SUBSEQUENCES AND REARRANGEMENTS OF SEQUENCES IN $FK$ SPACES

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The purpose of this paper is to study $FK$ spaces which contain all subsequences or all rearrangements of a given sequence. Using a result of Bennett and Kalton we are able to show that if a separable $FK$ space contains all subsequences or all rearrangements of a sequence with two or more finite cluster points, then it contains $m$. We are also able to show that if $\ell^p$ contains all rearrangements of some sequence not in $\ell^p$, then it is a wedge space. This leads to proofs that if $X$ is a solid symmetric $FK$ space, $X \setminus \ell^p \neq \phi$, $X \neq s$, then $X \neq \ell^p_A$ for any matrix $A$ and if in addition $X$ is not wedge then $X$ and $\ell^p$ are not linearly homeomorphic, via a matrix, hence extending a result of Banach.

1. Recently there has been a large number of papers [8], [9], [11], [13], [14] and [15] considering subsequences and rearrangements of sequences in $c_A$ and $\ell_A$. In this paper we consider these operations in an $FK$ space setting and are able to generalize many of these results.

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Let $s$ denote the space of all complex-valued sequences. An $FK$ space is a vector subspace of $s$ which is also a Fréchet space, (complete linear metric) with continuous coordinates. A $BK$ space is a normed $FK$ space. Some discussion of $FK$ spaces is given in [19]. Well-known examples of $BK$ spaces are the spaces $m$, $c$, $c_0$ of bounded, convergent, null sequences respectively, all with $\|x\|_\infty = \sup |x_k|$, $E\times$ the set of all finite sequences; that is, sequences all but finitely many of whose terms are zero. We shall assume that all $FK$ spaces contain $E\times$. Let $m_0$ be the linear span of all sequences of 0’s and 1’s and $E\times$ the set of all finite sequences; that is, sequences all but finitely many of whose terms are zero. We shall assume that all $FK$ spaces contain $E\times$. Let $A$ be a matrix, $E$ an $FK$ space, $E_A = \{x \in s: Ax \in E\}$ is well known to be an $FK$ space.

Let $e = (1, 1, 1, \cdots)$, $e^i = (0, \cdots, 0, 1, 0, \cdots)$ (with 1 in rank $j$). We denote the $n$th section of an element $x \in E$ by $P_n x = \sum_{i=1}^n x_i e^i$ and say
that $x$ has AK provided that $P_n x \to x$ in $E$. The FK space $E$ is called wedge when $e^n \to 0$ in $E$.

The $\alpha$ and $\beta$ duals of a subset $X$ of $s$ are defined by

$$X^\alpha = \left\{ y \in s : \sum_{j=1}^{x} |x_j| < \infty \text{ for each } x \in X \right\}$$

$$X^\beta = \left\{ y \in s : \sum_{j=1}^{x} x_j \text{ converges for each } x \in X \right\}.$$  

$E$ is solid if $x \in E$ implies $(a, x) \in E$ for each $a \in m$. Let $\Sigma$ denote all permutations (rearrangements) of the positive integers. $E$ is symmetric if $x \in E$ implies $x_\sigma = (x_{\sigma(i)}) \in E$ for each $\sigma \in \Sigma$.

In [6], R. C. Buck proved the Tauberian theorem that if $x$ is nonconvergent, then no regular summability matrix can sum every subsequence of $x$. I. J. Maddox in [15] improved Buck's theorem by showing that if $A$ sums every subsequence of a divergent real sequence then $c_A \supset m$.

In [11], J. A. Fridy proved a theorem analogous to Buck's, in which subsequence is replaced by rearrangement. T. A. Keagy in [13] extends Fridy's theorem as Maddox extended Buck's.

In the following two theorems, we consider subsequences and rearrangements of a sequence in an FK space. Theorem 2, along with the facts

(i) $c_A$ is always separable;
(ii) if $x \in m$ and every subsequence (rearrangement) of $x$ is in $c_A$ then $\exists N$ such that $a_n = 0$ for $n \geq N$, and this implies that $c_A = s$; gives us their results.

**Theorem 1.** Let $E$ be an FK space $\supset E^\circ$. The following are equivalent.

(a) There exists an $x \in E$ with the properties:

(i) for some $p, q$ real numbers, $p \neq q$, $pe$ and $qe$ are subsequences of $x$, 
(ii) $E$ contains all subsequences of $x$.
(b) $E \supset m$
(c) $E \supset m_0$
(d) $e \in E$ and there exists a $y \in E$ with the properties:
(i) for some $p, q$ real numbers, $p \neq q$, $pe$ and $qe$ are subsequences of $y$, 
(ii) $E$ contains all rearrangements of $y$.

**Proof.** Clearly (b) $\Rightarrow$ (a), (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d).
(c) ⇒ (b) Bennett and Kalton's extension of Seevers results Theorem 1, p. 513 of [5].
(a) ⇒ (c) \( E \) contains all sequences of \( p \)'s and \( q \)'s hence \( E \) contains all sequences of 0's and 1's.
(d) ⇒ (c) Let \( z \) be a sequence of 0's and 1's such that only finitely many \( z_i = 1 \) or \( = 0 \). Since \( e \in E \) and \( E^* \subseteq E \) then \( z \in E \). Let \( z \) be a sequence of 0's and 1's with an infinite number of \( z_i = 0 \) and an infinite number of \( z_i = 1 \).

Let \( r(k) \) and \( s(k) \) be such that \( z_{r(k)} = 1, z_{s(k)} = 0 \) for all \( k \) and \( \{r(k)\} \cup \{s(k)\} = \mathbb{Z}^+ \).

Let \( y^1, y^2, y^3, y^4 \) be rearrangements of \( y \) such that

\[
\begin{align*}
y^1_{r(2k)} &= p, & y^2_{s(2k)} &= q \\
y^2_{r(2k)} &= q, & y^2_{s(2k)} &= p, \\
y^3_{r(2k-1)} &= p, & y^3_{s(k)} &= q \\
y^4_{r(2k-1)} &= q, & y^4_{s(k)} &= p, \\
\end{align*}
\]

Hence

\[
\frac{1}{3(p-q)} [(y^1 - y^2) + (p-q)e + (y^3 - y^4) + (p-q)e] = z
\]

and so \( z \in E \). Since \( z \) was arbitrary it follows that \( E \supseteq m_0 \).

Using a form of the closed graph theorem due to Kalton, Bennett and Kalton as Theorem 25 p. 577 of [4] prove

**Theorem (Bennett-Kalton).** If \( E \) is a separable FK space \( \supseteq E^* \) and \( E + c_0 \supseteq m_0 \) then \( E \supseteq m \).

Using this theorem and arguments similar to those of Theorem 1, we have

**Theorem 2.** Let \( E \) be a separable FK space \( \supseteq E^* \). The following are equivalent.
(a) \( \exists x \in E \) with at least two distinct finite cluster points and \( E \) contains all subsequences of \( x \).
(b) \( E \supseteq m \).
(c) \( E \supseteq m_0 \).
(d) \( \exists y \in E \) with at least two distinct finite cluster points, \( E \) contains all rearrangements of \( y \) and \( e \in E \).

**Lemma 1.** Let \( Y \) be a linear sequence space, \( x \in Y \setminus \ell^p \) such that every rearrangement of \( x \) belongs to \( Y \). Then there exists a \( z \in Y \setminus \ell^p \) such...
that every rearrangement of \( z \) belongs to \( Y \) and \( |z_i| = 0 \) for an infinite number of subscripts.

**Proof.** Let \( y \) be a rearrangement of \( x \) such that the even coordinates form a sequence which is not in \( \ell^p \) and the sequence \( (y_{4n} - y_{4n-2}) \notin \ell^p \). Let \( y' \) be the rearrangement of \( x \) which permutes the \( 4n \)th and the \( 4n-2 \)nd slots of \( y \). Let \( z = y - y' \). The odd coordinates of \( z \) are 0 and \( z \in Y \setminus \ell^p \). Clearly any rearrangement of \( z \) belongs to \( Y \).

**Theorem 3.** Let \( A = (a_{ij}) \) be a matrix, \( a^n \) the \( n \)th column of \( A \) and \( 1 \leq p < \infty \). If there exists an \( x \in \ell^p_A \cap \ell^p \) such that every rearrangement of \( x \) belongs to \( \ell^p_A \) then \( \|a^n\|_p \to 0 \).

**Proof.** By a Lemma in [11], each row of \( A \) is in \( c_0 \). If \( x \notin m \) then the rows of \( A \) are in \( E^\infty \), for if \( \exists p \) such that \( (a_{pn})_{n=1}^\infty \notin E^\infty \) then \( \exists \) a rearrangement of \( x \) such that \( \Sigma a_{p,k}x^\sigma(i) \) is not convergent. Let \( \beta^n \) be the \( n \)th row. If \( \exists N \) such that \( P_N \beta^n - \beta^n = 0 \) for all \( n \) then \( \ell^p_A = s \) and \( \|a^n\|_p = 0 \) for \( s \geq N \). If \( N \) does not exist then \( \exists \) a monotonic increasing sequence of positive integers \( (p(k)) \) and a rearrangement \( x_\sigma \) of \( x \) such that

\[
\left| \sum a_{p(k),i}x_{\sigma(i)} \right| \geq 1,
\]

which implies \( x_\sigma \notin \ell^p_A \), a contradiction; so \( N \) exists. If \( x \in m \), we may assume \( \|x\|_\infty \leq \frac{1}{2} \). Suppose \( \|a^n\|_p \not\to 0 \), then there exists \( \epsilon > 0 \) and an increasing sequence of integers \( r \) such that \( \|a^n\|_p \geq \epsilon \), for all \( i \). We now define a subsequence \( (\ell(k)) \) of \( r \) and \( (m(k)) \) of positive integers. Let \( \ell(1) = r_1 \), \( m(0) = 0 \) and \( m(1) \) be such that \( \|a^{\ell(1)} - P_{m(1)}a^{\ell(1)}\|_p < \frac{1}{2}\epsilon \). Since the rows are in \( c_0 \), pick \( \ell(2) > \ell(1) \) such that \( \|P_{m(1)}a^{\ell(2)}\|_p < \frac{1}{4}\epsilon \). Pick \( m(2) > m(1) \) such that \( \|a^{\ell(2)} - P_{m(2)}a^{\ell(2)}\|_p < \frac{1}{4}\epsilon \).

Proceeding in this manner we inductively define increasing sequences \( (\ell(k)) \) (a subsequence of \( r \)) and \( (m(k)) \) such that

\[
\|a^{\ell(k)}\|_p \geq \epsilon
\]

\[
\|P_{m(k)}a^{\ell(k+1)}\|_p < \frac{1}{2^{k+1}} \epsilon
\]

\[
\|P_{m(k)}a^{\ell(k)} - a^{\ell(k)}\|_p < \frac{1}{2^k} \epsilon.
\]

Hence

\[
\|P_{m(k)} - P_{m(k-1)}a^{\ell(k)}\|_p \geq \frac{1}{2} \epsilon. \quad (k \geq 2)
\]
By Lemma 1, \( \exists \ z \in \ell_p^\lambda \setminus \ell^p \) such that \( |z_i| = 0 \) for \( i \neq \ell(k) \) for some \( k \) and \( \|z\|_\infty \leq 1 \) since \( \|x\|_\infty \leq \frac{1}{2} \). Hence

\[
\left( \left| \sum_{k=1}^\infty a_{n, \ell(k)} z_{\ell(k)} \right| \right) \in \ell^p
\]
call it \( \gamma^0 \). Let

\[
\gamma^1 = |\alpha^{\ell(1)} - P_m(\alpha^{\ell(1)})|
\]
(i.e. the absolute value of each term)

\[
\gamma^n = |\alpha^{\ell(n)} - (P_m(\alpha^{\ell(n)}) - P_m(\alpha^{\ell(n-1)}))\alpha^{\ell(n)}| \quad \text{for} \quad n \geq 1
\]

\[
\|\gamma^n\|_p \leq \frac{1}{2^n} \varepsilon + \frac{1}{2^n} \varepsilon = \frac{1}{2^{n-1}} \varepsilon.
\]

Let \( \delta = \sum_{i=0}^\infty \gamma^i \). Since \( \sum_{i=0}^\infty \|\gamma^i\|_p < \infty \), it follows that \( \delta \in \ell^p \). Let \( m(s-1) < q \leq m(s) \)

\[
|a_{q, \ell(s)} z_{\ell(s)}| \leq \left| \sum_{k=1}^\infty a_{q, \ell(k)} z_{\ell(k)} \right| + \sum_{k=1}^\infty \left| a_{q, \ell(k)} z_{\ell(k)} \right|
\]

\[
\leq \left| \sum_{k=1}^\infty a_{q, \ell(k)} z_{\ell(k)} \right| + \sum_{k=1}^\infty \left| a_{q, \ell(k)} \right|
\]

\[
\leq \delta_q.
\]

Hence the sequence

\[
\delta' = z_{\ell(1)} P_m(\alpha^{\ell(1)}) + \sum_{k=2}^\infty z_{\ell(k)} (P_m(\alpha^{\ell(k)}) - P_m(\alpha^{\ell(k-1)})) \in \ell^p.
\]

But

\[
\|\delta'\|_p = \|z_{\ell(1)} P_m(\alpha^{\ell(1)})\|_p + \sum_{k=2}^\infty \|z_{\ell(k)}\|_p \| (P_m(\alpha^{\ell(k)}) - P_m(\alpha^{\ell(k-1)})) \|_p
\]

\[
\geq |z_{\ell(1)}|^p \left( \frac{\varepsilon}{2} \right)^p + \sum_{k=2}^\infty |z_{\ell(k)}|^p \left( \frac{\varepsilon}{2} \right)^p
\]

which implies \( z \in \ell^p \), a contradiction. Hence \( \|\alpha^n\|_p \to 0 \).

This theorem was stated for \( p = 1 \) in the Notices by Keagy [14]. In [2] Bennett defined the concept of a wedge space. He then proves several equivalent conditions one of them being \( E \supset z^\alpha \) for some \( z \in c_0 \). As Theorems 36 and 41, he shows \( \ell^p_\lambda \) is wedge iff \( \|\alpha^n\|_p \to 0 \) where \( \alpha^n \) is the \( n \)th column of \( A \).
**Corollary 1.** Let $X$ be a non-wedge FK space, $y \in X \setminus \ell^p$ such that $y_\sigma \in X$ for all $\sigma \in \Sigma$. Then $X \neq \ell^p_\Lambda$ for any matrix $A$.

**Corollary 2.** Let $X \neq s$ be a solid symmetric FK space $X \setminus \ell^p \neq \phi$. Then $X \neq \ell^p_\Lambda$ for any matrix $A$.

**Proof.** In [12] Garling proves that $X \subseteq m$; but all wedge spaces contain unbounded sequences hence $X$ is nonwedge. Since $\ell^q$ is always solid symmetric we have

**Corollary 3.** If $q > p$ then $\ell^q \neq \ell^p_\Lambda$ for any matrix $A$.

This was proved using wedge spaces by Bennett in [2] and other techniques by DeVos in [10].

**Theorem 4.** Let $X$ be a non-wedge FK space with $AK$, $y \in X \setminus \ell^p$ such that $y_\sigma \in X$ for all $\sigma \in \Sigma$. Then $X$ cannot equal $\ell^p_\Lambda$ nor can it be a closed subspace of $\ell^p_\Lambda$ for any matrix $A$.

**Proof.** Let $z \in m_0$ be chosen such that $z_{n(k)} = 1$ and $z_i = 0$ for $i \neq n(k)$ where $(n(k))$ is an increasing sequence of positive integers such that !$e^{n(k)!} \geq c > 0$ where ! is the paranorm of $X$ and $\|a^{n(k)}\|_p < 1/2^k$ where $a^{n(k)}$ is the $n(k)$ column of the matrix $A$. $z \notin X$ and $z \in \ell^p_\Lambda$ with $AK$ hence $z$ is the closure of $X$ in $\ell^p_\Lambda$. Hence $X$ is not closed in $\ell^p_\Lambda$.

Garling in [11] defines the spaces

$$\mu_z = \left\{ x \in s: \sup_{\sigma \in \Sigma} \sum_{i=1}^{\infty} |x_{\sigma(i)}z_i| < \infty \right\}$$

and shows that $\mu_z$ is a symmetric solid BK space. As Proposition 11 he shows for $z \in c_0$, $\mu_z \nsubseteq \ell'$. Combining these results we add another condition to Bennett's Theorem 36.

**Theorem 5.** The following conditions are equivalent for any matrix $A$.

(i) $\ell_A$ is a (weak) wedge space

(ii) $\|a^n\|_1 \to 0$

(iii) $\exists x \in \ell_A \setminus \ell$ such that $x_\sigma \in \ell_A$ for all $\sigma \in \Sigma$.

For $p > 1$, the converse of Theorem 3 is false. For the following example let all sequences be real. In [16] Ruckle defines the sequence $h$ such that $h_n = n^{1/p} - (n - 1)^{1/p}$ and shows that $\mu_h \nsubseteq \ell^p$. Let $A$ be the matrix such that

$$a_{1n} = h_n \quad \text{and} \quad a_{pn} = 0 \quad \text{for} \quad p > 1;$$
Thus, $\ell^p_A = s_A = h^b \supset \mu_h$. Let $x \in h^b$ such that $x_\sigma \in h^b$ for all permutations $\sigma$. Then $x_\sigma \in h^a$ for all permutations $\sigma$. Hence $x \in \mu_h$ which implies $x \in \ell^p$.

Banach in [1] shows that if $p \neq q, q \geq 1$ then $\ell^p$ and $\ell^q$ are not linearly homeomorphic. He does this by showing that their linear dimensions are incomparable. If $X$ and $Y$ are linear topological spaces then $\dim X \leq \dim Y$ iff $X$ is isomorphic to a closed subspace of $Y$. The following theorems which follow easily from Theorem 3 are extensions of these results.

**Theorem 6.** Let $X$ be a nonwedge FK space such that $\exists x \in X \setminus \ell^p$ with $x_\sigma \in X$ for all $\sigma \in \Sigma$. Then $X$ and $\ell^p$ are not linearly homeomorphic via a matrix.

**Theorem 7.** Let $X$ be a nonwedge FK space with AK such that $\exists x \in X \setminus \ell^p$ with $x_\sigma \in X$ for all $\sigma \in \Sigma$. Then $\dim_* X \nleq \dim_* \ell^p$.

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