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Two characterizations of a Banach space with the Radon–Nikodym property are proved here. The first shows its equivalence with a condition on the dual space which is somewhat weaker than that of being an Asplund space. This leads to a second characterization by a renorming property.

A convex function f on a Banach space X will be assumed to take its values in $(-\infty, +\infty]$ and to be finite at some point. The *domain of continuity* of f is the convex open set of all points at which f is finite and continuous. The space X is called an *Asplund space* if each convex function on X is Fréchet differentiable on a dense G_δ subset of its domain of continuity. If X is the dual of a Banach space Y, then it will be called a *weak*-Asplund space* if each weak* lower semi-continuous (w^*-lsc) convex function on X is Fréchet differentiable on a dense G_δ subset of its domain of continuity. The terms "G_δ" and "domain of continuity" here still refer to the norm topology on X. Thus a dual space which is an Asplund space is also a weak*-Asplund space. A Banach space may be said to have the *Radon–Nikodym property* (RNP) if each closed bounded convex subset is the closed convex hull of its strongly exposed points **[6]**. A point x in a set C is said to be *strongly exposed* by a linear functional y if the supremum of y over C is finite and attained at x and $\|x_i - x\| \to 0$ whenever $\{x_i\}$ is a sequence in C for which $y(x_i) \to y(x)$.

Using the same method as in **[3]**, we characterize the dual of a space with the RNP by its differentiability properties. This allows us to give an alternate proof of a result of Huff and Morris **[4]** concerning the density of strongly exposing functionals and to observe that weak*-Asplund spaces enjoy some of the permanence properties that Asplund spaces do.

THEOREM 1. *A Banach space X has the* RNP *if and only if X^* is a weak*-Asplund space.*

Proof. Assume X has the RNP and let f be a w^*-lsc convex function on X^* with nonempty domain of continuity D. Choose any point $w \in D$ and an $\varepsilon > 0$ so that f is bounded on $N = \{y : \|y - w\| \leqq \epsilon\}$ and $N \subseteq D$. We use the dual norm so that N is weak* closed. Define g

on X^* by $g(y)=f(y)$ if $y \in N$ and $g(y)= +\infty$ otherwise. Then g is a w^*-lsc convex function on X^*, bounded on N, and whose domain of continuity is the interior of N. We may assume without loss of generality that the unit ball B of X^* is contained in N and $-1 \leqq g(y) \leqq 0$ for all $y \in N$. Choose some $\lambda > 1$ such that $N \subseteq \lambda B$.

Define p on X^* by $p(y)=0$ if $y \in B$ and $p(y)= +\infty$ otherwise. Let $q(y)=p(y/\lambda)-1$. Then p and q are w^*-lsc convex functions on X^*. For any convex function h on X^*, the conjugate of h on X is $h^*(x)=\sup\{y(x)-h(y)\colon\ y \in X^*\}$ for each $x \in X$. Thus $p^*(x)=\|x\|$ and $q^*(x)=\lambda\|x\|+1$. Since $q(y) \leqq g(y) \leqq p(y)$ for all $y \in X^*$, $p^*(x) \leqq g^*(x) \leqq q^*(x)$ for all $x \in X$, and hence $\|x\| \leqq g^*(x) \leqq \lambda\|x\|+1$ for all $x \in X$. This implies that the closed convex set $C=\{x \in X\colon\ g^*(x) \leqq 2\}$ is bounded and has nonempty interior.

Let epi $g^*=\{(x,r)\colon\ x \in X,\ g^*(x) \leqq r\}$, $H=\{(x,r)\colon\ x \in X,\ r \geqq 2\}$ and $K=\text{epi } g^* \cap H$. Since $K \subseteq C \times [0,2]$, K is a closed bounded convex subset of $X \times \mathbf{R}$ with nonempty interior. It is well-known that the RNP is preserved under products; hence $X \times \mathbf{R}$ has the RNP and K must be the closed convex hull of its strongly exposed points. As a consequence, there must be a point a in the interior of C such that $g^*(a)<2$ and $(a, g(a))$ is strongly exposed as a point of K by some functional $(b,-1) \in X^* \times \mathbf{R}$. Since $g^*(a)<2$ and epi g is convex, $(a, g(a))$ is also strongly exposed as a point of epi g by $(b,-1)$. Because g is w^*-lsc, Theorem 1 in [**2**, p. 450] together with the Lemma in [**3**] implies that g is Fréchet differentiable at b with gradient a. Since b lies in the interior of N, f is also Fréchet differentiable at b and $\|w-b\|<\epsilon$. Since the choice of $w \in D$ and $\epsilon>0$ was arbitrary, the set G of points at which f is Fréchet differentiable is dense in D. Lemma 6 in [**1**, p. 43] implies that G must in fact be a dense G_δ subset of D and therefore X^* is a weak*-Asplund space.

Assume now that X^* is a weak*-Asplund space and C is a closed bounded convex set in X. Define $f(x)=0$ if $x \in C$ and $f(x)= +\infty$ otherwise. Then $f^*(y)=\sup\{y(x)-f(x)\colon\ x \in X\}$ is a w^*-lsc convex function on X^* whose domain of continuity is X^*. Since X^* is a weak*-Asplund space, f^* is Fréchet differentiable on a dense G_δ subset G of X^*. From Theorem 1 in [**2**, p. 450] it follows that each functional in G strongly exposes a point of C. The density of G implies that C is the closed convex hull of its strongly exposed points and hence X has the RNP.

The last part of the proof of Theorem 1 actually proves the following result of Huff and Morris [**4**]:

COROLLARY 2. *If X has the* RNP *and C is a closed bounded convex subset, then the set of linear functionals which strongly expose some point of C is a dense G_δ subset of X^*.*

The convexity restriction on C which occurs in this proof is easily dropped by observing that a linear functional strongly exposes a point of a closed bounded set A whenever it strongly exposes a point of the closed convex hull of A.

Since the dual norm on a dual space is a w^*-lsc convex function, the following is also an immediate consequence of Theorem 1:

COROLLARY 3. *If X has the* RNP, *then the dual norm on X^* is Fréchet differentiable on a dense G_δ subset of X^*.*

The differentiability of the dual norm can be used to characterize spaces with the RNP. Let A be a nonempty bounded subset of a Banach space X. A slice of A will be any set of the form $S(A, y, \epsilon) = \{x \in A\colon y(x) + \epsilon > \sup y[A]\}$ where $y \in X^*$ and $\epsilon > 0$. We can show the following:

THEOREM 4. *A Banach space X fails to have the* RNP *if and only if there is an equivalent norm on X for which the dual norm on X^* is Fréchet differentiable nowhere.*

Proof. If such a renorming exists, then Corollary 3 implies that X cannot have the RNP. In order to prove the other direction assume that for each equivalent norm on X, the dual norm on X^* is Fréchet differentiable at some point. Let C be any closed bounded convex subset of X and let B be the unit ball. Let D be the closure of $C + B$ and let E be the closure of $D + (-D)$, then E is the unit ball of an equivalent norm on X.

Define f on X by $f(x) = 0$ is $x \in E$ and $f(x) = +\infty$ otherwise, then the conjugate of f, $f^*(y) = \sup\{y(x) - f(x)\colon x \in X\}$, is the corresponding dual norm on X^*. By hypothesis, f^* is Fréchet differentiable at some point b with gradient $a \in X^{**}$. Since f^* is w^*-lsc, Corollary 5 in **[2]** implies that a actually belongs to X. By Theorem 1 in **[2]** f is norm rotund at a relative to b and therefore E is strongly exposed at a by b. Thus diam $S(E, b, \epsilon) \to 0$ as $\epsilon \to 0$. By the construction of E it follows that diam $S(E, b, \epsilon) \geqq$ diam $S(C, b, \epsilon)$ and hence diam $S(C, b, \epsilon) \to 0$ as $\epsilon \to 0$ also. Since each closed bounded convex set C is dentable in the sense defined by Rieffel **[7]**, X must have the RNP and the theorem follows.

A number of permanence properties for weak*-Asplund spaces may be proved using Theorem 1 in very much the same fashion as the permanence properties for Asplund spaces were proved in **[5]**.

THEOREM 5. *If X^* and Y^* are weak*-Asplund spaces, then $X^* \times Y^*$ is weak*-Asplund.*

Proof. Theorem 1 implies that both X and Y have the RNP and hence $X \times Y$ has the RNP. Therefore $(X \times Y)^*$, which is isomorphic to $X^* \times Y^*$, is weak*-Asplund.

THEOREM 6. *If X^* is a weak*-Asplund space and M is a weak* closed subspace of X^*, then X^*/M is weak*-Asplund.*

Proof. Let $M^{\perp} = \{x \in X: y(x) = 0 \text{ for all } y \in M\}$ be a closed subspace of X. Theorem 1 implies that X has the RNP and hence $M^{\perp}$ has the RNP also. Therefore $(M^{\perp})^*$, which is isomorphic to X^*/M, is weak*-Asplund.

Namioka and Phelps **[5]** raised the question of whether a Banach space X is an Asplund space whenever X^* has the RNP. This may now be restated in the following way: If X^{**} is a weak*-Asplund space, is X an Asplund space? The converse is known to be true. If we consider X to be a (norm) closed subspace of X^{**} by the usual embedding, then each continuous convex function defined on an open convex subset of X is the restriction of a w^*-lsc convex function on X^{**}. We note, however, that since the RNP is not preserved under quotients, we cannot expect even a weak* closed subspace of a weak*-Asplund space to thave good differentiability properties.

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