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A RATIONAL OCTIC RECIPROCITY LAW

KENNETH S. WILLIAMS

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A rational octic reciprocity theorem analogous to the rational biquadratic reciprocity theorem of Burde is proved.

Let p and q be distinct primes $\equiv 1 \pmod{4}$ such that $(p/q) = (q/p) = 1$. For such primes there are integers a, b, A, B with

$$(1) \quad \begin{cases} p = a^2 + b^2, a \equiv 1 \pmod{2}, b \equiv 0 \pmod{2}, \\ q = A^2 + B^2, A \equiv 1 \pmod{2}, B \equiv 0 \pmod{2}. \end{cases}$$

Moreover it is well-known that $(A/q) = 1, (B/q) = (-1)^{(q-1)/4}$. If k is a quadratic residue \pmod{q} we set

$$\left(\frac{k}{q}\right)_4 = \begin{cases} +1, & \text{if } k \text{ is a biquadratic residue } \pmod{q}, \\ -1, & \text{otherwise.} \end{cases}$$

In 1969 Burde [2] proved the following

THEOREM (Burde).

$$\left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = (-1)^{(q-1)/4} \left(\frac{aB - bA}{q}\right).$$

Recently Brown [1] has posed the problem of finding an octic reciprocity law analogous to Burde's biquadratic law for distinct primes p and q with $p \equiv q \equiv 1 \pmod{8}$ and $(p/q)_4 = (q/p)_4 = 1$. It is the purpose of this paper to give such a law. From this point on we assume that p and q satisfy these conditions and set for any biquadratic residue $k \pmod{q}$

$$\left(\frac{k}{q}\right)_8 = \begin{cases} +1, & \text{if } k \text{ is an octic residue } \pmod{q}, \\ -1, & \text{otherwise.} \end{cases}$$

It is a familiar result that there are integers c, d, C, D with

$$(2) \quad \begin{cases} p = c^2 + 2d^2, c \equiv 1 \pmod{2}, d \equiv 0 \pmod{2}, \\ q = C^2 + 2D^2, C \equiv 1 \pmod{2}, D \equiv 0 \pmod{2}. \end{cases}$$

Moreover we have $(D/q) = 1$. Also from Burde's theorem we have

$$(3) \quad \left(\frac{aB - bA}{q}\right) = 1,$$

and from the law of biquadratic reciprocity after a little calculation we find that $(B/q)_4 = +1$. We prove

THEOREM. *Let p and q be distinct primes $\equiv 1 \pmod{8}$ such that*

$$\left(\frac{p}{q}\right)_4 = \left(\frac{q}{p}\right)_4 = 1. \quad \text{Then } \left(\frac{p}{q}\right)_8 \left(\frac{q}{p}\right)_8 = \left(\frac{aB - bA}{q}\right)_4 \left(\frac{cD - dC}{q}\right).$$

We note that it is easy to show that

$$\left(\frac{\pm aB \pm bA}{q}\right)_4 = \left(\frac{aB - bA}{q}\right)_4, \quad \left(\frac{\pm cD \pm dC}{q}\right) = \left(\frac{cD - dC}{q}\right),$$

so that the expression on the right-hand side of the theorem is independent of the particular choices of a, b, c, d, A, B, C, D made in (1) and (2). In the course of the proof it is convenient to make a particular choice of a, b, c, d (see (9) and (10)).

We begin by proving three lemmas.

$$\text{LEMMA 1. } (c + d\sqrt{-2})^{(q-1)/2} \equiv ((cD - dC)/q) \pmod{q}.$$

Proof. As $(p/q) = 1$ we can define an integer u by $p \equiv u^2 \pmod{q}$. Next we define integers l and m by

$$l \equiv \frac{cD - dC + Du}{2}, \quad m \equiv \frac{C}{D} \cdot \frac{cD - dC - Du}{4} \pmod{q},$$

so that

$$l^2 - 2m^2 \equiv cD(cD - dC) \pmod{q}$$

and

$$2lm \equiv dD(cD - dC) \pmod{q},$$

giving

$$D(cD - dC)(c + d\sqrt{-2}) \equiv (l + m\sqrt{-2})^2 \pmod{q},$$

and so

$$D^{(q-1)/2}(cD - dC)^{(q-1)/2}(c + d\sqrt{-2})^{(q-1)/2} \equiv (l + m\sqrt{-2})^{q-1} \pmod{q}.$$

Now working modulo q we have

$$\begin{aligned} (l + m\sqrt{-2})^{q-1} &\equiv \frac{(l + m\sqrt{-2})^q}{l + m\sqrt{-2}} \equiv \frac{l^q + m^q(\sqrt{-2})^q}{l + m\sqrt{-2}} \\ &\equiv \frac{l + mi^q 2^{q/2}}{l + m\sqrt{-2}} \equiv \frac{l + mi\sqrt{2}}{l + m\sqrt{-2}} \\ &\equiv 1, \end{aligned}$$

also

$$D^{(q-1)/2} \equiv \left(\frac{D}{q}\right) = 1,$$

and

$$(cD - dC)^{(q-1)/2} \equiv \left(\frac{cD - dC}{q}\right),$$

from which the required result follows immediately.

LEMMA 2. $(a + b\sqrt{-1})^{(q-1)/4} \equiv ((aB - bA)/q)_4 \pmod{q}.$

Proof. As $(p/q) = 1$ we define an integer u by $p \equiv u^2 \pmod{q}$ as in Lemma 1. Next we define integers r and s by

$$r \equiv \frac{aB - bA + Bu}{2}, s \equiv \frac{A}{B} \cdot \frac{aB - bA - Bu}{2} \pmod{q}$$

so that

$$r^2 - s^2 \equiv aB(aB - bA) \pmod{q}$$

and

$$2rs \equiv bB(aB - bA) \pmod{q}$$

giving

$$B(aB - bA)(a + b\sqrt{-1}) \equiv (r + s\sqrt{-1})^2 \pmod{q},$$

and so

$$B^{(q-1)/4}(aB - bA)^{(q-1)/4}(a + b\sqrt{-1})^{(q-1)/4} \equiv (r + s\sqrt{-1})^{(q-1)/2} \pmod{q}.$$

Thus as $(B/q)_4 = ((aB - bA)/q) = 1$ we obtain

$$(a + b\sqrt{-1})^{(q-1)/4} \equiv \left(\frac{aB - bA}{q}\right)_4 (r + s\sqrt{-1})^{(q-1)/2} \pmod{q}.$$

Next we note that $r^2 + s^2 \equiv uB(aB - bA) \pmod{q}$ so that

$$\left(\frac{r^2 + s^2}{q}\right) = \left(\frac{p}{q}\right) \left(\frac{B}{q}\right) \left(\frac{aB - bA}{q}\right) = 1.$$

Hence we may define an integer w by $w^2 \equiv r^2 + s^2 \pmod{q}$. Then we define integers e and f by

$$e \equiv \frac{rB - sA + Bw}{2}, f \equiv \frac{A}{B} \cdot \frac{rB - sA - Bw}{2} \pmod{q}$$

so that

$$e^2 - f^2 \equiv rB(rB - sA) \pmod{q}$$

and

$$2ef \equiv sB(rB - sA) \pmod{q}$$

giving

$$B(rB - sA)(r + s\sqrt{-1}) \equiv (e + f\sqrt{-1})^2 \pmod{q},$$

and so

$$B^{(q-1)/2}(rB - sA)^{(q-1)/2}(r + s\sqrt{-1})^{(q-1)/2} \equiv (e + f\sqrt{-1})^{q-1} \pmod{q}.$$

Now working modulo q we have

$$\begin{aligned} (e + f\sqrt{-1})^{q-1} &\equiv \frac{(e + f\sqrt{-1})^q}{(e + f\sqrt{-1})} \equiv \frac{e^q + f^q(\sqrt{-1})^q}{e + f\sqrt{-1}} \\ &\equiv \frac{e + f\sqrt{-1}}{e + f\sqrt{-1}} \equiv 1, \end{aligned}$$

and

$$B^{(q-1)/2} \equiv \left(\frac{B}{q}\right) = 1, \quad (rB - sA)^{(q-1)/2} \equiv \left(\frac{rB - sA}{q}\right),$$

so

$$(r + s\sqrt{-1})^{(q-1)/2} \equiv \left(\frac{rB - sA}{q}\right),$$

giving

$$(a + b\sqrt{-1})^{(q-1)/4} \equiv \left(\frac{aB - bA}{q}\right)_4 \left(\frac{rB - sA}{q}\right) \pmod{q}.$$

The required result now follows as modulo q we have

$$\begin{aligned} rB - sA &\equiv \frac{B(aB - bA + Bu)}{2} - \frac{A^2}{B} \frac{(aB - bA - Bu)}{2} \\ &\equiv \frac{B}{2} \{(aB - bA + Bu) + (aB - bA - Bu)\} \\ &\equiv B(aB - bA), \end{aligned}$$

that is

$$\left(\frac{rB - sA}{q}\right) = \left(\frac{B}{q}\right) \left(\frac{aB - bA}{q}\right) = +1.$$

Before proving the final lemma we state some results we shall need. Let $w = \exp(2\pi i/8) = (\sqrt{2} + \sqrt{-2})/2$ and let R be the ring

of integers of the cyclotomic field $Q(w) = Q(\sqrt[4]{2}, \sqrt{-1})$. R is a unique factorization domain. Let π be any prime factor of p in R , fixed once and for all. For integers $x \not\equiv 0 \pmod{p}$ we define an octic character $(\text{mod } p)$ by

$$\left(\frac{x}{\pi}\right)_8 = w^\lambda \text{ if } x^{(p-1)/8} \equiv w^\lambda \pmod{\pi}, 0 \leq \lambda \leq 7.$$

If $x \equiv 0 \pmod{p}$ we set $(x/\pi)_8 = 0$. In terms of this character we define the corresponding Jacobi and Gauss sums for arbitrary integers k and l as follows:

$$J(k, l) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^k \left(\frac{1-x}{\pi}\right)_8^l,$$

$$G(k) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^k \exp(2\pi i x/p).$$

These sums have the following well-known properties (see for example [4], Chapter 8):

- (4) $J(k, l)\overline{J(k, l)} = p,$ if $k, l \not\equiv 0 \pmod{8},$
- (5) $J(k, l) = \frac{G(k)G(l)}{G(k+l)},$ if $k, l, k+l \not\equiv 0 \pmod{8},$
- (6) $G(k)G(-k) = (-1)^{k(p-1)/8}p,$ if $k \not\equiv 0 \pmod{8}.$

We shall also need the evaluation of the familiar sum

$$(7) \quad G(4) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^4 \exp(2\pi i x/p) = \sum_{x=0}^{p-1} \left(\frac{x}{p}\right) \exp(2\pi i x/p) = p^{1/2}$$

and the result

$$(8) \quad J(2, 2) = \pm J(1, 2).$$

A more precise form of (8) follows from a theorem of Jacobi (see for Example [3], page 411, equation (99)). Finally we let $\sigma_k (k = 1, 3, 5, 7)$ be the automorphism of $Q(w)$ defined by $\sigma_k(w) = w^k$.

Now from (5) and (6) we have

$$\sigma_3(J(1, 4)) = J(3, 12) = J(3, 4) = \frac{G(3)G(4)}{G(7)} = \frac{G(1)G(4)}{G(5)} = J(1, 4),$$

so that $J(1, 4) \in Z[\sqrt{-2}]$. Moreover from (4) we have $J(1, 4)\overline{J(1, 4)} = p$ so we may choose the signs of c and d in (2) so that

$$(9) \quad J(1, 4) = c + d\sqrt{-2}.$$

Also from (5) and (6) we have

$$\sigma_5(J(1, 2)) = J(5, 10) = J(5, 2) = \frac{G(5)G(2)}{G(7)} = \frac{G(1)G(2)}{G(3)} = J(1, 2),$$

so that $J(1, 2) \in Z[\sqrt{-1}]$. Moreover from (4) we have $J(1, 2)\overline{J(1, 2)} = p$ so we may choose the signs of a and b in (1) so that

$$(10) \quad J(1, 2) = a + b\sqrt{-1},$$

since it is easy to prove (and well-known) that $J(1, 2) \equiv 1 \pmod{2}$.

$$\text{LEMMA 3. } G(1)^8 = p(a + b\sqrt{-1})^2(c + d\sqrt{-2})^4.$$

Proof. From (5), (9), (10) have

$$c + d\sqrt{-2} = J(1, 4) = \frac{G(1)G(4)}{G(5)}$$

and

$$a + b\sqrt{-1} = J(1, 2) = \frac{G(1)G(2)}{G(3)}.$$

Multiplying these together we obtain

$$(a + b\sqrt{-1})(c + d\sqrt{-2}) = \frac{G(1)^2G(2)G(4)}{G(3)G(5)} = \frac{G(1)^2G(2)}{(-1)^{(p-1)/8}p^{1/2}}$$

by (6) and (7). Hence taking the fourth power of both sides we get

$$(11) \quad G(1)^8G(2)^4 = p^2 (a + b\sqrt{-1})^4(c + d\sqrt{-2})^4.$$

Now from (5) and (7) we have

$$J(2, 2) = \frac{G(2)^2}{G(4)} = \frac{G(2)^2}{p^{1/2}},$$

so that from (8) and (10) we obtain

$$(12) \quad G(2)^4 = p\{J(2, 2)\}^2 = p\{J(1, 2)\}^2 = p(a + b\sqrt{-1})^2,$$

and the required result now follows from (11) and (12).

Proof of theorem. Raising $G(1)$ to the q th power we obtain modulo q ,

$$G(1)^q \equiv \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^q \exp(2\pi i x q/p) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8 \exp(2\pi i x q/p),$$

since $q \equiv 1 \pmod{q}$, giving

$$G(1)^q \equiv \left(\frac{q}{\pi}\right)_8^{-1} \sum_{x=0}^{p-1} \left(\frac{xq}{\pi}\right)_8 \exp(2\pi i(xq)/p) = \left(\frac{q}{\pi}\right)_8^{-1} G(1),$$

since $(q, p) = 1$ implies that

$$\sum_{x=0}^{p-1} \left(\frac{xq}{\pi}\right)_8 \exp(2\pi i(xq)/p) = \sum_{y=0}^{p-1} \left(\frac{y}{\pi}\right)_8 \exp(2\pi iy/p) = G(1).$$

Hence

$$G(1)^q \equiv \left(\frac{q}{\pi}\right)_8^{-1} G(1) = \left(\frac{q}{p}\right)_8 G(1),$$

that is

$$G(1)^{q-1} \equiv \left(\frac{q}{p}\right)_8 \pmod{q}.$$

Hence by Lemmas 1, 2, 3 we have modulo q

$$\begin{aligned} \left(\frac{q}{p}\right)_8 &\equiv (G(1)^8)^{(q-1)/8} \\ &\equiv p^{(q-1)/8} (a + b\sqrt{-1})^{(q-1)/4} (c + d\sqrt{-2})^{(q-1)/2} \\ &\equiv \left(\frac{p}{q}\right)_8 \left(\frac{aB - bA}{q}\right)_4 \left(\frac{cD - dC}{q}\right), \end{aligned}$$

from which the theorem follows.

EXAMPLE. We take $p = 17 \equiv 1 \pmod{8}$ and $q = 409 \equiv 1 \pmod{8}$ so that we may choose

$$\begin{aligned} a &= 1, b = 4, c = 3, d = 2, \\ A &= 3, B = 20, C = 11, D = 12. \end{aligned}$$

Since $q \equiv 1 \pmod{p}$ we clearly have

$$\left(\frac{q}{p}\right) = \left(\frac{q}{p}\right)_4 = \left(\frac{q}{p}\right)_8 = 1.$$

As $((aB - bA)/q) = (8/409) = +1$ by Burde's theorem we have $(p/q)_4 = 1$. Finally

$$\begin{aligned} \left(\frac{aB - bA}{q}\right)_4 &= \left(\frac{8}{409}\right)_4 = \left(\frac{194}{409}\right) = -1, \\ \left(\frac{cD - dC}{q}\right) &= \left(\frac{14}{409}\right) = -1, \end{aligned}$$

so by the theorem of this paper we have $(p/q)_8 = 1$, which is easily verified directly.

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Pacific Journal of Mathematics

Vol. 63, No. 2

April, 1976

Joseph Anthony Ball and Arthur R. Lubin, <i>On a class of contractive perturbations of restricted shifts</i>	309
Joseph Becker and William C. Brown, <i>On extending higher derivations generated by cup products to the integral closure</i>	325
Andreas Blass, <i>Exact functors and measurable cardinals</i>	335
Joseph Eugene Collison, <i>A variance property for arithmetic functions</i>	347
Craig McCormack Cordes, <i>Quadratic forms over nonformally real fields with a finite number of quaternion algebras</i>	357
Freddy Delbaen, <i>Weakly compact sets in H^1</i>	367
G. D. Dikshit, <i>Absolute Nörlund summability factors for Fourier series</i>	371
Edward Richard Fadell, <i>Nielsen numbers as a homotopy type invariant</i>	381
Josip Globevnik, <i>Analytic extensions of vector-valued functions</i>	389
Robert Gold, <i>Genera in normal extensions</i>	397
Solomon Wolf Golomb, <i>Formulas for the next prime</i>	401
Robert L. Griess, Jr., <i>The splitting of extensions of $SL(3, 3)$ by the vector space F_3^3</i>	405
Thomas Alan Keagy, <i>Matrix transformations and absolute summability</i>	411
Kazuo Kishi, <i>Analytic maps of the open unit disk onto a Gleason part</i>	417
Kwangil Koh, Jiang Luh and Mohan S. Putcha, <i>On the associativity and commutativity of algebras over commutative rings</i>	423
James C. Lillo, <i>Asymptotic behavior of solutions of retarded differential difference equations</i>	431
John Alan MacBain, <i>Local and global bifurcation from normal eigenvalues</i>	445
Anna Maria Mantero, <i>Sets of uniqueness and multiplicity for L^p</i>	467
J. F. McClendon, <i>Embedding metric families</i>	481
L. Robbiano and Giuseppe Valla, <i>Primary powers of a prime ideal</i>	491
Wolfgang Ruess, <i>Generalized inductive limit topologies and barrelledness properties</i>	499
Judith D. Sally, <i>Bounds for numbers of generators of Cohen-Macaulay ideals</i>	517
Helga Schirmer, <i>Mappings of polyhedra with prescribed fixed points and fixed point indices</i>	521
Cho Wei Sit, <i>Quotients of complete multipartite graphs</i>	531
S. Sznajder and Zbigniew Zielezny, <i>Solvability of convolution equations in \mathfrak{K}'_p, $p > 1$</i>	539
Mitchell Herbert Taibleson, <i>The existence of natural field structures for finite dimensional vector spaces over local fields</i>	545
William Yslas Vélez, <i>A characterization of completely regular fields</i>	553
P. S. Venkatesan, <i>On right unipotent semigroups</i>	555
Kenneth S. Williams, <i>A rational octic reciprocity law</i>	563
Robert Ross Wilson, <i>Lattice orderings on the real field</i>	571
Harvey Eli Wolff, <i>V-localizations and V-monads. II</i>	579