

# Pacific Journal of Mathematics

**A GENERALIZATION OF A THEOREM OF CHACON**

ROBERT CHEN

# A GENERALIZATION OF A THEOREM OF CHACON

ROBERT CHEN

**A generalization of a theorem of Chacon is proved simply by an application of a maximal inequality. A pointwise convergence theorem and the submartingale convergence theorem are immediate consequences.**

Let  $(\Omega, \mathcal{F}, P)$  be a probability space,  $\{X_n\}$  be a sequence of integrable random variables adapted to the increasing sequence  $\{\mathcal{F}_n\}$  of sub  $\sigma$ -fields of  $\mathcal{F}$ ,  $B$  be the collection of all bounded stopping times (with respect to  $\{\mathcal{F}_n\}$ ), and  $D$  be the collection of random variables  $Y$  which are measurable with respect to  $\mathcal{F}_\infty = \sigma(\{\mathcal{F}_n\})$  and, for each  $w$  in  $\Omega$ ,  $Y(w)$  is a cluster value of the sequence  $\{X_n(w)\}$ .

The main purpose of this note is to generalize (in Theorem 1) the result stated as Corollary 1, due to Chacon ([3]). The result is a reformulation of a result due to Baxter ([2]) but our method of proof is much simpler than that in ([2]) and ([3]), and is just a simple application of a maximal inequality due to Chacon and Sucheston ([4]). A pointwise convergence theorem and the submartingale convergence theorem are immediate consequences ([1] and [5]).

**THEOREM 1.** *Suppose that  $\sup_{t \in B} E(|X_t|) < \infty$  and  $Y_1, Y_2$  are any two random variables in  $D$ . Then there exist  $\tau_n^*, t_n^*$  in  $B$  such that  $\tau_n^* \geq n$ ,  $t_n^* \geq n$ , and*

$$(1) \quad \lim_{n \rightarrow \infty} E\{|(X_{\tau_n^*} - X_{t_n^*}) - (Y_1 - Y_2)|\} = 0.$$

*Proof.* By Lemma 1 of [1] and the Borel-Cantelli lemma, for any two random variables  $Y_1, Y_2$  in  $D$ , there exist two strictly increasing sequences  $\{\tau_n\}$  and  $\{t_n\}$  in  $B$  such that  $\lim_{n \rightarrow \infty} X_{\tau_n} = Y_1$  almost surely and  $\lim_{n \rightarrow \infty} X_{t_n} = Y_2$  almost surely. By the condition that  $\sup_{t \in B} E(|X_t|) < \infty$  and the Fatou lemma,  $Y_1$  and  $Y_2$  are integrable.

To prove (1), we need a maximal inequality, which I learned from Chacon and Sucheston.

$$(2) \quad \lambda P\left(\left[\sup_n |X_n| \geq \lambda\right]\right) \leq \sup_{t \in B} E(|X_t|) \text{ for each positive constant } \lambda.$$

To see (2), let  $M$  be a fixed positive integer and define a bounded

stopping time  $\tau$  by  $\tau(w) = \inf\{n \mid 1 \leq n \leq M, |X_n(w)| \geq \lambda\}$ ,  $\tau(w) = M + 1$  if no such  $n$  exists,  $w \in \Omega$ . Then

$$\lambda P\left(\left[\sup_{1 \leq n \leq M} |X_n| \geq \lambda\right]\right) \leq E(|X_\tau|) \leq \sup_{t \in B} E(|X_t|).$$

(2) follows immediately on letting  $M \rightarrow \infty$ .

Now, for each positive integer  $k$  and each positive constant  $d$ , define  $j(k, d) = \inf\{n \mid k \leq n, |X_n| \geq d\}$ ,  $j(k, d) = \infty$  if no such  $n$  exists. Let  $A(k, d) = [j(k, d) < \infty]$ . Since, by (2), for fixed  $k$ ,  $P(A(k, d)) \rightarrow 0$  as  $d \rightarrow \infty$ ,  $E\{|(Y_1 - Y_2)\chi_{A(k, d)}|\} \rightarrow 0$  as  $d \rightarrow \infty$ . Therefore, for each positive integer  $k$ , there exists a  $d_k$  such that  $E\{|(Y_1 - Y_2)\chi_{A(k, d_k)}|\} \leq 1/k$ . Next, for each fixed  $k$ , let  $Z = \max\{|X_1|, |X_2|, \dots, |X_{k-1}|, d_k\chi_{A^c(k, d_k)} + |X_{j(k, d_k)}\chi_{A(k, d_k)}|\}$ ,  $Z_n = X_{n \wedge j(k, d_k)}$  for all  $n \geq 1$ . Then it is easy to see that  $|Z_n| \leq Z$  for all  $n \geq 1$  and  $E\{Z\} < \infty$ . Since  $\lim_{n \rightarrow \infty} (X_{\tau_n} - X_{t_n}) = (Y_1 - Y_2)$  almost surely and, on  $A(k, d_k)$ ,  $\lim_{n \rightarrow \infty} (Z_{\tau_n} - Z_{t_n}) = 0$  (since  $\{\tau_n\}$  and  $\{t_n\}$  are strictly increasing).  $\lim_{n \rightarrow \infty} (Z_{\tau_n} - Z_{t_n}) = (Y_1 - Y_2)\chi_{A^c(k, d_k)}$  almost surely. Therefore, by the Lebesgue dominated convergence theorem,  $E\{|(Z_{\tau_n} - Z_{t_n}) - (Y_1 - Y_2)\chi_{A^c(k, d_k)}|\} \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $j(k, d_k) \geq k$  and  $\{\tau_n\}, \{t_n\}$  are strictly increasing, we can and do choose, for each positive integer  $k$ , two bounded stopping times  $\tau_k^*$  and  $t_k^*$  in  $B$  such that  $\tau_k^* \geq k$ ,  $t_k^* \geq k$ , and  $E\{|(X_{\tau_k^*} - X_{t_k^*}) - (Y_1 - Y_2)\chi_{A^c(k, d_k)}|\} \leq 1/k$ . Therefore,  $\tau_k^* \geq k$ ,  $t_k^* \geq k$ , and  $E\{|(X_{\tau_k^*} - X_{t_k^*}) - (Y_1 - Y_2)\chi_{A^c(k, d_k)}|\} \leq 2/k$  for all  $k \geq 1$ . (1) follows on letting  $k \rightarrow \infty$  and the proof of Theorem 1 now is complete.

**COROLLARY 1 (Chacon).** *Let  $\{X_n\}$  be a sequence of integrable random variables such that  $\liminf_{n \rightarrow \infty} E(|X_n|) < \infty$ . Then,*

$$(3) \quad \limsup_{\tau, t \in B} E(X_\tau - X_t) \geq E(X^* - X_*), \text{ where } X^* = \limsup_{n \rightarrow \infty} X_n, \text{ and}$$

$$X_* = \liminf_{n \rightarrow \infty} X_n.$$

*Further, if  $\sup_{t \in B} E(|X_t|) < \infty$ , then  $X^*$  and  $X_*$  are integrable.*

*Proof.* If  $\sup_{t \in B} E(|X_t|) < \infty$ , then, by Theorem 1,  $X^*, X_*$  are integrable and  $\limsup_{\tau, t \in B} E(X_\tau - X_t) \geq E(X^* - X_*)$ . If  $\sup_{t \in B} E(|X_t|) = \infty$ , without loss of generality, we can and do assume that  $\sup_{t \in B} E(X_t^+) = \infty$ . Since  $\liminf_{n \rightarrow \infty} E(|X_n|) < \infty$ , there exists a strictly increasing sequence  $\{n_j\}$  of positive integers such that  $E(|X_{n_j}|) \leq M$  for all  $j \geq 1$  and some constant  $M$ . Now, for each bounded stopping time  $t$  in  $B$ , let  $t' = t$  on  $\{X_t^+ > 0\}$  and  $t' = n$  on  $\{X_t^+ = 0\}$  where  $n = \inf\{n_j \mid n_j \geq \sup\{t(w) \mid w \in \{X_t^+ = 0\}\}\}$ . We then have  $E(X_{t'} - X_n) \geq E(X_t^+) - M$  and

$\sup_{\tau, t} E(X_\tau - X_t) = \infty \cong E(X^* - X_*)$  and (3) follows immediately from this fact. The proof of Corollary 1 now is complete.

**COROLLARY 2** (Theorem 2 of [1]). *Under the conditions of Corollary 1 and consider the following two assertions:*

- (a) *The generalized sequence  $\{E(X_t) | t \in B\}$  is convergent.*
- (b)  *$X_n$  converges almost surely to a finite limit.*

*Then (a) implies (b).*

**COROLLARY 3** (the submartingale convergence theorem). *Suppose that  $\{X_n\}$  is a sequence of  $L_1$ -bounded random variables adapted to the increasing sequence  $\{\mathcal{F}_n\}$  of  $\sigma$ -fields. Suppose that  $E(X_{n+1} | \mathcal{F}_n) \cong X_n$  almost surely for all  $n \cong 1$ . Then  $X_n$  converges almost surely to a finite limit.*

**REMARK.** Corollaries 1 and 2 also hold under any one of the following two conditions.

- (i)  $\sup_n E(X_n^+) < \infty$ .
- (ii)  $\sup_n E(X_n^-) < \infty$ .

**ACKNOWLEDGEMENTS.** I would like to thank Professors Chacon and Sucheston for their valuable suggestions and comments.

#### REFERENCES

1. D. G. Austin, G. A. Edgar, and A. Ionescu Tulcea, *Pointwise convergence in terms of expectations*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete, **30** (1974), 17-26.
2. J. R. Baxter, *Convergence of randomly stopped variables*, (to appear).
3. R. V. Chacon, *A "stopped" proof of convergence*, Advances in Mathematics, **14** (1974), 365-368.
4. R. V. Chacon and L. Sucheston, *On convergence of vector-valued asymptotic Martingales*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete, **33** (1975), 55-59.
5. Ch. W. Lamb, *A short proof of the Martingale convergence theorem*, Proc. Amer. Math. Soc., **38** (1973), 215-217.

Received November 25, 1975.

UNIVERSITY OF MIAMI





Walter Allegretto, <i>Nonoscillation theory of elliptic equations of order <math>2n</math></i> . . . . .	1
Bruce Allen Anderson, <i>Sequencings and starters</i> . . . . .	17
Friedrich-Wilhelm Bauer, <i>A shape theory with singular homology</i> . . . . .	25
John Kelly Beem, <i>Characterizing Finsler spaces which are pseudo-Riemannian of constant curvature</i> . . . . .	67
Dennis K. Burke and Ernest A. Michael, <i>On certain point-countable covers</i> . . . . .	79
Robert Chen, <i>A generalization of a theorem of Chacon</i> . . . . .	93
Francis H. Clarke, <i>On the inverse function theorem</i> . . . . .	97
James Bryan Collier, <i>The dual of a space with the Radon-Nikodým property</i> . . . . .	103
John E. Cruthirds, <i>Infinite Galois theory for commutative rings</i> . . . . .	107
Artatrana Dash, <i>Joint essential spectra</i> . . . . .	119
Robert M. DeVos, <i>Subsequences and rearrangements of sequences in FK spaces</i> . . . . .	129
Geoffrey Fox and Pedro Morales, <i>Non-Hausdorff multifunction generalization of the Kelley-Morse Ascoli theorem</i> . . . . .	137
Richard Joseph Fleming, Jerome A. Goldstein and James E. Jamison, <i>One parameter groups of isometries on certain Banach spaces</i> . . . . .	145
Robert David Gulliver, II, <i>Finiteness of the ramified set for branched immersions of surfaces</i> . . . . .	153
Kenneth Hardy and István Juhász, <i>Normality and the weak cb property</i> . . . . .	167
C. A. Hayes, <i>Derivation of the integrals of <math>L^{(q)}</math>-functions</i> . . . . .	173
Frederic Timothy Howard, <i>Roots of the Euler polynomials</i> . . . . .	181
Robert Edward Jamison, II, Richard O'Brien and Peter Drummond Taylor, <i>On embedding a compact convex set into a locally convex topological vector space</i> . . . . .	193
Andrew Lelek, <i>An example of a simple triod with surjective span smaller than span</i> . . . . .	207
Janet E. Mills, <i>Certain congruences on orthodox semigroups</i> . . . . .	217
Donald J. Newman and A. R. Reddy, <i>Rational approximation of <math>e^{-x}</math> on the positive real axis</i> . . . . .	227
John Robert Quine, Jr., <i>Homotopies and intersection sequences</i> . . . . .	233
Nambury Sitarama Raju, <i>Periodic Jacobi-Perron algorithms and fundamental units</i> . . . . .	241
Herbert Silverman, <i>Convexity theorems for subclasses of univalent functions</i> . . . . .	253
Charles Frederick Wells, <i>Centralizers of transitive semigroup actions and endomorphisms of trees</i> . . . . .	265
Volker Wrobel, <i>Spectral approximation theorems in locally convex spaces</i> . . . . .	273
Hidenobu Yoshida, <i>On value distribution of functions meromorphic in the whole plane</i> . . . . .	283