

# Pacific Journal of Mathematics

**A RATIONAL OCTIC RECIPROCITY LAW**

KENNETH S. WILLIAMS

## A RATIONAL OCTIC RECIPROCITY LAW

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**A rational octic reciprocity theorem analogous to the rational biquadratic reciprocity theorem of Burde is proved.**

Let  $p$  and  $q$  be distinct primes  $\equiv 1 \pmod{4}$  such that  $(p/q) = (q/p) = 1$ . For such primes there are integers  $a, b, A, B$  with

$$(1) \quad \begin{cases} p = a^2 + b^2, a \equiv 1 \pmod{2}, b \equiv 0 \pmod{2}, \\ q = A^2 + B^2, A \equiv 1 \pmod{2}, B \equiv 0 \pmod{2}. \end{cases}$$

Moreover it is well-known that  $(A/q) = 1, (B/q) = (-1)^{(q-1)/4}$ . If  $k$  is a quadratic residue  $\pmod{q}$  we set

$$\left(\frac{k}{q}\right)_4 = \begin{cases} +1, & \text{if } k \text{ is a biquadratic residue } \pmod{q}, \\ -1, & \text{otherwise.} \end{cases}$$

In 1969 Burde [2] proved the following

**THEOREM (Burde).**

$$\left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = (-1)^{(q-1)/4} \left(\frac{aB - bA}{q}\right).$$

Recently Brown [1] has posed the problem of finding an octic reciprocity law analogous to Burde's biquadratic law for distinct primes  $p$  and  $q$  with  $p \equiv q \equiv 1 \pmod{8}$  and  $(p/q)_8 = (q/p)_8 = 1$ . It is the purpose of this paper to give such a law. From this point on we assume that  $p$  and  $q$  satisfy these conditions and set for any biquadratic residue  $k \pmod{q}$

$$\left(\frac{k}{q}\right)_8 = \begin{cases} +1, & \text{if } k \text{ is an octic residue } \pmod{q}, \\ -1, & \text{otherwise.} \end{cases}$$

It is a familiar result that there are integers  $c, d, C, D$  with

$$(2) \quad \begin{cases} p = c^2 + 2d^2, c \equiv 1 \pmod{2}, d \equiv 0 \pmod{2}, \\ q = C^2 + 2D^2, C \equiv 1 \pmod{2}, D \equiv 0 \pmod{2}. \end{cases}$$

Moreover we have  $(D/q) = 1$ . Also from Burde's theorem we have

$$(3) \quad \left(\frac{aB - bA}{q}\right) = 1,$$

and from the law of biquadratic reciprocity after a little calculation we find that  $(B/q)_4 = +1$ . We prove

**THEOREM.** *Let  $p$  and  $q$  be distinct primes  $\equiv 1(\text{mod } 8)$  such that*

$$\left(\frac{p}{q}\right)_4 = \left(\frac{q}{p}\right)_4 = 1. \quad \text{Then } \left(\frac{p}{q}\right)_8 \left(\frac{q}{p}\right)_8 = \left(\frac{aB - bA}{q}\right)_4 \left(\frac{cD - dC}{q}\right).$$

We note that it is easy to show that

$$\left(\frac{\pm aB \pm bA}{q}\right)_4 = \left(\frac{aB - bA}{q}\right)_4, \quad \left(\frac{\pm cD \pm dC}{q}\right) = \left(\frac{cD - dC}{q}\right),$$

so that the expression on the right-hand side of the theorem is independent of the particular choices of  $a, b, c, d, A, B, C, D$  made in (1) and (2). In the course of the proof it is convenient to make a particular choice of  $a, b, c, d$  (see (9) and (10)).

We begin by proving three lemmas.

**LEMMA 1.**  $(c + d\sqrt{-2})^{(q-1)/2} \equiv ((cD - dC)/q) \pmod{q}.$

*Proof.* As  $(p/q) = 1$  we can define an integer  $u$  by  $p \equiv u^2(\text{mod } q)$ . Next we define integers  $l$  and  $m$  by

$$l \equiv \frac{cD - dC + Du}{2}, \quad m \equiv \frac{C}{D} \cdot \frac{cD - dC - Du}{4} \pmod{q},$$

so that

$$l^2 - 2m^2 \equiv cD(cD - dC) \pmod{q}$$

and

$$2lm \equiv dD(cD - dC) \pmod{q},$$

giving

$$D(cD - dC)(c + d\sqrt{-2}) \equiv (l + m\sqrt{-2})^2 \pmod{q},$$

and so

$$D^{(q-1)/2}(cD - dC)^{(q-1)/2}(c + d\sqrt{-2})^{(q-1)/2} \equiv (l + m\sqrt{-2})^{q-1} \pmod{q}.$$

Now working modulo  $q$  we have

$$\begin{aligned} (l + m\sqrt{-2})^{q-1} &\equiv \frac{(l + m\sqrt{-2})^q}{l + m\sqrt{-2}} \equiv \frac{l^q + m^q(\sqrt{-2})^q}{l + m\sqrt{-2}} \\ &\equiv \frac{l + mi^q 2^{q/2}}{l + m\sqrt{-2}} \equiv \frac{l + mi\sqrt{2}}{l + m\sqrt{-2}} \\ &\equiv 1, \end{aligned}$$

also

$$D^{(q-1)/2} \equiv \left(\frac{D}{q}\right) = 1,$$

and

$$(cD - dC)^{(q-1)/2} \equiv \left(\frac{cD - dC}{q}\right),$$

from which the required result follows immediately.

LEMMA 2.  $(a + b\sqrt{-1})^{(q-1)/4} \equiv ((aB - bA)/q)_4 \pmod{q}.$

*Proof.* As  $(p/q) = 1$  we define  $ar$  integer  $u$  by  $p \equiv u^2 \pmod{q}$  as in Lemma 1. Next we define integers  $r$  and  $s$  by

$$r \equiv \frac{aB - bA + Bu}{2}, s \equiv \frac{A}{B} \cdot \frac{aB - bA - Bu}{2} \pmod{q}$$

so that

$$r^2 - s^2 \equiv aB(aB - bA) \pmod{q}$$

and

$$2rs \equiv bB(aB - bA) \pmod{q}$$

giving

$$B(aB - bA)(a + b\sqrt{-1}) \equiv (r + s\sqrt{-1})^2 \pmod{q},$$

and so

$$B^{(q-1)/4}(aB - bA)^{(q-1)/4}(a + b\sqrt{-1})^{(q-1)/4} \equiv (r + s\sqrt{-1})^{(q-1)/2} \pmod{q}.$$

Thus as  $(B/q)_4 = ((aB - bA)/q) = 1$  we obtain

$$(a + b\sqrt{-1})^{(q-1)/4} \equiv \left(\frac{aB - bA}{q}\right)_4 (r + s\sqrt{-1})^{(q-1)/2} \pmod{q}.$$

Next we note that  $r^2 + s^2 \equiv uB(aB - bA) \pmod{q}$  so that

$$\left(\frac{r^2 + s^2}{q}\right) = \left(\frac{p}{q}\right)_4 \left(\frac{B}{q}\right) \left(\frac{aB - bA}{q}\right) = 1.$$

Hence we may define an integer  $w$  by  $w^2 \equiv r^2 + s^2 \pmod{q}$ . Then we define integers  $e$  and  $f$  by

$$e \equiv \frac{rB - sA + Bw}{2}, f \equiv \frac{A}{B} \cdot \frac{rB - sA - Bw}{2} \pmod{q}$$

so that

$$e^2 - f^2 \equiv rB(rB - sA) \pmod{q}$$

and

$$2ef \equiv sB(rB - sA) \pmod{q}$$

giving

$$B(rB - sA)(r + s\sqrt{-1}) \equiv (e + f\sqrt{-1})^2 \pmod{q},$$

and so

$$B^{(q-1)/2}(rB - sA)^{(q-1)/2}(r + s\sqrt{-1})^{(q-1)/2} \equiv (e + f\sqrt{-1})^{q-1} \pmod{q}.$$

Now working modulo  $q$  we have

$$\begin{aligned} (e + f\sqrt{-1})^{q-1} &\equiv \frac{(e + f\sqrt{-1})^q}{(e + f\sqrt{-1})} \equiv \frac{e^q + f^q(\sqrt{-1})^q}{e + f\sqrt{-1}} \\ &\equiv \frac{e + f\sqrt{-1}}{e + f\sqrt{-1}} \equiv 1, \end{aligned}$$

and

$$B^{(q-1)/2} \equiv \left(\frac{B}{q}\right) = 1, \quad (rB - sA)^{(q-1)/2} \equiv \left(\frac{rB - sA}{q}\right),$$

so

$$(r + s\sqrt{-1})^{(q-1)/2} \equiv \left(\frac{rB - sA}{q}\right),$$

giving

$$(a + b\sqrt{-1})^{(q-1)/4} \equiv \left(\frac{aB - bA}{q}\right) \left(\frac{rB - sA}{q}\right) \pmod{q}.$$

The required result now follows as modulo  $q$  we have

$$\begin{aligned} rB - sA &\equiv \frac{B(aB - bA + Bu)}{2} - \frac{A^2}{B} \frac{(aB - bA - Bu)}{2} \\ &\equiv \frac{B}{2} \{(aB - bA + Bu) + (aB - bA - Bu)\} \\ &\equiv B(aB - bA), \end{aligned}$$

that is

$$\left(\frac{rB - sA}{q}\right) = \left(\frac{B}{q}\right) \left(\frac{aB - bA}{q}\right) = +1.$$

Before proving the final lemma we state some results we shall need. Let  $w = \exp(2\pi i/8) = (\sqrt{2} + \sqrt{-2})/2$  and let  $R$  be the ring

of integers of the cyclotomic field  $Q(w) = Q(\sqrt{2}, \sqrt{-1})$ .  $R$  is a unique factorization domain. Let  $\pi$  be any prime factor of  $p$  in  $R$ , fixed once and for all. For integers  $x \not\equiv 0 \pmod{p}$  we define an octic character  $(\text{mod } p)$  by

$$\left(\frac{x}{\pi}\right)_8 = w^\lambda \text{ if } x^{(p-1)/8} \equiv w^\lambda \pmod{\pi}, \quad 0 \leq \lambda \leq 7.$$

If  $x \equiv 0 \pmod{p}$  we set  $(x/\pi)_8 = 0$ . In terms of this character we define the corresponding Jacobi and Gauss sums for arbitrary integers  $k$  and  $l$  as follows:

$$J(k, l) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^k \left(\frac{1-x}{\pi}\right)_8^l,$$

$$G(k) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^k \exp(2\pi i x/p).$$

These sums have the following well-known properties (see for example [4], Chapter 8):

$$(4) \quad J(k, l)\overline{J(k, l)} = p, \quad \text{if } k, l \not\equiv 0 \pmod{8},$$

$$(5) \quad J(k, l) = \frac{G(k)G(l)}{G(k+l)}, \quad \text{if } k, l, k+l \not\equiv 0 \pmod{8},$$

$$(6) \quad G(k)G(-k) = (-1)^{k(p-1)/8} p, \quad \text{if } k \not\equiv 0 \pmod{8}.$$

We shall also need the evaluation of the familiar sum

$$(7) \quad G(4) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^4 \exp(2\pi i x/p) = \sum_{x=0}^{p-1} \left(\frac{x}{p}\right) \exp(2\pi i x/p) = p^{1/2}$$

and the result

$$(8) \quad J(2, 2) = \pm J(1, 2).$$

A more precise form of (8) follows from a theorem of Jacobi (see for Example [3], page 411, equation (99)). Finally we let  $\sigma_k (k = 1, 3, 5, 7)$  be the automorphism of  $Q(w)$  defined by  $\sigma_k(w) = w^k$ .

Now from (5) and (6) we have

$$\sigma_3(J(1, 4)) = J(3, 12) = J(3, 4) = \frac{G(3)G(4)}{G(7)} = \frac{G(1)G(4)}{G(5)} = J(1, 4),$$

so that  $J(1, 4) \in Z[\sqrt{-2}]$ . Moreover from (4) we have  $J(1, 4)\overline{J(1, 4)} = p$  so we may choose the signs of  $c$  and  $d$  in (2) so that

$$(9) \quad J(1, 4) = c + d\sqrt{-2}.$$

Also from (5) and (6) we have

$$\sigma_s(J(1, 2)) = J(5, 10) = J(5, 2) = \frac{G(5)G(2)}{G(7)} = \frac{G(1)G(2)}{G(3)} = J(1, 2) ,$$

so that  $J(1, 2) \in Z[\sqrt{-1}]$ . Moreover from (4) we have  $J(1, 2)\overline{J(1, 2)} = p$  so we may choose the signs of  $a$  and  $b$  in (1) so that

$$(10) \quad J(1, 2) = a + b\sqrt{-1} ,$$

since it is easy to prove (and well-known) that  $J(1, 2) \equiv 1 \pmod{2}$ .

LEMMA 3.  $G(1)^8 = p(a + b\sqrt{-1})^2(c + d\sqrt{-2})^4 .$

*Proof.* From (5), (9), (10) have

$$c + d\sqrt{-2} = J(1, 4) = \frac{G(1)G(4)}{G(5)}$$

and

$$a + b\sqrt{-1} = J(1, 2) = \frac{G(1)G(2)}{G(3)} .$$

Multiplying these together we obtain

$$(a + b\sqrt{-1})(c + d\sqrt{-2}) = \frac{G(1)^2G(2)G(4)}{G(3)G(5)} = \frac{G(1)^2G(2)}{(-1)^{(p-1)/8}p^{1/2}}$$

by (6) and (7). Hence taking the fourth power of both sides we get

$$(11) \quad G(1)^8G(2)^4 = p^2 (a + b\sqrt{-1})^4(c + d\sqrt{-2})^4 .$$

Now from (5) and (7) we have

$$J(2, 2) = \frac{G(2)^2}{G(4)} = \frac{G(2)^2}{p^{1/2}} ,$$

so that from (8) and (10) we obtain

$$(12) \quad G(2)^4 = p\{J(2, 2)\}^2 = p\{J(1, 2)\}^2 = p(a + b\sqrt{-1})^2 ,$$

and the required result now follows from (11) and (12).

*Proof of theorem.* Raising  $G(1)$  to the  $q$ th power we obtain modulo  $q$ ,

$$G(1)^q \equiv \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8^q \exp(2\pi i x q/p) = \sum_{x=0}^{p-1} \left(\frac{x}{\pi}\right)_8 \exp(2\pi i x q/p) ,$$

since  $q \equiv 1 \pmod{q}$ , giving

$$G(1)^q \equiv \left(\frac{q}{\pi}\right)_s^{-1} \sum_{x=0}^{p-1} \left(\frac{xq}{\pi}\right)_s \exp(2\pi i(xq)/p) = \left(\frac{q}{\pi}\right)_s^{-1} G(1),$$

since  $(q, p) = 1$  implies that

$$\sum_{x=0}^{p-1} \left(\frac{xq}{\pi}\right)_s \exp(2\pi ixq/p) = \sum_{y=0}^{p-1} \left(\frac{y}{\pi}\right)_s \exp(2\pi iy/p) = G(1).$$

Hence

$$G(1)^q \equiv \left(\frac{q}{\pi}\right)_s^{-1} G(1) = \left(\frac{q}{p}\right)_s G(1),$$

that is

$$G(1)^{q-1} \equiv \left(\frac{q}{p}\right)_s (\text{mod } q).$$

Hence by Lemmas 1, 2, 3 we have modulo  $q$

$$\begin{aligned} \left(\frac{q}{p}\right)_s &\equiv (G(1)^8)^{(q-1)/8} \\ &\equiv p^{(q-1)/8} (a + b\sqrt{-1})^{(q-1)/4} (c + d\sqrt{-2})^{(q-1)/2} \\ &\equiv \left(\frac{p}{q}\right)_s \left(\frac{aB - bA}{q}\right)_4 \left(\frac{cD - dC}{q}\right), \end{aligned}$$

from which the theorem follows.

EXAMPLE. We take  $p = 17 \equiv 1 \pmod{8}$  and  $q = 409 \equiv 1 \pmod{8}$  so that we may choose

$$\begin{aligned} a &= 1, b = 4, c = 3, d = 2, \\ A &= 3, B = 20, C = 11, D = 12. \end{aligned}$$

Since  $q \equiv 1 \pmod{p}$  we clearly have

$$\left(\frac{q}{p}\right) = \left(\frac{q}{p}\right)_4 = \left(\frac{q}{p}\right)_8 = 1.$$

As  $((aB - bA)/q) = (8/409) = +1$  by Burde's theorem we have  $(p/q)_4 = 1$ . Finally

$$\begin{aligned} \left(\frac{aB - bA}{q}\right)_4 &= \left(\frac{8}{409}\right)_4 = \left(\frac{194}{409}\right) = -1, \\ \left(\frac{cD - dC}{q}\right) &= \left(\frac{14}{409}\right) = -1, \end{aligned}$$

so by the theorem of this paper we have  $(p/q)_8 = 1$ , which is easily verified directly.



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