Pacific Journal of Mathematics

SUBSEQUENCES AND REARRANGEMENTS OF SEQUENCES IN FK SPACES

ROBERT M. DEVOS

Vol. 64, No. 1 May 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 64, No. 1, 1976

SUBSEQUENCES AND REARRANGEMENTS OF SEQUENCES IN *FK* SPACES

ROBERT DEVOS

The purpose of this paper is to study *FK* spaces which contain all subsequences or all rearrangements of a given sequence. Using a result of Bennett and Kalton we are able to show that if a separable *FK* space contains all subsequences or all rearrangements of a sequence with two or more finite cluster points, then it contains m. We are also able to show that if ℓ^p contains all rearrangements of some sequence not in ℓ^p, then it is a wedge space. This leads to proofs that if X is a solid symmetric *FK* space, $X\setminus\ell^p \neq \phi$, $X \neq s$, then $X \neq \ell^p_A$ for any matrix A and if in addition X is not wedge then X and ℓ^p are not linearly homeomorphic, via a matrix, hence extending a result of Banach.

1. Recently there has been a large number of papers **[8]**, **[9]**, **[11]**, **[13]**, **[14]** and **[15]** considering subsequences and rearrangements of sequences in c_A and ℓ_A. In this paper we consider these operations in an *FK* space setting and are able to generalize many of these results.

The author would like to thank G. Bennett, F. W. Hartmann, A. K. Snyder and A. Wilansky for inspiration and many valuable conversations.

Let s denote the space of all complex-valued sequences. An *FK* space is a vector subspace of s which is also a Fréchet space, (complete linear metric) with continuous coordinates. A *BK* space is a normed *FK* space. Some discussion of *FK* spaces is given in **[19]**. Well-known examples of *BK* spaces are the spaces m, c, c_0 of bounded, convergent, null sequences respectively, all with $\|x\|_\infty = \sup|x_k|$,

$$\ell^p = \left\{x \in s \colon \|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < \infty\right\} \qquad (1 \leqq p < \infty)$$

(and we write $\ell = \ell^1$.)

Let m_0 be the linear span of all sequences of 0's and 1's and E^∞ the set of all finite sequences; that is, sequences all but finitely many of whose terms are zero. We shall assume that all *FK* spaces contain E^∞. Let A be a matrix, E an *FK* space, $E_A = \{x \in s\colon Ax \in E\}$ is well known to be an *FK* space.

Let $e = (1,1,1,\cdots)$, $e^j = (0,\cdots,0,1,0,\cdots)$ (with 1 in rank j). We denote the nth section of an element $x \in E$ by $P_nx = \Sigma_{i=1}^n x_ie^i$ and say that x has AK provided that $P_n x \to x$ in E. The FK space E is called wedge when $e^n \to 0$ in E.

The α and β duals of a subset X of s are defined by

$$X^\alpha = \left\{ y \in s\colon \sum_{j=1}^{\infty} |x_i y_i| < \infty \quad \text{for each} \quad x \in X \right\}$$

$$X^\beta = \left\{ y \in s\colon \sum_{j=1}^{\infty} x_i y_i \quad \text{converges for each} \quad x \in X \right\}.$$

E is solid if $x \in E$ implies $(a_i x_i) \in E$ for each $a \in m$. Let Σ denote all permutations (rearrangements) of the positive integers. E is symmetric if $x \in E$ implies $x_\sigma = (x_{\sigma(i)}) \in E$ for each $\sigma \in \Sigma$.

In **[6]**, R. C. Buck proved the Tauberian theorem that if x is nonconvergent, then no regular summability matrix can sum every subsequence of x. I. J. Maddox in **[15]** improved Buck's theorem by showing that if A sums every subsequence of a divergent real sequence then $c_A \supset m$.

In **[11]**, J. A. Fridy proved a theorem analogous to Buck's, in which subsequence is replaced by rearrangement. T. A. Keagy in **[13]** extends Fridy's theorem as Maddox extended Buck's.

In the following two theorems, we consider subsequences and rearrangements of a sequence in an FK space. Theorem 2, along with the facts

(i) c_A is always separable;

(ii) if $x \notin m$ and every subsequence (rearrangement) of x is in c_A then $\exists\ N$ such that $a_{jn} = 0$ for $n \geqq N$, and this implies that $c_A = s$; gives us their results.

THEOREM 1. *Let E be an FK space $\supseteq E^\infty$. The following are equivalent.*

(a) *There exists an $x \in E$ with the properties:*
  (i) *for some p, q real numbers, $p \neq q$, pe and qe are subsequences of x.*
  (ii) *E contains all subsequences of x.*

(b) $E \supseteq m$

(c) $E \supseteq m_0$

(d) *$e \in E$ and there exists a $y \in E$ with the properties:*
  (i) *for some p, q real numbers, $p \neq q$, pe and qe are subsequences of y.*
  (ii) *E contains all rearrangements of y.*

Proof. Clearly (b) $\Rightarrow$ (a), (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d).

(c) $\Rightarrow$ (b) Bennett and Kalton's extension of Seevers results Theorem 1, p. 513 of **[5]**.

(a) $\Rightarrow$ (c) E contains all sequences of p's and q's hence E contains all sequences of 0's and 1's.

(d) $\Rightarrow$ (c) Let z be a sequence of 0's and 1's such that only finitely many $z_i = 1$ or $= 0$. Since $e \in E$ and $E^\infty \subseteq E$ then $z \in E$. Let z be a sequence of 0's and 1's with an infinite number of $z_i = 0$ and an infinite number of $z_i = 1$.

Let $r(k)$ and $s(k)$ be such that $z_{r(k)} = 1$, $z_{s(k)} = 0$ for all k and $\{r(k)\} \cup \{s(k)\} = \mathbf{Z}^+$.

Let y^1, y^2, y^3, y^4 be rearrangements of y such that

$$\begin{array}{lll} y^1_{r(2k)} = p, & y^1_{s(k)} = q & \\ y^2_{r(2k)} = q, & y^2_{s(k)} = p, & y^2_{r(2k-1)} = y^1_{r(2k-1)} \\ y^3_{r(2k-1)} = p, & y^3_{s(k)} = q & \\ y^4_{r(2k-1)} = q, & y^4_{s(k)} = p, & y^4_{r(2k)} = y^3_{r(2k)}. \end{array}$$

Hence

$$\frac{1}{3(p-q)}\left[(y^1 - y^2) + (p-q)e + (y^3 - y^4) + (p-q)e\right] = z$$

and so $z \in E$. Since z was arbitrary it follows that $E \supseteq m_0$.

Using a form of the closed graph theorem due to Kalton, Bennett and Kalton as Theorem 25 p. 577 of **[4]** prove

THEOREM (BENNETT-KALTON). *If E is a separable FK space $\supseteq E^\infty$ and $E + c_0 \supseteq m_0$ then $E \supseteq m$.*

Using this theorem and arguments similar to those of Theorem 1, we have

THEOREM 2. *Let E be a separable FK space $\supseteq E^\infty$. The following are equivalent.*

(a) *$\exists\ x \in E$ with at least two distinct finite cluster points and E contains all subsequences of x.*

(b) *$E \supseteq m$.*

(c) *$E \supseteq m_0$.*

(d) *$\exists\ y \in E$ with at least two distinct finite cluster points, E contains all rearrangements of y and $e \in E$.*

LEMMA 1. *Let Y be a linear sequence space, $x \in Y \backslash \ell^p$ such that every rearrangement of x belongs to Y. Then there exists a $z \in Y \backslash \ell^p$ such* *that every rearrangement of z belongs to Y and $|z_i| = 0$ for an infinite number of subscripts.*

Proof. Let y be a rearrangement of x such that the even coordinates form a sequence which is not in ℓ^p and the sequence $(y_{4n} - y_{4n-2}) \notin \ell^p$. Let y' be the rearrangement of x which permutes the $4n$th and the $4n-2$nd slots of y. Let $z = y - y'$. The odd coordinates of z are 0 and $z \in Y \backslash \ell^p$. Clearly any rearrangement of z belongs to Y.

THEOREM 3. *Let $A = (a_{ij})$ be a matrix, α^n the nth column of A and $1 \leqq p < \infty$. If there exists an $x \in \ell^p_A \backslash \ell^p$ such that every rearrangement of x belongs to ℓ^p_A then $\|\alpha^n\|_p \to 0$.*

Proof. By a Lemma in **[11]**, each row of A is in c_0. If $x \notin m$ then the rows of A are in E^∞, for if $\exists$ p such that $(a_{pn})_{n=1}^\infty \notin E^\infty$ then $\exists$ a rearrangement of x such that $\Sigma\, a_{p,k} x_{\sigma(i)}$ is not convergent. Let β^n be the nth row. If $\exists$ N such that $P_N\beta^n - \beta^n = 0$ for all n then $\ell^p_A = s$ and $\|\alpha^n\|_p = 0$ for $n \geqq N$. If N does not exist then $\exists$ a monotonic increasing sequence of positive integers $(p(k))$ and a rearrangement x_σ of x such that

$$\left| \sum_i a_{p(k),\, i} x_{\sigma(i)} \right| \geqq 1,$$

which implies $x_\sigma \notin \ell^p_A$, a contradiction; so N exists. If $x \in m$, we may assume $\|x\|_\infty \leqq \frac{1}{2}$. Suppose $\|\alpha^n\|_p \not\to 0$, then there exists $\epsilon > 0$ and an increasing sequence of integers r such that $\|\alpha^{r_i}\|_p \geqq \epsilon$, for all i. We now define a subsequence $(\ell(k))$ of r and $(m(k))$ of positive integers. Let $\ell(1) = r_1$, $m(0) = 0$ and $m(1)$ be such that $\|\alpha^{\ell(1)} - P_{m(1)}\alpha^{\ell(1)}\|_p < \frac{1}{2}\epsilon$. Since the rows are in c_0, pick $\ell(2) > \ell(1)$ such that $\|P_{m(1)}\alpha^{\ell(2)}\|_p < \frac{1}{4}\epsilon$. Pick $m(2) > m(1)$ such that $\|\alpha^{\ell(2)} - P_{m(2)}\alpha^{\ell(2)}\|_p < \frac{1}{4}\epsilon$.

Proceeding in this manner we inductively define increasing sequences $(\ell(k))$ (a subsequence of r) and $(m(k))$ such that

$$\|\alpha^{\ell(k)}\|_p \geqq \epsilon$$

$$| P_{m(k)}\alpha^{\ell(k+1)}\|_p < \frac{1}{2^{k+1}}\,\epsilon$$

$$| P_{m(k)}\alpha^{\ell(k)} - \alpha^{\ell(k)}\|_p < \frac{1}{2^k}\,\epsilon.$$

Hence

$$| (P_{m(k)} - P_{m(k-1)})\alpha^{\ell(k)}\|_p \geqq \tfrac{1}{2}\epsilon. \qquad (k \geqq 2)$$

By Lemma 1, $\exists\ z \in \ell^p_A \backslash \ell^p$ such that $|z_i| = 0$ for $i \neq \ell(k)$ for some k and $\|z\|_\infty \leqq 1$ since $\|x\|_\infty \leqq \frac{1}{2}$. Hence

$$\left(\left| \sum_{k=1}^{\infty} a_{n,\ell(k)} z_{\ell(k)} \right| \right) \in \ell^p$$

call it γ^0. Let

$$\gamma^1 = |\alpha^{\ell(1)} - P_{m(1)} \alpha^{\ell(1)}|$$

(i.e. the absolute value of each term)

$$\gamma^n = |\alpha^{\ell(n)} - (P_{m(n)} - P_{m(n-1)}) \alpha^{\ell(n)}| \quad \text{for} \quad n \geqq 1$$

$$\|\gamma^n\|_p \leqq \frac{1}{2^n}\epsilon + \frac{1}{2^n}\epsilon = \frac{1}{2^{n-1}}\epsilon.$$

Let $\delta = \Sigma_{i=0}^{\infty} \gamma^i$. Since $\Sigma_{i=0}^{\infty} \|\gamma^i\|_p < \infty$, it follows that $\delta \in \ell^p$. Let $m(s-1) < q \leqq m(s)$

$$\begin{aligned} |a_{q,\ell(s)} z_{\ell(s)}| &\leqq \left| \sum_{k=1}^{\infty} a_{q,\ell(k)} z_{\ell(k)} \right| + \sum_{\substack{k=1 \\ k \neq s}}^{\infty} |a_{q,\ell(k)} z_{\ell(k)}| \\ &\leqq \left| \sum_{k=1}^{\infty} a_{q,\ell(k)} z_{\ell(k)} \right| + \sum_{\substack{k=1 \\ k \neq s}}^{\infty} |a_{q,\ell(k)}| \\ &\leqq \delta_q. \end{aligned}$$

Hence the sequence

$$\delta' = z_{\ell(1)} P_{m(1)} \alpha^{\ell(1)} + \sum_{k=2}^{\infty} z_{\ell(k)} (P_{m(k)} - P_{m(k-1)}) \alpha^{\ell(k)} \in \ell^p.$$

But

$$\begin{aligned} \|\delta'\|_p^p &= \|z_{\ell(1)} P_{m(1)} \alpha^{\ell(1)}\|_p^p + \sum_{k=2}^{\infty} |z_{\ell(k)}|^p \, \|(P_{m(k)} - P_{m(k-1)}) \alpha^{\ell(k)}\|_p^p \\ &\geqq |z_{\ell(1)}|^p \left(\frac{\epsilon}{2}\right)^p + \sum_{k=2}^{\infty} |z_{\ell(k)}|^p \left(\frac{\epsilon}{2}\right)^p \end{aligned}$$

which implies $z \in \ell^p$, a contradiction. Hence $\|\alpha^n\|_p \to 0$.

This theorem was stated for $p = 1$ in the Notices by Keagy **[14]**. In **[2]** Bennett defined the concept of a wedge space. He then proves several equivalent conditions one of them being $E \supset z^\alpha$ for some $z \in c_0$. As Theorems 36 and 41, he shows ℓ^p_A is wedge iff $\|\alpha^n\|_p \to 0$ where α^n is the nth column of A.

COROLLARY 1. *Let X be a non-wedge FK space, $y \in X \backslash \ell^p$ such that $y_\sigma \in X$ for all $\sigma \in \Sigma$. Then $X \neq \ell^p_A$ for any matrix A.*

COROLLARY 2. *Let $X \neq s$ be a solid symmetric FK space $X \backslash \ell^p \neq \phi$. Then $X \neq \ell^p_A$ for any matrix A.*

Proof. In **[12]** Garling proves that $X \subseteq m$; but all wedge spaces contain unbounded sequences hence X is nonwedge.

Since ℓ^q is always solid symmetric we have

COROLLARY 3. *If $q > p$ then $\ell^q \neq \ell^p_A$ for any matrix A.*

This was proved using wedge spaces by Bennett in **[2]** and other techniques by DeVos in **[10]**.

THEOREM 4. *Let X be a non-wedge FK space with AK, $y \in X \backslash \ell^p$ such that $y_\sigma \in X$ for all $\sigma \in \Sigma$. Then X cannot equal ℓ^p_A nor can it be a closed subspace of ℓ^p_A for any matrix A.*

Proof. Let $z \in m_0$ be chosen such that $z_{n(k)} = 1$ and $z_i = 0$ for $i \neq n(k)$ where $(n(k))$ is an increasing sequence of positive integers such that $!e^{n(k)}! \geqq c > 0$ where !! is the paranorm of X and $\|\alpha^{n(k)}\|_p < 1/2^k$ where $\alpha^{n(k)}$ is the $n(k)$ column of the matrix A. $z \notin X$ and $z \in \ell^p_A$ with AK hence z is the closure of X in ℓ^p_A. Hence X is not closed in ℓ^p_A.

Garling in **[11]** defines the spaces

$$\mu_z = \left\{ x \in s : \sup_{\sigma \in \Sigma} \sum_{i=1}^{\infty} |x_{\sigma(i)} z_i| < \infty \right\}$$

and shows that μ_z is a symmetric solid BK space. As Proposition 11 he shows for $z \in c_0$, $\mu_z \supsetneqq \ell'$. Combining these results we add another condition to Bennett's Theorem 36.

THEOREM 5. *The following conditions are equivalent for any matrix A.*

(i) *ℓ_A is a* (weak) *wedge space*

(ii) $\|\alpha^n\|_1 \to 0$

(iii) *$\exists\ x \in \ell_A \backslash \ell$ such that $x_\sigma \in \ell_A$ for all $\sigma \in \Sigma$.*

For $p > 1$, the converse of Theorem 3 is false. For the following example let all sequences be real. In **[16]** Ruckle defines the sequence h such that $h_n = n^{1/p} - (n-1)^{1/p}$ and shows that $\mu_h \subsetneqq \ell^p$. Let A be the matrix such that

$$a_{1n} = h_n \quad \text{and} \quad a_{pn} = 0 \quad \text{for} \quad p > 1;$$

Thus, $\ell^p_A = s_A = h^\beta \supset \mu_h$. Let $x \in h^\beta$ such that $x_\sigma \in h^\beta$ for all permutations σ. Then $x_\sigma \in h^\alpha$ for all permutations σ. Hence $x \in \mu_h$ which implies $x \in \ell^p$.

Banach in [1] shows that if $p \neq q$, $q \geqq 1$ then ℓ^p and ℓ^q are not linearly homeomorphic. He does this by showing that their linear dimensions are incomparable. If X and Y are linear topological spaces then $\dim_\ell X \leqq \dim_\ell Y$ iff X is isomorphic to a closed subspace of Y. The following theorems which follow easily from Theorem 3 are extensions of these results.

THEOREM 6. *Let X be a nonwedge FK space such that $\exists\ x \in X \backslash \ell^p$ with $x_\sigma \in X$ for all $\sigma \in \Sigma$. Then X and ℓ^p are not linearly homeomorphic via a matrix.*

THEOREM 7. *Let X be a nonwedge FK space with AK such that $\exists$ $x \in X \backslash \ell^p$ with $x_\sigma \in X$ for all $\sigma \in \Sigma$. Then* $\dim_\ell X \not\leqq \dim_\ell \ell^p$.

REFERENCES

1. S. Banach, *Theorie des opérations linéaires*, Warszawa, 1932.
2. G. Bennett, *A new class of sequence spaces with applications in Summability Theory*, J. Reine Angew. Math., **266** (1974), 49–75.
3. ———, *Some inclusion theorems for sequence spaces*, Pacific J. Math., **46** (1973), 17–30.
4. G. Bennett and N. J. Kalton, *FK-spaces containing c_0*, Duke Math. J., **39** (1972), 561–582.
5. ———, *Inclusion theorems for K-spaces*, Canad. J. Math., **25** (1973), 511–524.
6. R. C. Buck, *A note on subsequences*, Bull. Amer. Math. Soc., **49** (1943), 898–899.
7. ———, *An addendum to "A note on subsequences"*, Proc. Amer. Math. Soc., **7** (1956), 1074–1075.
8. D. F. Dawson, *Summability of subsequences and other regular transformations of a sequence*, Boll. Unione Mat. Ital. (4), **8** (1973), 449–455.
9. ———, *Summability of subsequences and stretching of sequences*, Pacific J Math., **44** (1973), 455–460.
10. R. DeVos, *θ maps between FK spaces and summability*, Math. Z., **129** (1972), 287–298.
11. J. A. Fridy, *Summability of rearrangements of sequences*, to appear.
12. D. J. H. Garling, *On Symmetric Sequence Spaces*, Proc. London Math. Soc., **16** (1966), 85–106.
13. T. A. Keagy, *A Tauberian theorem for rearrangements*, Notices Amer. Math. Soc., **22** (1975), A384.
14. ———, *Matrix transformations and absolute summability*, Notices Amer. Math. Soc., **22** (1975), A462.
15. I. J. Maddox, *A Tauberian theorem for subsequences*, Bull. London Math. Soc., **2** (1970), 63–65.
16. W. Ruckle, *On perfect symmetric BK spaces*, Math. Ann., **175** (1968), 121–126.
17. G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc., **133** (1968), 267–280.
18. A. K. Snyder and A. Wilansky, *Inclusion theorems and semiconservative FK spaces*, Rocky Mountain J. Math., **2** (1972), 595–603.
19. A Wilansky, *Functional Analysis*, Blaisdell, 1964.

Received September 8, 1975 and in revised form March 30, 1976.

VILLANOVA UNIVERSITY

Pacific Journal of Mathematics
Vol. 64, No. 1 May, 1976