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It is known that if a derivation basis $\mathscr{B}$ possesses Vitali-like covering properties, with covering families having arbitrarily small $L^{(p)}(\mu)$-overlap, where $1 \leqq p < +\infty$ and μ is a σ-finite measure in an abstract measure space, then $\mathscr{B}$ derives the μ-integrals of all functions $f \in L^{(q)}(\mu)$ where $p^{-1} + q^{-1} = 1$ if $p > 1$; $q = +\infty$ if $p = 1$. The converse is well known for the case $q = +\infty$, $p = 1$, and a partial converse is known for the case $p > 1$, if $\mathscr{B}$ is a $[\mathfrak{U}, \delta]$-basis. The present paper offers a converse for $p > 1$ under general hypotheses and, simultaneously, removes the necessity that $\mathscr{B}$ be a $[\mathfrak{U}, \delta]$-basis.

1. General definitions and terminology.

Our universe is a set of points S. We shall agree that if $A \subseteq S$ and $B \subseteq S$, then $A - B = \{x : (x \in A) \wedge (x \notin B)\}$; thus $A - B = A - (A \cap B)$. If $A \subseteq S$, we shall denote the complement of A in S by $\tilde{A}$. $\mathfrak{M}$ denotes a fixed Boolean σ-algebra of subsets of S, with S as its unit; μ denotes a fixed σ-finite measure defined on $\mathfrak{M}$, and μ^* is the completion of μ defined on the class $\mathfrak{M}^*$ of subsets of S. We let $\mathfrak{N}$ denote the family of μ-nullsets and $\mathfrak{N}^*$ the family of μ^*-nullsets. We let $\bar{\mu}$ denote the outer measure derived from μ. If $X \subseteq S$, then $\bar{X}$ denotes a measure cover of X; it is well known that $\bar{\mu}(X \cap M) = \mu(\bar{X} \cap M)$ holds for each set $M \in \mathfrak{M}$ and each μ-cover $\bar{X}$ of X. For any set $X \subseteq S$ we let χ_X denote the characteristic function of X.

A *derivation basis* $\mathfrak{B}$ is defined as follows. We assume that to each point x of a fixed subset E of S, called the *domain* of $\mathfrak{B}$, there correspond Moore–Smith sequences of $\mathcal{M}$-sets of positive μ-measure, called *constituents*, which are said to *converge* to x, and are denoted generically by $\{M_\iota(x)\}$. We further assume (Fréchet's convergence axiom) that each cofinal subsequence of an x-converging sequence also converges to x. The elements of $\mathfrak{B}$ are thus converging sequences together with corresponding convergence points. We denote by $\mathscr{D}$ the family of all $\mathfrak{B}$-constituents; i.e., the family of all sets belonging to one or more of the sequences $\{M_\iota(x)\}$ for some $x \in E$. This family $\mathscr{D}$ is called the *spread* of $\mathfrak{B}$.

If λ is a real-valued function defined on $\mathscr{D}$ and $x \in E$, then we define $D^*\lambda(x)$ and $D_*\lambda(x)$ by

$$D^*\lambda(x) = \sup\left[\limsup \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right]$$

and

$$D_*\lambda(x) = \inf\left[\liminf \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right],$$

where the expressions in brackets mean, respectively, the limit superior and inferior of any fixed x-converging sequence $\{M_\iota(x)\}$, and then the supremum and infimum of these values are taken among all such sequences. $D^*\lambda(x)$ and $D_*\lambda(x)$ are called, respectively, the *upper* and *lower* $\mathfrak{B}$*-derivates* of λ *at* x. If $D^*\lambda(x) = D_*\lambda(x)$ (whether finite or infinite), then their common value is denoted by $D\lambda(x)$, and is called the $\mathfrak{B}$*-derivative of* λ *at* x.

We say that λ is a μ*-finite* μ*-integral* iff there exists a μ-measurable function f such that $-\infty < \lambda(M) = \int_M f d\mu < +\infty$ whenever $M \in \mathcal{M}$ and $\mu(M)$ is finite. We say that λ is $\mathfrak{B}$*-derivable* iff $D\lambda(x)$ exists and coincides with $f(x)$ for μ^*-almost all $x \in E$.

By a *subbasis* of $\mathfrak{B}$ we mean any basis $\mathfrak{B}^*$ whose associated sequences belong to $\mathfrak{B}$ and which associates with these sequences the same convergence points as does $\mathfrak{B}$. Clearly, the spread of $\mathfrak{B}^*$ is a subfamily of the spread of $\mathfrak{B}$. The domain of $\mathfrak{B}^*$ is the set of its associated points, which is a subset of E.

If $X \subseteq E$ and $\mathcal{B}^*$ is any subbasis of $\mathfrak{B}$ such that the domain of $\mathfrak{B}^*$ includes the set X (mod $\mathcal{N}^*$), then the spread $\mathcal{V}$ of $\mathfrak{B}^*$ is called a $\mathfrak{B}$*-fine* covering of X. Sometimes a $\mathfrak{B}$-fine covering is defined as any family $\mathcal{V}$ of $\mathfrak{B}$-constituents that contains, for μ^*-almost all $x \in X$, the sets of at least one sequence $\{M_\iota(x)\}$. Although these definitions differ slightly, in their applications they have the same effect, so we may use them interchangeably.

If $\mathcal{H}$ is any finite or countably infinite subfamily of $\mathcal{M}$, then for any $x \in S$, we define $n_{\mathcal{H}}(x)$ as the number of members of $\mathcal{H}$ to which x belongs. We denote the union of the family $\mathcal{H}$ by $\cup\mathcal{H}$; it is clear that $n_{\mathcal{H}}(x) = 0$ if $x \in (S - (\cup\mathcal{H}))$. We define $e_{\mathcal{H}}(x) = n_{\mathcal{H}}(x) - 1$ if $x \in \cup\mathcal{H}$, $e_{\mathcal{H}}(x) = 0$ for all other points $x \in S$. It is clear that $e_{\mathcal{H}}(x) > 0$ iff x belongs to at least two members of $\mathcal{H}$. We note that both $n_{\mathcal{H}}$ and $e_{\mathcal{H}}$ are μ-measurable functions.

Henceforth, we let p denote an arbitrary but fixed real number such that $1 < p < +\infty$, and we define q so that $p^{-1} + q^{-1} = 1$; we have $1 < q < +\infty$. We say that the derivation basis $\mathfrak{B}$ is $L^{(p)}(\mu)$*-strong* iff for each set $X \subseteq E$ of finite outer $\bar{\mu}$-measure, each $\mathfrak{B}$-fine covering $\mathcal{V}$ of X, and each $\epsilon > 0$, there exists a finite or countably infinite subfamily $\mathcal{H}$ of $\mathcal{V}$ such that, putting $H = \cup\mathcal{H}$, we have

(i) $X - H \in \mathcal{N}^*$ ($\mathcal{H}$ covers μ^*-almost all of X);

(ii) $\mu(H - \bar{X}) < \epsilon$ (the μ-overflow of $\mathcal{H}$ with respect to X is less than ϵ),

(iii) $\|e_{\mathcal{H}}\|_p < \epsilon$; i.e., $\left(\int_S e^p_{\mathcal{H}}(x)\,d\mu(x)\right)^{1/p} < \epsilon$ (the $L^{(p)}(\mu)$ overlap of $\mathcal{H}$ is less than ϵ).

2. The main theorem. Throughout this section, we assume that $\mathfrak{B}$ is a derivation basis with domain $E \subseteq S$, that derives the μ-integrals of all functions $f \in L^{(q)}(\mu)$. We note that this implies, in particular, that $\mathfrak{B}$ has the density property for all $\mathcal{M}$-sets of finite μ-measure, and hence also for the complements of such sets.

We begin by proving some needed lemmas.

LEMMA 2.1. *If $\mathcal{H}$ is any finite or countably infinite family of $\mathcal{M}$-sets, then*

$$0 \leqq \int_S n^p_{\mathcal{H}}(x)\,d\mu(x) \leqq 2^p \int_S e^p_{\mathcal{H}}(x)\,d\mu(x) + \mu(\cup \mathcal{H}).$$

Proof. Let $A = \{x : n_{\mathcal{H}}(x) = 1\}$, $B = \{x : n_{\mathcal{H}}(x) \geqq 2\}$. Clearly, $A \cup B = \cup \mathcal{H}$ and, for $x \in B$, $n_{\mathcal{H}}(x) = e_{\mathcal{H}}(x) + 1 \leqq 2e_{\mathcal{H}}(x)$. Thus

$$\begin{aligned} 0 \leqq \int_S n^p_{\mathcal{H}}(x)\,d\mu(x) &= \int_B n^p_{\mathcal{H}}(x)\,d\mu(x) + \int_A n^p_{\mathcal{H}}(x)\,d\mu(x) \\ &\leqq 2^p \int_S e^p_{\mathcal{H}}(x)\,d\mu(x) + \mu(\cup \mathcal{H}). \end{aligned}$$

LEMMA 2.2. *Suppose that $\mathcal{H}$ is any finite or countably infinite family of $\mathcal{M}$-sets for which $\int_S n^p_{\mathcal{H}}(x)\,d\mu(x)$ is finite. If W is any $\mathcal{M}$-set and $\mathcal{G} = \mathcal{H} \cup \{W\}$, then*

$$0 \leqq \int_S e^p_{\mathcal{G}}(x)\,d\mu(x) \leqq \int_S e^p_{\mathcal{H}}(x)\,d\mu(x) + p \int_W n^{p-1}_{\mathcal{H}}(x)\,d\mu(x).$$

Proof. We observe that $e_{\mathcal{G}}(x) = e_{\mathcal{H}}(x)$ if $x \in (H - W)$, where $H = \cup \mathcal{H}$, $e_{\mathcal{G}}(x) = 0$ if $x \in (W - H)$, and $e_{\mathcal{G}}(x) = n_{\mathcal{H}}(x)$ if $x \in W \cap H$. Thus, because all the following integrals are finite owing to our hypotheses, we may write

$$\begin{aligned} (1) \qquad 0 \leqq \int_S e^p_{\mathcal{G}}(x)\,d\mu(x) &= \int_{\cup \mathcal{G}} e^p_{\mathcal{G}}(x)\,d\mu(x) \\ &= \int_{H-W} e^p_{\mathcal{G}}(x)\,d\mu(x) + \int_{W-H} e^p_{\mathcal{G}}(x)\,d\mu(x) + \int_{W \cap H} e^p_{\mathcal{G}}(x)\,d\mu(x) \\ &= \int_{H-W} e^p_{\mathcal{H}}(x)\,d\mu(x) + \int_{W \cap H} n^p_{\mathcal{H}}(x)\,d\mu(x) \end{aligned}$$

$$= \int_H e^p_{\mathscr{H}}(x)d\mu(x) - \int_{W\cap H} e^p_{\mathscr{H}}(x)d\mu(x) + \int_{W\cap H} n^p_{\mathscr{H}}(x)d\mu(x)$$

$$= \int_H e^p_{\mathscr{H}}(x)d\mu(x) + \int_W (n^p_{\mathscr{H}}(n) - e^p_{\mathscr{H}}(x))d\mu(x).$$

Because $0 \leqq \int_S n^p_{\mathscr{H}}(x)d\mu(x) < +\infty$, it follows that $n_{\mathscr{H}}$ and $e_{\mathscr{H}}$ are finite μ-almost everywhere. Hence, for μ-almost all points $x \in W \cap H$, we have $n^p_{\mathscr{H}}(x) = n^p$ and $e^p_{\mathscr{H}}(x) = (n-1)^p$, where n is some positive integer. By the mean-value theorem, we can write

$$0 \leqq n^p_{\mathscr{H}}(x) - e^p_{\mathscr{H}}(x) = n^p - (n-1)^p = p\xi^{p-1},$$

where $n - 1 < \xi < n$; and so

$$0 \leqq n^p_{\mathscr{H}}(x) - e^p_{\mathscr{H}}(x) \leqq pn^{p-1} = pn^{p-1}_{\mathscr{H}}(x). \tag{2}$$

The desired result is obtained by substituting (2) into the final term of (1).

LEMMA 2.3. *Suppose that* $X \subseteq E$, $\bar{X}$ *is any* μ*-cover of* X, $0 < \mu(\bar{X}) < +\infty$, *and* $\mathscr{V}$ *is a* $\mathfrak{B}$*-fine covering of* X. *Suppose also that* $0 < \alpha < 1$ *and that* $\mathscr{H}$ *is a finite or countably infinite subfamily of* $\mathscr{M}$ *subject to the conditions*

(i) $\int_S e^p_{\mathscr{H}}(x)d\mu(x) \leqq \alpha\mu(\bar{X} \cap H)$, *where* $H = \cup \mathscr{H}$;
(ii) $(1-\alpha)\Sigma_{V\in\mathscr{H}}\mu(V) \leqq \mu(\bar{X} \cap H)$;
(iii) $\mu(\bar{X} - H) > 0$.

Then there exists at least one set W *such that*

(iv) $W \in \mathscr{V}$ *and* $\int_W n^{p-1}_{\mathscr{H}}(x)d\mu(x) + \mu(W - \bar{X}) \leqq (\alpha/2p)\mu(W)$.

Moreover, if W *is any set satisfying* (iv), *and if we set* $\mathscr{G} = \mathscr{H} \cup \{W\}$, $G = \cup \mathscr{G}$, *then*

(v) $\int_S e^p_{\mathscr{G}}(x)d\mu(x) \leqq \alpha\mu(\bar{X} \cap G)$ *and*
(vi) $(1-\alpha)\Sigma_{V\in\mathscr{G}}\mu(V) \leqq \mu(\bar{X} \cap G)$.

Proof. From (i) and (ii) and the finiteness of $\mu(\bar{X})$, we infer the finiteness of $\int_S e^p_{\mathscr{H}}(x)d\mu(x)$ and $\mu(\cup \mathscr{H})$. These facts and Lemma 2.1 tell us that $0 \leqq \int_S n^p_{\mathscr{H}}(x)d\mu(x) < +\infty$; hence, because $(p-1)q = p$, we have $n^{p-1}_{\mathscr{H}} \in L^{(q)}(\mu)$. Thus $\mathfrak{B}$ derives the μ-integral of $n^{p-1}_{\mathscr{H}}$ as well as the integral of the characteristic function of $\bar{\bar{X}} = S - \bar{X}$. Accordingly, if we define

$$\lambda(M) = \int_M n_{\mathscr{H}}^{p-1}(x)\,d\mu(x) + \mu(M - \bar{X})$$

for each $M \in \mathscr{M}$, then it follows that $\mathfrak{B}$ derives λ. From this fact and (iii) we infer the existence of at least one point $z \in (X - H)$ for which

$$D\lambda(z) = n_{\mathscr{H}}^{p-1}(z) + \chi_{\bar{\bar{X}}}(z) = 0. \tag{1}$$

The existence of a set W satisfying (iv) follows at once from (1) and the fact that $\mathscr{V}$ is a $\mathfrak{B}$-fine covering of X.

Next, we consider an arbitrary set W satisfying (iv). We observe that

$$\begin{aligned}\mu(W - (\bar{X} - H)) = \mu(W \cap (\bar{\bar{X}} \cup H)) &\leqq \mu(W - \bar{X}) + \mu(W \cap H) \\ &\leqq \mu(W - \bar{X}) + \int_W n_{\mathscr{H}}^{p-1}(x)\,d\mu(x) \leqq \frac{\alpha}{2p}\,\mu(W),\end{aligned}$$

from which it follows easily that

$$\left(1 - \frac{\alpha}{2p}\right)\mu(W) \leqq \mu(W \cap (\bar{X} - H); \quad \mu(W) \leqq 2\mu(W \cap (\bar{X} - H)). \tag{2}$$

From (iv) and (2) we obtain

$$\int_W n_{\mathscr{H}}^{p-1}(x)\,d\mu(x) \leqq \frac{\alpha}{2p}\,\mu(W) \leqq \frac{\alpha}{p}\,\mu(W \cap (\bar{X} - H)). \tag{3}$$

Using (i), (3), and Lemma 2.2, we see that

$$\begin{aligned}\int_S e_{\mathscr{G}}^{p}(x)\,d\mu(x) &\leqq \int_S e_{\mathscr{H}}^{p}(x)\,d\mu(x) + p\int_W n_{\mathscr{H}}^{p-1}(x)\,d\mu(x) \\ &\leqq \alpha[\mu(\bar{X} \cap H) + \mu(W \cap (\bar{X} - H))] = \alpha\mu(\bar{X} \cap G),\end{aligned}$$

which establishes (v).

From (ii) and (2) we obtain

$$\begin{aligned}(1 - \alpha)\sum_{V \in \mathscr{G}} \mu(V) &= (1 - \alpha)\sum_{V \in \mathscr{H}} \mu(V) + (1 - \alpha)\mu(W) \\ &\leqq \mu(\bar{X} \cap H) + \left(1 - \frac{\alpha}{2p}\right)\mu(W) \\ &\leqq \mu(\bar{X} \cap H) + \mu(W \cap (\bar{X} - H)) = \mu(\bar{X} \cap G),\end{aligned}$$

and this completes the proof of the lemma.

THEOREM 2.4. *$\mathfrak{B}$ is $L^{(p)}(\mu)$-strong.*

Proof. We choose any set $X \subseteq E$ with $0 < \bar{\mu}(X) < +\infty$, select any μ-cover $\bar{X}$ of X, let $\mathcal{V}$ denote an arbitrary $\mathfrak{B}$-fine covering of X, and fix an arbitrary number α, $0 < \alpha < 1$.

Because $\mathfrak{B}$ derives the μ-integral of the characteristic function of $\tilde{\bar{X}}$, there exists at least one point $z \in X$ for which $D\lambda(z) = \chi_{\tilde{\bar{X}}}(z) = 0$, where $\lambda(M) = \int_M \chi_{\tilde{\bar{X}}}(x)\,d\mu(x) = \mu(M - \bar{X})$ for each set $M \in \mathcal{M}$. Thus, because $\mathcal{V}$ is a $\mathfrak{B}$-fine covering of X, there must be at least one set $W \in \mathcal{V}$ such that

$$(1) \qquad \mu(W - \bar{X}) \leqq \frac{\alpha}{2p}\,\mu(W).$$

Let $\mathcal{F}_1$ denote the family of those sets $W \in \mathcal{V}$ that satisfy the relation (1). Then $\mathcal{F}_1 \neq \emptyset$; also, it follows easily from (1) that $0 < (1-\alpha)\mu(W) \leqq \mu(\bar{X} \cap W) \leqq \mu(\bar{X})$ if $W \in \mathcal{F}_1$. Thus, if we set $\zeta_1 = \sup_{W \in \mathcal{F}_1} \mu(W)$, it follows that $0 < \zeta_1 < +\infty$. We choose a member V_1 of $\mathcal{F}_1$ with $\mu(V_1) > \frac{1}{2}\zeta_1$. We set $\mathcal{H}_1 = \{V_1\}$, $H_1 = \cup\,\mathcal{H}_1$, and observe that $\mathcal{H}_1$ satisfies the conditions (i) and (ii) of Lemma 2.3.

We proceed inductively. We suppose $k \geqq 1$ and that we have a family $\mathcal{H}_k = \{V_1, V_2, \cdots, V_k\} \subseteq \mathcal{V}$, satisfying the conditions (i) and (ii) of Lemma 2.3, with $H_k = \cup\,\mathcal{H}_k$. If $\mu(\bar{X} - H_k) = 0$, we define $\mathcal{H}_{k+1} = \mathcal{H}_k$, $\cup\,\mathcal{H}_{k+1} = H_{k+1} = H_k$. It is obvious that $\mathcal{H}_{k+1}$ satisfies the conditions (i) and (ii) of Lemma 2.3 because they hold for $\mathcal{H}_k$.

If $\mu(\bar{X} - H_k) > 0$, we use Lemma 2.3 to infer that the family $\mathcal{F}_{k+1}$, consisting of those sets $W \in \mathcal{V}$ satisfying the relation

$$(2) \qquad \int_W n_{\mathcal{H}_k}^{p-1}(x)\,d\mu(x) + \mu(W - \bar{X}) \leqq \frac{\alpha}{2p}\,\mu(W),$$

is nonempty. From (2), it follows easily that $(1 - \alpha/2p)\mu(W) \leqq \mu(W \cap \bar{X})$, whence $\mu(W) \leqq 2\mu(W \cap \bar{X})$, whenever $W \in \mathcal{F}_{k+1}$. Thus, setting $\zeta_{k+1} = \sup_{W \in \mathcal{F}_{k+1}} \mu(W)$, it follows that $0 < \zeta_{k+1} < +\infty$. We select a member V_{k+1} of $\mathcal{F}_{k+1}$ such that $\mu(V_{k+1}) > \frac{1}{2}\zeta_{k+1}$, and we define $\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{V_{k+1}\}$, $H_{k+1} = \cup\,\mathcal{H}_{k+1}$. Lemma 2.3 now tells us that

$$(3) \qquad \int_S e_{\mathcal{H}_{k+1}}^{p}(x)\,d\mu(x) \leqq \alpha\mu(\bar{X} \cap H_{k+1}) \quad \text{and}$$

$$(1-\alpha)\left(\sum_{V \in \mathcal{H}_{k+1}} \mu(V)\right) \leqq \mu(\bar{X} \cap H_{k+1}).$$

Thus, whether $\mu(\bar{X} - H_k) = 0$ or $\mu(\bar{X} - H_k) > 0$, we obtain a family $\mathscr{H}_{k+1} \subset \mathscr{V}$ satisfying the relations (3).

In this way, we obtain inductively a sequence $\{\mathscr{H}_k\}$ of finite subfamilies of $\mathscr{V}$, satisfying (3). We let $\mathscr{H} = \bigcup_{k=1}^{\infty} \mathscr{H}_k$, $H = \bigcup \mathscr{H}$. The monotone convergence theorem applied to (3) yields

$$(4) \qquad \int_S e^p_{\mathscr{H}}(x)\,d\mu(x) \leqq \alpha\mu(\bar{X} \cap H) \leqq \alpha\mu(\bar{X}) < +\infty \quad \text{and}$$

$$(1-\alpha)\mu(H) \leqq (1-\alpha) \sum_{V \in \mathscr{H}} \mu(V) \leqq \mu(\bar{X} \cap H) \leqq \mu(\bar{X}) < +\infty,$$

from which it follows that

$$(5) \qquad \mu(H - \bar{X}) \leqq \alpha\mu(H) \leqq \frac{\alpha}{1-\alpha}\,\mu(\bar{X}) < +\infty.$$

Because α is arbitrary, $0 < \alpha < 1$, it is clear from (4) and (5) that $\mathscr{H}$ can be chosen to satisfy conditions (ii) and (iii) of our definition of $L^{(p)}(\mu)$-strength in §1. It remains to be shown that $\mathscr{H}$ covers μ^*-almost all of X. Suppose, on the contrary, that $\mu(\bar{X} - H) = \bar{\mu}(X - H) > 0$. Thus $\mu(\bar{X} - H_k) \geqq \mu(\bar{X} - H) > 0$ for $k = 1, 2, \cdots$, which means that the inductive process does not stop producing new sets, and so $\mathscr{H}$ consists of a countably infinite family of sets $\{V_1, V_2, \cdots, V_k, \cdots\}$ chosen from $\mathscr{V}$. The conditions (i), (ii) and (iii) of Lemma 2.3 are satisfied by $\mathscr{H}$; hence, according to that lemma, there is a set $W \in \mathscr{V}$ such that

$$(6) \qquad \int_W n^{p-1}_{\mathfrak{H}}(x)\,d\mu(x) + \mu(W - \bar{X}) \leqq \frac{\alpha}{2p}\,\mu(W).$$

From (6) and the fact that $n_{\mathfrak{H}_k} \uparrow n_{\mathfrak{H}}$ as $k \to +\infty$, it follows that

$$\int_W n^{p-1}_{\mathfrak{H}_k}(x)\,d\mu(x) + \mu(W - \bar{X}) \leqq \frac{\alpha}{2p}\,\mu(W)$$

for each positive integer k, and therefore $W \in \mathfrak{F}_{k+1}$ for each such k. Hence $0 < \mu(W) \leqq \zeta_{k+1} < 2\mu(V_{k+1})$ for $k = 1, 2, \cdots$. However, from (4) we have

$$\sum_{V \in \mathfrak{H}} \mu(V) = \sum_{\iota=1}^{\infty} \mu(V_\iota) \leqq \frac{\mu(\bar{X})}{1-\alpha} < +\infty,$$

which implies that $\mu(V_{k+1}) \to 0$ as $k \to +\infty$. This contradiction forces us to conclude that $\mu(\bar{X} - H) = 0$, and completes the proof of the theorem.

In **[4]** it is shown, under a relatively mild pre-topological (actually dispensable) condition, that $L^{(p)}(\mu)$-strength is sufficient for a basis to derive all the μ-integrals of $L^{(q)}(\mu)$-functions. Accordingly, we now can assert that $L^{(p)}(\mu)$-strength is both necessary and sufficient for this purpose. One may still question whether or not there exists any basis at all with exactly $L^{(p)}(\mu)$-strength; i.e., one that is $L^{(p)}(\mu)$-strong but not $L^{(p')}(\mu)$-strong for any $p' > p$. Such a basis is known **[3]** with μ = plane Lebesgue measure, p any given real number greater than 1.

The technique used herein appears to be applicable to dual Orlicz spaces of more general character than the $L^{(p)}$- and $L^{(q)}$-spaces here considered. However, a preliminary study indicates that some conditions will have to be imposed on the Orlicz spaces. The writer is investigating this problem. Recently, A. Cordoba obtained the result of the present paper for the special case of a Euclidean derivation basis that is invariant under translation, using methods of functional analysis. His proof is given in **[1]**.

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