Pacific Journal of Mathematics

JOINT ESSENTIAL SPECTRA

Artatrana Dash

Vol. 64, No. 1 May 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 64, No 1, 1976

JOINT ESSENTIAL SPECTRA

A. T. DASH

There are many pretty and significant results concerning the essential spectrum of a single operator. The purpose of this note is to introduce the concept of essential spectrum for a sequence of operators and prove analogous results.

1. Introduction. For a nice discussion relating to the essential spectrum of a single operator (bounded linear transformation) A in a complex separable infinite dimensional Hilbert space $\mathcal{H}$, the reader is referred to Fillmore, Stampfli and Williams **[5]**. The purpose of this article is to discuss and prove analogous results concerning the *joint essential spectrum* [Def. 2.3] of n-tuples of operators which were announced in **[4]**. The results of this paper can also be extended to sequences of operators $\{A_n\}$ with very little modifications in the proofs and definitions. However, for brevity we have chosen to discuss them for n-tuples of operators on $\mathcal{H}$.

2. Joint essential spectra. Before we formally define the notion of joint essential spectrum of an n-tuple of operators, we first review the various definitions of joint spectrum existing in the literature.

DEFINITION 2.1. Let $a = (a_1, \cdots, a_n)$ be an n-tuple of commuting elements in a Banach algebra $\mathfrak{A}$ with identity. The *double commutant joint spectrum* $\sigma''(a)$ of a is defined as the set of all $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$ (n-fold Cartesian product of all complex numbers $\mathbf{C}$) such that the closed ideal generated by the set $\{a_j - z_j\}_{1 \leq j \leq n}$ is a proper ideal in a''. Here a'' is the set of all elements of $\mathfrak{A}$ which commute with the elements of $\mathfrak{A}$ that commute with $a_1, a_2, \cdots, a_n$. Consult **[3]**.

Similarly, the *commutant joint spectrum* $\sigma'(a)$ of a is the set of all points $z = (z_1, \cdots, z_n)$ in $\mathbf{C}^n$ such that the set $\{a_j - z_j\}_{1 \leq j \leq n}$ is contained in a proper (two sided) ideal of a', where a' is the set of all elements of $\mathfrak{A}$ that commute with $a_1, a_2, \cdots, a_n$. See Taylor **[8]**.

There is also another notion of joint spectrum which exists in the literature and is the following:

DEFINITION 2.2. Let $a = (a_1, \cdots, a_n)$ be a commuting n-tuple of elements in $\mathfrak{A}$. Then the *joint spectrum* $\sigma(a)$ is defined as

$$\sigma(a) = \sigma^l(a) \cup \sigma^r(a);$$

where the *left* (*right*) *joint spectrum* $\sigma^l(a)$ ($\sigma^r(a)$) is defined as the set of all points $z = (z_1, \cdots, z_n)$ in $\mathbf{C}^n$ such that $\{a_j - z_j\}_{1 \leqq j \leqq n}$ generates a proper left (right) ideal in the algebra $\mathfrak{A}$. For this notion of joint spectrum, the reader is referred to Bonsall and Duncan **[1]** and Harte **[6]**.

Clearly, we have

$$\sigma(a) \subseteq \sigma'(a) \subseteq \sigma''(a).$$

Note that if a consists of a single element, then all these notions of spectra coincide with its usual spectrum. If $a = (a_1, \cdots, a_n)$ is an n-tuple of commuting elements in $\mathfrak{A}$, then $\sigma'(a)$, $\sigma''(a)$, $\sigma(a)$, $\sigma^l(a)$ and $\sigma^r(a)$ are nonempty compact subsets of $\mathbf{C}^n$, otherwise (if a_j's do not commute) each of these sets may be empty. Here in what follows we shall mainly be concerned with the Definition 2.2 for joint spectrum of an n-tuple of elements. Moreover, Definition 2.2 can very well be considered for the definition of joint spectrum of arbitrary (not necessarily commuting) n-tuple of elements in $\mathfrak{A}$.

REMARK 1. The definition of joint spectrum given above is valid for any complex algebra with identity. Furthermore, in a Banach algebra with identity it is well known that a left (right) ideal is proper if and only if its closure is a proper left (right) ideal. Thus there is no ambiguity if the ideals considered in the above definitions are closed or not.

In the sequel $\mathcal{L}(\mathcal{H})$ denotes the algebra of all operators on $\mathcal{H}$ and $\mathcal{K}$ denotes the ideal of compact operators on $\mathcal{H}$. Let ν be the canonical homomorphism from $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K} = \mathcal{C}$. If $A = (A_1, \cdots, A_n)$ is an n-tuple of operators on $\mathcal{H}$, then we write $\nu(A_j) = a_j$, the coset containing A_j for each j, $1 \leqq j \leqq n$. Now we are ready to introduce the following

DEFINITION 2.3. The *joint essential spectrum* of an n-tuple of operators $A = (A_1, \cdots, A_n)$ on $\mathcal{H}$ denoted by $\sigma_e(A)$ is defined as the joint spectrum $\sigma(a)$ of $a = (a_1, \cdots, a_n)$.

Thus $z = (z_1, \cdots, z_n)$ belongs to $\sigma(a)$ if and only if $\{a_j - z_j\}_{1 \leqq j \leqq n}$ is contained in a proper left or right ideal of the Calkin algebra $\mathcal{C}$. We call the set $\sigma^l(a)$ ($\sigma^r(a)$) as the *left* (*right*) *joint essential spectrum* and denote it by $\sigma_e^l(A)$ ($\sigma_e^r(A)$). Clearly, $\sigma_e^l(A) \subseteq \sigma^l(A)$, $\sigma_e^r(A) \subseteq \sigma^r(A)$; and hence $\sigma_e(A) = \sigma_e^l(A) \cup \sigma_e^r(A) \subseteq \sigma(A)$. Moreover, if $A^* = (A_1^*, \cdots, A_n^*)$ and $z^* = (z_1^*, \cdots, z_n^*)$ (here, z_j^* is the complex conjugate of z_j and A_j^* is the adjoint of the operator A_j), then one can easily show that $\sigma_e(A^*) = \sigma_e(A)^*$ and $\sigma_e(A_1 + K_1, \cdots, A_n + K_n) = \sigma_e(A)$ for all $K_1, K_2, \cdots, K_n$ in $\mathcal{K}$.

If we denote by F, the set of all Fredholm operators (an operator T is called *Fredholm* if and only if $\nu(T)$ is invertible in the Calkin algebra $\mathcal{C}$) on $\mathcal{H}$, then the joint essential spectrum $\sigma_e(A)$ of $A = (A_1, \cdots, A_n)$ is equivalently defined as

$$\sigma_e(A) = \sigma_e^l(A) \cup \sigma_e^r(A),$$

where

$$\sigma_e^l(A) = \left\{ z = (z_1, \cdots, z_n) \colon \sum_{j=1}^{n} B_j(A_j - z_j) \notin F \text{ for all } B_j \in \mathcal{L}(\mathcal{H}) \right\}$$

and

$$\sigma_e^r(A) = \left\{ z = (z_1, \cdots, z_n) \colon \sum_{j=1}^{n} (A_j - z_j)B_j \notin F \text{ for all } B_j \in \mathcal{L}(\mathcal{H}) \right\}.$$

If there is no ambiguity, we shall write Σ for $\Sigma_{j=1}^n$. Furthermore, if $A = (A_1, \cdots, A_n)$ is an n-tuple of *essentially commuting* (pairwise commuting modulo the compacts) operators, then $\sigma_e^l(A)$, $\sigma_e^r(A)$ and $\sigma_e(A)$ are nonempty compact subsets of $\mathbf{C}^n$.

LEMMA 2.4. *Let $b = (b_1, \cdots, b_n)$ be an n-tuple of elements in a unital C^*-algebra $\mathfrak{A}$. Then*:

(a) $(z_1, \cdots, z_n) \in \sigma^l(b_1, \cdots, b_n)$ *if and only if*

$$0 \in \sigma(\Sigma(b_j - z_j)^*(b_j - z_j))$$

(b) $(z_1, \cdots, z_n) \in \sigma^r(b_1, \cdots, b_n)$ *if and only if*

$$0 \in \sigma(\Sigma(b_j - z_j)(b_j - z_j)^*).$$

Proof of (a). If $(z_1, \cdots, z_n) \in \sigma^l(b_1, \cdots, b_n)$, then it follows easily from the definition of joint spectrum that

$$0 \in \sigma^l(\Sigma(b_j - z_j)^*(b_j - z_j)) = \sigma(\Sigma(b_j - z_j)^*(b_j - z_j)).$$

Conversely, suppose that $0 \in \sigma(\Sigma(b_j - z_j)^*(b_j - z_j))$. Then there exists a sequence $\{u_k\}$ in $\mathfrak{A}$ with $\|u_k\| = 1$ such that $\|\Sigma(b_j - z_j)^*(b_j - z_j)u_k\| \to 0$ as $k \to \infty$. Therefore we have

$$\begin{aligned}\|(b_j - z_j)u_k\|^2 &= \|u_k^*(b_j - z_j)^*(b_j - z_j)u_k\| \\ &\leqq \|\Sigma u_k^*(b_j - z_j)^*(b_j - z_j)u_k\| \to 0 \quad \text{as} \quad k \to \infty,\end{aligned}$$

for each j, $1 \leqq j \leqq n$. Thus it follows that $(z_1, \cdots, z_n) \in \sigma^l(b_1, \cdots, b_n)$. This proves Lemma 2.4(a) and part (b) follows by taking adjoints.

COROLLARY 2.5. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of operators on $\mathcal{H}$. Then*:

(a) $(z_1, \cdots, z_n) \in \sigma_e^l(A)$ *if and only if* $0 \in \sigma_e(\Sigma(A_j - z_j)^*(A_j - z_j))$
(b) $((z_1, \cdots, z_n) \in \sigma_e^r(A)$ *if and only if* $0 \in \sigma_e(\Sigma(A_j - z_j)(A_j - z_j)^*)$.

The following theorem characterizes the left joint essential spectrum and right joint essential spectrum of an n-tuple of operators on $\mathscr{H}$.

THEOREM 2.6. *Let* $A = (A_1, \cdots, A_n)$ *be an* n*-tuple of operators on* $\mathscr{H}$. *Then*:

(a) $z = (z_1, \cdots, z_n) \in \sigma_e^l(A)$ *if and only if there exists a sequence* $\{x_k\}$ *of unit vectors in* $\mathscr{H}$ *with* $x_k \to 0$ *weakly such that*

$$\|(A_j - z_j)x_k\| \to 0 \quad as \quad k \to \infty,$$

for each j, $1 \leqq j \leqq n$.

(b) $z \in \sigma_e^r(A)$ *if and only if there exists a sequence* $\{x_k\}$ *of unit vectors in* $\mathscr{H}$ *with* $x_k \to 0$ *weakly such that*

$$\|(A_j^* - z_j^*)x_k\| \to 0 \quad as \quad k \to \infty,$$

for each j, $1 \leqq j \leqq n$.
Moreover, the sequence $\{x_k\}$ *can be chosen orthonormal.*

Proof of (a). Let $z \in \sigma_e^l(A) = \sigma^l(a)$. Then $\Sigma\, b_j(a_j - z_j) \neq 1$ for all b_j in $\mathscr{C}$. In particular, this implies that $\Sigma(a_j - z_j)^*(a_j - z_j) = \nu(\Sigma(A_j - z_j)^*(A_j - z_j))$ does not have a left inverse in $\mathscr{C}$. Hence there exists a sequence $\{x_k\}$ of unit vectors with $x_k \to 0$ weakly such that

$$\|\Sigma(A_j - z_j)^*(A_j - z_j)x_k\| \to 0 \quad \text{as} \quad k \to \infty \quad [\mathbf{5}, \text{Thm. } 1.1].$$

Thus,

$$\begin{aligned}\Sigma\|(A_j - z_j)x_k\|^2 &= \langle \Sigma(A_j - z_j)^*(A_j - z_j)x_k, x_k\rangle \\ &\leqq \|\Sigma(A_j - z_j)^*(A_j - z_j)x_k\| \to 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

This implies that $\|(A_j - z_j)x_k\| \to 0$ as $k \to \infty$, for each j, $1 \leqq j \leqq n$.

Conversely, suppose $\{x_k\}$ is a sequence of unit vectors with $x_k \to 0$ weakly such that $\|(A_j - z_j)x_k\| \to 0$ as $k \to \infty$ for each j, $1 \leqq j \leqq n$. This implies that

$$\|\Sigma(A_j - z_j)^*(A_j - z_j)x_k\| \leqq \Sigma\|(A_j - z_j)^*(A_j - z_j)x_k\| \to 0.$$

Therefore we have

$$0 \in \sigma_e^l(\Sigma(A_j - z_j)^*(A_j - z_j)) = \sigma_e(\Sigma(A_j - z_j)^*(A_j - z_j)).$$

Thus it follows from Corollary 2.5 that $z \in \sigma_e^l(A)$. Furthermore, it is not hard to see that $z \in \sigma_e^r(A)$ if and only if $z^* \in \sigma_e^l(A^*)$. Thus the proof of (b) reduces to that of (a) just discussed above. Therefore the proof of the theorem is complete.

The above theorem may be restated as follows: $z \in \sigma_e(A)$ if and only if there exists a sequence $\{x_k\}$ of unit vectors with $x_k \to 0$ weakly such that either

$$\|(A_j - z_j)x_k\| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for each } j, \quad 1 \leqq j \leqq n,$$

or

$$\|(A_j - z_j)^* x_k\| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for each } j, \quad 1 \leqq j \leqq n.$$

REMARK 2. As indicated in the introduction all the results of this paper can be extended to an infinite sequence of operators with very little effort. For example, as suggested by the referee the proof of Theorem 2.6(a) for an infinite sequence of operators $A = \{A_j\}$ may be briefly outlined as follows:

It can be easily shown that $z = (z_1, \cdots, z_n, \cdots)$ belongs to $\sigma_e^l(A)$ if and only if $0 \in \sigma_e(\Sigma \gamma_j (A_j - z_j)^*(A_j - z_j))$, where $\gamma_j = (2^j \|A_j - z_j\|^2)^{-1}$. Now it is not hard to see that both conditions are equivalent to the condition that there exist unit vectors x_k with $x_k \to 0$ weakly such that $\|(A_j - z_j)x_k\| \to 0$ for each j.

By now it is clear that in order to prove the rest of the results of this paper for infinite sequences of operators (or elements in the Calkin algebra) one may use the same weight factors γ_j while all other arguments remain the same. Therefore there is no loss of generality in describing the rest of the results for n-tuples.

To the best of the author's knowledge the use of the weight factor γ_j is the idea of Professor J. P. Williams which the author learned from him during the A.M.S. meeting held at Washington, D.C., 1975, although the same is being suggested by the referee.

DEFINITION 2.7. Let $A = (A_1, \cdots, A_n)$ be an n-tuple of operators on $\mathcal{H}$. A point $z = (z_1, \cdots, z_n)$ of $\mathbf{C}^n$ is called a *joint eigenvalue* of A if and only if there exists a nonzero vector f in $\mathcal{H}$ such that

$$(A_j - z_j)f = 0, \quad \text{for each } j, \quad 1 \leqq j \leqq n.$$

The set of all joint eigenvalues of A is denoted by $\sigma_p(A)$ **[3]**. Moreover, it is easy to see that $(z_1, \cdots, z_n) \in \sigma_p(A)$ if and only if $0 \in \sigma_p(\Sigma(A_j - z_j)^*(A_j - z_j))$.

Next we describe the relationship between the joint spectrum and the joint essential spectrum.

THEOREM 2.8. *Let* $A = (A_1, \cdots, A_n)$ *be an n-tuple of operators on* $\mathcal{H}$. *Then*:

(a) $\sigma^l(A) = \sigma_e^l(A) \cup \sigma_p(A)$

(b) $\sigma^r(A) = \sigma_e^r(A) \cup \sigma_p(A^*)^*$,

and hence we have

(c) $\sigma(A) = \sigma_e(A) \cup \sigma_p(A) \cup \sigma_p(A^*)^*$.

Proof. First we shall prove (a). Suppose $z = (z_1, \cdots, z_n) \in \sigma^l(A)$ and $z \notin \sigma_p(A)$. Thus it follows from Lemma 2.4 and the above remarks that $0 \in \sigma(\Sigma(A_j - z_j)^*(A_j - z_j))$ and $0 \notin \sigma_p(\Sigma(A_j - z_j)^*(A_j - z_j))$. Hence the operator $\Sigma(A_j - z_j)^*(A_j - z_j)$ is 1–1 and has dense range. Since $\Sigma(A_j - z_j)^*(A_j - z_j)$ is not invertible, the range of $\Sigma(A_j - z_j)^*(A_j - z_j)$ is not closed. Thus the operator $\Sigma(A_j - z_j)^*(A_j - z_j)$ is not Fredholm; and hence $0 \in \sigma_e(\Sigma(A_j - z_j)^*(A_j - z_j))$. Therefore, it follows from Corollary 2.5 that $z \in \sigma_e^l(A)$. Moreover, from Theorem 2.8(a) and earlier remarks we have

$$\sigma^r(A) = \sigma^l(A^*)^* = (\sigma_e^l(A^*) \cup \sigma_p(A^*))^* = \sigma_e^l(A^*)^* \cup \sigma_p(A^*)^*$$
$$= \sigma_e^r(A) \cup \sigma_p(A^*)^*.$$

This proves Theorem 2.8(b). The proof of (c) follows immediately from (a) and (b). Thus the proof of the theorem is complete.

DEFINITION 2.9. Let $A = (A_1, \cdots, A_n)$ be an n-tuple of operators on $\mathcal{H}$. Then $z = (z_1, \cdots, z_n)$ is a *joint approximate point spectrum* of A or $z \in \sigma_\pi(A)$, if there exists a sequence $\{f_k\}$ of unit vectors such that

$$\|(A_j - z_j)f_k\| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for each } j, \quad 1 \leqq j \leqq n.$$

We say that $z \in \sigma_\delta(A)$, the *joint approximate defect spectrum* of A if and only if $z^* \in \sigma_\pi(A^*)$.

It is well known that $\sigma^l(A) = \sigma_\pi(A)$ and $\sigma^r(A) = \sigma_\delta(A)$. Consult **[3]**. Clearly, $z = (z_1, \cdots, z_n) \in \sigma_\pi(A)$ if and only if $0 \in \sigma_\pi(\Sigma(A_j - z_j)^*(A_j - z_j))$ and $z \in \sigma_\delta(A)$ if and only if $z^* \in \sigma_\pi(A^*)$ if and only if $0 \in \sigma_\pi(\Sigma(A_j - z_j)(A_j - z_j)^*)$.

THEOREM 2.10. *Let* $A = (A_1, \cdots, A_n)$ *be an n-tuple of operators on* $\mathcal{H}$. *Then*:

(a) *The joint approximate point spectrum* $\sigma_\pi(A)$ *of* A *consists of* $\sigma_e^l(A)$ *together with the joint eigenvalues of finite multiplicity.*

(b) *The joint approximate defect spectrum* $\sigma_\delta(A)$ *of* A *consists of* $\sigma_e^r(A)$ *together with the set of all* $z \in \mathbf{C}^n$ *such that* z^* *is a joint eigenvalue of finite multiplicity of* A^*.

Proof. Suppose $z = (z_1, \cdots, z_n) \in \sigma_\pi(A)$ and $z \notin \sigma_e^l(A)$. This means that $0 \in \sigma_\pi(\Sigma(A_j - z_j)^*(A_j - z_j))$ and $0 \notin \sigma_e^l(\Sigma(A_j - z_j)^*(A_j - z_j))$. Therefore, the operator $\Sigma(A_j - z_j)^*(A_j - z_j)$ is not bounded below but has closed range and finite dimensional null space. Hence the null space of $\Sigma(A_j - z_j)^*(A_j - z_j)$ is nontrivial so that $0 \in \sigma_p(\Sigma(A_j - z_j)^*(A_j - z_j))$. Thus it follows that $z \in \sigma_p(A)$ and z is a joint eigenvalue of finite multiplicity. Recall that $z \in \sigma_\delta(A)$ if and only if $z^* \in \sigma_\pi(A^*)$. Thus the proof of (b) reduces to that of (a) by taking adjoints.

3. Hyponormal elements in the Calkin algebra.

We give here a simple proof of the BDF Lemma 2.1 [**2**].

LEMMA 3.1 [**2**, Lemma 2.1]. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of essentially normal operators $(A_jA_j^* - A_j^*A_j \in \mathcal{K}$, for each j, $j = 1, 2, \cdots, n)$. Then*

$$\sigma_e(A) = \sigma_e^l(A) = \sigma_e^r(A).$$

Proof. Suppose $z = (z_1, \cdots, z_n) \in \sigma_e(A)$. This means that either $\Sigma\, b_j(a_j - z_j) \neq 1$ for all $b_j \in \mathscr{C}$ or $\Sigma(a_j - z_j)b_j \neq 1$ for all $b_j \in \mathscr{C}$. In particular, this implies that $0 \in \sigma(\Sigma(a_j - z_j)^*(a_j - z_j))$. Hence $z \in \sigma^l(a_1, \cdots, a_n)$. Therefore $z \in \sigma_e^l(A)$, and hence $\sigma_e(A) = \sigma_e^l(A)$. Moreover, we have $\sigma_e(A) = \sigma_e(A^*)^* = \sigma_e^l(A^*)^* = \sigma_e^r(A)$. This proves the lemma.

Note that the proof of Lemma 3.1 for an infinite sequence of operators follows trivially from Remark 2.

COROLLARY 3.2. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of normal operators. Then $z = (z_1, \cdots, z_n) \in \sigma_e(A)$ if and only if there exists an orthonormal sequence $\{e_k\}$ such that*

$$\|(A_j - z_j)e_k\| \to 0 \quad \textit{and} \quad \|(A_j - z_j)^* e_k\| \to 0 \quad \textit{as} \quad k \to \infty,$$

and for each j, $1 \leqq j \leqq n$.

Such a point in the joint essential spectrum is called a *normal joint essential approximate eigenvalue.* The above corollary states that each point in the joint essential spectrum of an n-tuple of normal operators is a normal joint essential approximate eigenvalue.

THEOREM 3.3. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of essentially hyponormal operators $(a_ja_j^* \leqq a_j^*a_j$ for each j, $1 \leqq j \leqq n)$ on $\mathscr{H}$. Then*

$$\sigma_e(A) = \sigma_e^r(A).$$

Proof. Suppose $z = (z_1, \cdots, z_n) \in \sigma_e^l(A)$. Then there exists a sequence $\{u_k\}$ in $\mathscr{C}$ with $\|u_k\| = 1$ such that $\|(a_j - z_j)u_k\| \to 0$ as $k \to \infty$, for each j, $1 \leqq j \leqq n$. Consult the proof of Lemma 2.4. Since each a_j is hyponormal we have

$$\|(a_j^* - z_j^*)u_k\|^2 \leqq \|(a_j - z_j)u_k\|^2.$$

Thus $z \in \sigma_e^r(A)$. Therefore, $\sigma_e(A) = \sigma_e^l(A) \cup \sigma_e^r(A) = \sigma_e^r(A)$.

The above theorem can be restated as follows: If $A = (A_1, \cdots, A_n)$ is an n-tuple of essentially hyponormal operators on $\mathscr{H}$, then $z \in \sigma_e(A)$ if and only if there exists an orthonormal sequence $\{e_k\}$ such that $\|(A_j - z_j)^* e_k\| \to 0$ as $k \to \infty$, for each j, $1 \leqq j \leqq n$.

4. Joint eigenvalues in the Calkin algebra. It is well known that if $b \in \mathscr{C}$ and $z \in \sigma(b)$, then there is a projection $p \neq 0$ such that $bp = zp$ or $pb = zp$ **[5]**. The following theorem is an extension of this result to n-tuples.

THEOREM 4.1. *Let $a = (a_1, \cdots, a_n)$ be an n-tuple of elements in the Calkin algebra $\mathscr{C}$ and $z = (z_1, \cdots, z_n) \in \sigma(a)$. Then there is a nonzero projection p in $\mathscr{C}$ such that*

$$a_j p = z_j p \quad \text{for all } j, \quad 1 \leqq j \leqq n$$

or

$$p a_j = z_j p \quad \text{for all } j, \quad 1 \leqq j \leqq n.$$

Proof. Suppose $z = (z_1, \cdots, z_n) \in \sigma^l(a)$. This implies that $0 \in \sigma^l(\Sigma(a_j - z_j)^*(a_j - z_j)) = \sigma(\Sigma(a_j - z_j)^*(a_j - z_j))$. Thus it follows from **[5**, Thm. 4.1] that there exists a nonzero projection p in $\mathscr{C}$ such that $\Sigma(a_j - z_j)^*(a_j - z_j)p = 0$. This implies that $\Sigma\, p(a_j - z_j)^*(a_j - z_j)p = 0$, and hence $p(a_j - z_j)^*(a_j - z_j)p = 0$ for each j, $1 \leqq j \leqq n$. But $\|(a_j - z_j)p\|^2 = \|p(a_j - z_j)^*(a_j - z_j)p\|$. Thus it follows that $(a_j - z_j)p = 0$ for each j, $1 \leqq j \leqq n$, so that $a_j p = z_j p$ for each j, $1 \leqq j \leqq n$. In order to complete the proof we must consider the possibility that $z \in \sigma_e^r(A)$. But $\sigma_e^r(A) = \sigma_e^l(A^*)^*$. Thus this case reduces to the one just discussed. Hence the proof of the theorem is complete.

LEMMA 4.2. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of essentially commuting operators on $\mathscr{H}$. Then*

$$\sigma_e^l(A) \cap \sigma_e^r(A) \neq \phi.$$

Proof. It follows immediately from the definition of joint essential spectrum and the proof of Theorem 11.1 **[7]**.

COROLLARY 4.3. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of essentially commuting operators. Then there are orthogonal projections P and Q of infinite rank and nullity and an n-tuple of complex numbers $z = (z_1, \cdots, z_n)$ such that*

$$(A_j - z_j)P \text{ is compact for all } j, \qquad 1 \leqq j \leqq n$$

and

$$Q(A_j - z_j) \text{ is compact for all } j, \qquad 1 \leqq j \leqq n.$$

Proof. This follows immediately from the definition of joint essential spectrum, Lemma 4.2 and Theorem 4.1.

COROLLARY 4.4. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of essentially commuting operators on $\mathcal{H}$. Then:*

(a) *The operators $A_1, \cdots, A_n$ have a common invariant subspace "modulo the compacts".*

(*b*) *The operators $A_1^*, \cdots, A_n^*$ have a common invariant subspace "modulo the compacts".*

Proof. From Corollary 4.3 we have that for each j, $1 \leqq j \leqq n$, $PA_jP - A_jP = z_jP - A_jP = -(A_j - z_j)P$ is compact. Thus it follows that the operators $A_1, \cdots, A_n$ have a common invariant subspace "modulo the compacts". This proves (a); and the proof of (b) is now obvious.

We state the following without proof.

THEOREM 4.5. *Let $a = (a_1, \cdots, a_n)$ be an n-tuple of hyponormal elements in the Calkin algebra $\mathscr{C}$. Then:*

(a) *$z = (z_1, \cdots, z_n) \in \sigma(a)$ if and only if there is a nonzero projection p such that*

$$a_j^* p = z_j^* p, \quad \text{for each } j, \quad 1 \leqq j \leqq n.$$

(b) *If p is a nonzero projection such that*

$$a_j p = z_j p \ (1 \leqq j \leqq n), \text{ then}$$
$$a_j^* p = z_j^* p \quad \text{for each } j, \quad 1 \leqq j \leqq n.$$

COROLLARY 4.6. *If $A = (A_1, \cdots, A_n)$ be an n-tuple of essentially commuting essentially hyponormal operators, then $A_1, \cdots, A_n$ have a common reducing subspace "modulo the compacts".*

Proof. Since $\sigma_e^l(A) \cap \sigma_e^r(A) \neq \phi$ [Lemma 4.2], it follows that there exists $z = (z_1, \cdots, z_n)$ in $\sigma_e^l(A) \cap \sigma_e^r(A)$ and a nonzero projection p in the

Calkin algebra $\mathscr{C}$ such that $a_j p = z_j p$ $(1 \leqq j \leqq n)$. But $a_j - z_j$ is hyponormal for each j. This implies that $a_j^* p = z_j^* p$ for each j, $1 \leqq j \leqq n$. Thus we have $a_j p = z_j p = (z_j^* p)^* = (a_j^* p)^* = p a_j$, $1 \leqq j \leqq n$. Hence $A_j P - P A_j$ is compact for each j, where $\nu(P) = p$. This proves the result.

ACKNOWLEDGEMENT. I am grateful to the referee for helpful suggestions.

REFERENCES

1. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and elements of normed algebras*, Cambridge, 1971.

2. L. G. Brown, R. G. Douglas and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Proceedings of a conference on operator theory, Springer-Verlag **345** (1973), 58–128.

3. A. T. Dash, *Joint spectra*, Studia Math., **45** (1973), 225–237.

4. A. T. Dash, *Joint spectrum in the Calkin algebra*, Bull. Amer. Math. Soc., **81** (1975), 1083–1085.

5. P. A. Fillmore, J. G. Stampfli and J. P. Williams, *On the essential numerical range, the essential spectrum and a problem of Halmos*, Acta Sci. Math. Szeged, **33** (1972), 179–192.

6. R. Harte, *The spectral mapping theorem in several variables*, Bull. Amer. Math. Soc., **78** (1972), 871–875.

7. Z. Slodkowski and W. Zelazko, *On joint spectra of commuting families of operators*, Studia Math., **50** (1974), 127–148.

8. J. L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Analysis, **6** (1970), 172–191.

Received September 8, 1975 and in revised form February 24, 1976. This is supported by an NRC Grant A7545.

UNIVERSITY OF GUELPH

Pacific Journal of Mathematics

Vol. 64, No. 1 May, 1976