A RATIONAL OCTIC RECIPROCITY LAW

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A rational octic reciprocity theorem analogous to the rational biquadratic reciprocity theorem of Burde is proved.

Let \( p \) and \( q \) be distinct primes \( \equiv 1 \pmod{4} \) such that \((p/q) = (q/p) = 1\). For such primes there are integers \( a, b, A, B \) with

\[
\begin{aligned}
\left\{ 
& p = a^2 + b^2, \ a \equiv 1 \pmod{2}, \ b \equiv 0 \pmod{2}, \\
&q = A^2 + B^2, \ A \equiv 1 \pmod{2}, \ B \equiv 0 \pmod{2}.
\end{aligned}
\]

Moreover it is well-known than \((A/q) = 1, (B/q) = (-1)^{(q-1)/4}\). If \( k \) is a quadratic residue \( \pmod{q} \) we set

\[
\left( \frac{k}{q} \right)_4 = \begin{cases} 
+1, & \text{if } k \text{ is a biquadratic residue } \pmod{q}, \\
-1, & \text{otherwise}.
\end{cases}
\]

In 1969 Burde [2] proved the following

**Theorem (Burde).**

\[
\left( \frac{p}{q} \right)_4 \left( \frac{q}{p} \right)_4 = (-1)^{(q-1)/4} \left( \frac{aB - bA}{q} \right).
\]

Recently Brown [1] has posed the problem of finding an octic reciprocity law analogous to Burde's biquadratic law for distinct primes \( p \) and \( q \) with \( p \equiv q \equiv 1 \pmod{8} \) and \((p/q)_4 = (q/p)_4 = 1\). It is the purpose of this paper to give such a law. From this point on we assume that \( p \) and \( q \) satisfy these conditions and set for any biquadratic residue \( k \pmod{q} \)

\[
\left( \frac{k}{q} \right)_8 = \begin{cases} 
+1, & \text{if } k \text{ is an octic residue } \pmod{q}, \\
-1, & \text{otherwise}.
\end{cases}
\]

It is a familiar result that there are integers \( c, d, C, D \) with

\[
\begin{aligned}
\left\{ 
&p = c^2 + 2d^2, \ c \equiv 1 \pmod{2}, \ d \equiv 0 \pmod{2}, \\
&q = C^2 + 2D^2, \ C \equiv 1 \pmod{2}, \ D \equiv 0 \pmod{2}.
\end{aligned}
\]

Moreover we have \((D/q)_4 = 1\). Also from Burde's theorem we have

\[
\left( \frac{aB - bA}{q} \right) = 1,
\]

and from the law of biquadratic reciprocity after a little calculation we find that \((B/q)_4 = +1\). We prove
THEOREM. Let \( p \) and \( q \) be distinct primes \( \equiv 1(\text{mod} \ 8) \) such that
\[
\left( \frac{p}{q} \right)_4 = \left( \frac{q}{p} \right)_4 = 1.
\]
Then
\[
\left( \frac{p}{q} \right)_4 \left( \frac{q}{p} \right)_4 = \left( \frac{aB - bA}{q} \right)_4 \left( \frac{cD - dC}{q} \right)_4.
\]

We note that it is easy to show that the expression on the right-hand side of the theorem is independent of the particular choices of \( a, b, c, d, A, B, C, D \) made in (1) and (2). In the course of the proof it is convenient to make a particular choice of \( a, b, c, d \) (see (9) and (10)).

We begin by proving three lemmas.

**Lemma 1.** \( (c + d\sqrt{-2})^{(q-1)/2} \equiv ((cD - dC)/q) \) \( (\text{mod} \ q) \).

**Proof.** As \( (p/q) = 1 \) we can define an integer \( u \) by \( p = u^2(\text{mod} \ q) \). Next we define integers \( l \) and \( m \) by
\[
l \equiv \frac{cD - dC + Du}{2}, \quad m \equiv \frac{C \cdot cD - dC - Du}{4} \quad (\text{mod} \ q),
\]
so that
\[
l^2 - 2m^2 \equiv cD(cD - dC) \quad (\text{mod} \ q)
\]
and
\[
2lm \equiv dD(cD - dC) \quad (\text{mod} \ q),
\]
giving
\[
D(cD - dC)(c + d\sqrt{-2}) \equiv (l + m\sqrt{-2})^q \quad (\text{mod} \ q),
\]
and so
\[
D^{(q-1)/2}(cD - dC)^{(q-1)/2}(c + d\sqrt{-2})^{(q-1)/2} \equiv (l + m\sqrt{-2})^{q-1} \quad (\text{mod} \ q).
\]
Now working modulo \( q \) we have
\[
(l + m\sqrt{-2})^{q-1} = \frac{(l + m\sqrt{-2})^q}{l + m\sqrt{-2}} = \frac{l^q + m^q(\sqrt{-2})^q}{l + m\sqrt{-2}} = \frac{l + m\sqrt{2^{q/2}}}{l + m\sqrt{-2}} \equiv 1,
\]
also
\[ D^{(q-1)/2} \equiv \left( \frac{D}{q} \right) = 1, \]

and

\[ (cD - dC)^{(q-1)/2} \equiv \left( \frac{cD - dC}{q} \right), \]

from which the required result follows immediately.

**Lemma 2.** \((a + b\sqrt{-1})^{(q-1)/4} \equiv ((aB - bA)/q) \pmod{q} .\)

**Proof.** As \((p/q) = 1\) we define an integer \(u\) by \(p \equiv u \pmod{q}\) as in Lemma 1. Next we define integers \(r\) and \(s\) by

\[ r \equiv \frac{aB - bA + Bu}{2}, \quad s \equiv \frac{A \cdot aB - bA - Bu}{2} \pmod{q} \]

so that

\[ r^2 - s^2 \equiv aB(aB - bA) \pmod{q} \]

and

\[ 2rs \equiv bB(aB - bA) \pmod{q} \]

giving

\[ B(aB - bA)(a + b\sqrt{-1}) \equiv (r + s\sqrt{-1})^2 \pmod{q}, \]

and so

\[ B^{(q-1)/4}(aB - bA)^{(q-1)/4}(a + b\sqrt{-1})^{(q-1)/4} \equiv (r + s\sqrt{-1})^{(q-1)/2} \pmod{q}. \]

Thus as \((B/q)_4 = ((aB - bA)/q) = 1\) we obtain

\[ (a + b\sqrt{-1})^{(q-1)/4} \equiv \left( \frac{aB - bA}{q} \right) \left( r + s\sqrt{-1} \right)^{(q-1)/2} \pmod{q}. \]

Next we note that \(r^2 + s^2 \equiv uB(aB - bA) \pmod{q}\) so that

\[ \left( \frac{r^2 + s^2}{q} \right) = \left( \frac{p}{q} \right) \left( \frac{B}{q} \right) \left( \frac{aB - bA}{q} \right) = 1. \]

Hence we may define an integer \(w\) by \(w^2 \equiv r^2 + s^2 \pmod{q}\). Then we define integers \(e\) and \(f\) by

\[ e \equiv \frac{rB - sA + Bw}{2}, \quad f \equiv \frac{A \cdot rB - sA - Bw}{2} \pmod{q}, \]

so that
\[ e^2 - f^2 \equiv rB(rB - sA) \pmod{q} \]

and

\[ 2ef \equiv sB(rB - sA) \pmod{q} \]

giving

\[ B(rB - sA)(r + s\sqrt{-1}) \equiv (e + f\sqrt{-1})^2 \pmod{q}, \]

and so

\[ B^{(q-1)/2}(rB - sA)^{(q-1)/2}(r + s\sqrt{-1})^{(q-1)/2} \equiv (e + f\sqrt{-1})^{q-1} \pmod{q}. \]

Now working modulo \( q \) we have

\[
(e + f\sqrt{-1})^{q-1} \equiv \frac{(e + f\sqrt{-1})^q}{e + f\sqrt{-1}} \equiv \frac{e^q + f^q(\sqrt{-1})^q}{e + f\sqrt{-1}} \\
\equiv \frac{e + f\sqrt{-1}}{e + f\sqrt{-1}} = 1,
\]

and

\[ B^{(q-1)/2} = \left( \frac{B}{q} \right) = 1, \quad (rB - sA)^{(q-1)/2} \equiv \left( \frac{rB - sA}{q} \right), \]

so

\[ (r + s\sqrt{-1})^{(q-1)/2} \equiv \left( \frac{rB - sA}{q} \right), \]

giving

\[ (a + b\sqrt{-1})^{(q-1)/4} \equiv \left( \frac{aB - bA}{q} \right) \left( \frac{rB - sA}{q} \right) \pmod{q}. \]

The required result now follows as modulo \( q \) we have

\[ rB - sA \equiv \frac{B(aB - bA + Bu)}{2} - \frac{A^2}{B} \left( \frac{aB - bA - Bu}{2} \right) \]

\[ \equiv \frac{B}{2} \{(aB - bA + Bu) + (aB - bA - Bu)\} \]

\[ \equiv B(aB - bA), \]

that is

\[ \left( \frac{rB - sA}{q} \right) = \left( \frac{B}{q} \right) \left( \frac{aB - bA}{q} \right) = +1. \]

Before proving the final lemma we state some results we shall need. Let \( w = \exp(2\pi i/8) = (\sqrt{2} + \sqrt{-2})/2 \) and let \( R \) be the ring
of integers of the cyclotomic field \( Q(w) = Q(\sqrt{2}, \sqrt{-1}) \). \( R \) is a unique factorization domain. Let \( \pi \) be any prime factor of \( p \) in \( R \), fixed once and for all. For integers \( x \not\equiv 0(\text{mod } p) \) we define an octic character \( (\cdot/\pi)_8 \) by

\[
\left( \frac{x}{\pi} \right)_8 = w^{i} \text{ if } x^{(p-1)/8} = w^{\ell}(\text{mod } \pi), \quad 0 \leq \ell \leq 7.
\]

If \( x \equiv 0(\text{mod } p) \) we set \( (x/\pi)_8 = 0 \). In terms of this character we define the corresponding Jacobi and Gauss sums for arbitrary integers \( k \) and \( \ell \) as follows:

\[
J(k, \ell) = \sum_{x=0}^{p-1} \left( \frac{x}{\pi} \right)_8^k \left( \frac{1-x}{\pi} \right)_8^\ell,
\]

\[
G(k) = \sum_{x=0}^{p-1} \left( \frac{x}{\pi} \right)_8^k \exp(2\pi i x/p).
\]

These sums have the following well-known properties (see for example [4], Chapter 8):

(4) \( J(k, \ell)J(k, \ell) = p \), if \( k, \ell \not\equiv 0(\text{mod } 8) \),

(5) \( J(k, \ell) = \frac{G(k)G(\ell)}{G(k+\ell)} \), if \( k, \ell, k+\ell \not\equiv 0(\text{mod } 8) \),

(6) \( G(k)G(-k) = (-1)^{k(p-1)/8}p \), if \( k \not\equiv 0(\text{mod } 8) \).

We shall also need the evaluation of the familiar sum

(7) \( G(4) = \sum_{x=0}^{p-1} \left( \frac{x}{\pi}_8 \right)^4 \exp(2\pi i x/p) = \sum_{x=0}^{p-1} \left( \frac{x}{p} \right) \exp(2\pi i x/p) = p^{1/2} \)

and the result

(8) \( J(2, 2) = \pm J(1, 2) \).

A more precise form of (8) follows from a theorem of Jacobi (see for Example [3], page 411, equation (99)). Finally we let \( \sigma_k(k = 1, 3, 5, 7) \) be the automorphism of \( Q(w) \) defined by \( \sigma_k(w) = w^k \).

Now from (5) and (6) we have

\( \sigma_3(J(1, 4)) = J(3, 12) = J(3, 4) = \frac{G(3)G(4)}{G(7)} = \frac{G(1)G(4)}{G(5)} = J(1, 4) \),

so that \( J(1, 4) \in \mathbb{Z}[\sqrt{-2}] \). Moreover from (4) we have \( J(1, 4)J(1, 4) = p \) so we may choose the signs of \( c \) and \( d \) in (2) so that

(9) \( J(1, 4) = c + d\sqrt{-2} \).

Also from (5) and (6) we have
\[ \sigma_s(J(1, 2)) = J(5, 10) = J(5, 2) = \frac{G(5)G(2)}{G(7)} = \frac{G(1)G(2)}{G(3)} = J(1, 2) , \]

so that \( J(1, 2) \in \mathbb{Z}[\sqrt{-1}] \). Moreover from (4) we have \( J(1, 2)J(1, 2) = p \) so we may choose the signs of \( a \) and \( b \) in (1) so that

\begin{equation}
J(1, 2) = a + b\sqrt{-1} ,
\end{equation}

since it is easy to prove (and well-known) that \( J(1, 2) \equiv 1(\text{mod } 2) \).

**Lemma 3.** \( G(1)^s = p(a + b\sqrt{-1})^4(c + d\sqrt{-2})^4 . \)

**Proof.** From (5), (9), (10) have

\[ c + d\sqrt{-2} = J(1, 4) = \frac{G(1)G(4)}{G(5)} \]

and

\[ a + b\sqrt{-1} = J(1, 2) = \frac{G(1)G(2)}{G(3)} . \]

Multiplying these together we obtain

\[ (a + b\sqrt{-1})(c + d\sqrt{-2}) = \frac{G(1)^sG(2)G(4)}{G(3)G(5)} = \frac{G(1)^sG(2)}{(-1)^{(p-1)/8}p^{1/2}} \]

by (6) and (7). Hence taking the fourth power of both sides we get

\begin{equation}
G(1)^sG(2)^4 = p^2 (a + b\sqrt{-1})^4(c + d\sqrt{-2})^4 .
\end{equation}

Now from (5) and (7) we have

\[ J(2, 2) = \frac{G(2)^s}{G(4)} = \frac{G(2)^s}{p^{1/2}} , \]

so that from (8) and (10) we obtain

\begin{equation}
G(2)^4 = p[J(2, 2)]^2 = p[J(1, 2)]^2 = p(a + b\sqrt{-1})^2 ,
\end{equation}

and the required result now follows from (11) and (12).

**Proof of theorem.** Raising \( G(1) \) to the \( q \)th power we obtain modulo \( q \),

\[ G(1)^q = \sum_{x=0}^{q-1} \left( \frac{x}{\pi} \right)^q \exp \left( 2\pi ixq/p \right) = \sum_{x=0}^{q-1} \left( \frac{x}{\pi} \right)^q \exp \left( 2\pi ixq/p \right) , \]

since \( q \equiv (\text{mod } q) \), giving
\[ G(1)^q = \left( \frac{q}{\pi} \right)_8^{-1} \sum_{x=0}^{p-1} \left( \frac{xq}{\pi} \right)_8 \exp \left( 2\pi i(xq)/p \right) = \left( \frac{q}{\pi} \right)_8^{-1} G(1) , \]

since \((q, p) = 1\) implies that
\[ \sum_{x=0}^{p-1} \left( \frac{xq}{\pi} \right)_8 \exp \left( 2\pi i(xq)/p \right) = \sum_{y=0}^{p-1} \left( \frac{y}{\pi} \right)_8 \exp \left( 2\pi iy/p \right) = G(1) . \]

Hence
\[ G(1)^q = \left( \frac{q}{\pi} \right)_8^{-1} G(1) = \left( \frac{q}{p} \right)_8 G(1) , \]

that is
\[ G(1)^{q-1} = \left( \frac{q}{p} \right)_8 \mod q . \]

Hence by Lemmas 1, 2, 3 we have modulo \(q\)
\[ \left( \frac{q}{p} \right)_8 = (G(1))^{(q-1)/8} \]
\[ = p^{(q-1)/4}(a + b\sqrt{-1})^{(q-1)/4}(c + d\sqrt{-2})^{(q-1)/2} \]
\[ = \left( \frac{p}{q} \right)_8 \left( \frac{aB - bA}{q} \right)_8 \left( \frac{cD - dC}{q} \right)_8 , \]

from which the theorem follows.

**Example.** We take \(p = 17 \equiv 1 \mod 8\) and \(q = 409 \equiv 1 \mod 8\) so that we may choose
\[ a = 1, \ b = 4, \ c = 3, \ d = 2 , \]
\[ A = 3, \ B = 20, \ C = 11, \ D = 12 . \]

Since \(q \equiv 1 \mod p\) we clearly have
\[ \left( \frac{q}{p} \right)_8 = \left( \frac{q}{p} \right)_8 = \left( \frac{q}{p} \right)_8 = 1 . \]

As \(((aB - bA)/q) = (8/409) = +1\) by Burde’s theorem we have \((p/q)_s = 1\). Finally
\[ \left( \frac{aB - bA}{q} \right)_s = \left( \frac{8}{409} \right)_s = \left( \frac{194}{409} \right)_s = -1 , \]
\[ \left( \frac{cD - dC}{q} \right)_s = \left( \frac{14}{409} \right)_s = -1 , \]

so by the theorem of this paper we have \((p/q)_s = 1\), which is easily verified directly.
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