one-to-one. Let \( \gamma(g) = \gamma(h) \) for \( g, h \in E \). Then by Lemma 2, we have \( (g, h) \in \beta \). This, by Theorem 4, implies \( g = h \) and so \( \gamma \) is an isomorphism.

We now consider the quotient semigroup \( S/\beta \). The following theorem gives a necessary and sufficient condition for \( S/\beta \) to be an idempotent semigroup.

**Theorem 7.** Let \( S \) be a right unipotent semigroup and \( E = E(S) \). Then the quotient semigroup \( S/\beta \) is isomorphic with \( E \) if and only if the statement \((Qx)\) is satisfied for all \( x \) in \( S \). (The left-right dual holds for a left unipotent semigroup).

**Proof.** Let \( S/\beta \) be isomorphic with \( E \). As \( S/\beta \) is a homomorphic image of \( S \), each idempotent of \( S/\beta \) is the image an idempotent of \( S \) [10]. So each \( \beta \)-class of \( S \) contains at least one and hence exactly one idempotent of \( S \). Let \( x \in S \). Then there exists \( h \in E \) such that \( (x, h) \in \beta \). So for any \( e \in E \) and \( x' \in V(x) \) we have \( xex' = heh \). In particular taking \( e = x'x \) and \( e = h \) in turn we get \( xx' = hx'xh \) and \( xx'h = h \). The first equation gives \( xx'h = xx' \) and the second \( xx'h = h \). So \( xx' = h \). Hence for any \( e \in E \), by Lemma 1, we have \( xex' = heh = he = xx'e \) giving \((Qx)\).

Conversely let \((Qx)\) be satisfied for all \( x \in S \). Let \( x \in S \). Then for any \( e \in E \) and \( x' \in V(x) \), by Lemma 1, we have \( xex' = xx'e = xx'xx' \). Therefore \( (x, xx') \in \beta \) and hence each \( \beta \)-class contains a unique idempotent. Let \( \beta^* \) be the natural homomorphism of \( S \) upon \( S/\beta \). Then the mapping \( \beta^* \) restricted to \( E \) is an isomorphism of \( E \) upon \( S/\beta \). This completes the proof of the theorem.

**Remark.** One may appeal to Theorems 5 and 6 to prove Theorem 7. Clearly, for any \( x \in S \), the statements \((Qx)\), and \((C)\) of Theorem 5 are equivalent. Therefore \((Qx)\) is satisfied for all \( x \in S \) if and only if \( T = \theta(E) \) and hence if and only if \( S/\beta \) is isomorphic with \( E \).

From Theorems 1, 2, 7 and the corollary of Theorem 3 we have the following.

**Corollary [4].** Let \( S \) be an inverse semigroup. Then the following statements are equivalent.

(A) The quotient semigroup \( S/\beta \) is isomorphic with \( E \).

(B) \( S \) is a union groups.

(C) The idempotents of \( S \) are contained in the centre of \( S \).
REFERENCES


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