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The span of a metric space is the least upper bound of numbers α such that, roughly speaking, two points can move over the same portion of the space keeping a distance at least α from each other. The surjective span is obtained if it is required that, in addition, the whole space be covered by each of the moving points. These geometric ideas turn out to be important in continua theory. In the present paper, a simple triod is constructed such that the span of it is strictly greater than the surjective span.

Let X be a connected metric nonempty space. By p_1 and p_2 we denote the standard projections of the product $X \times X$ onto X, that is, $p_1(x, x') = x$ and $p_2(x, x') = x'$ for $(x, x') \in X \times X$. The *surjective span* $\sigma^*(X)$ [resp., the *surjective semispan* $\sigma_0^*(X)$] of X is defined to be the least upper bound of the set of real numbers α with the following property: there exist connected sets $C_\alpha \subset X \times X$ such that $\alpha \leqq \operatorname{dist}(x, x')$ for $(x, x') \in C_\alpha$ and $p_1(C_\alpha) = p_2(C_\alpha) = X$ [resp., $p_1(C_\alpha) = X$]. The *span* $\sigma(X)$ and the *semispan* $\sigma_0(X)$ of X are defined by the formulae:

$$\sigma(X) = \operatorname{Sup}\{\sigma^*(A) \colon \varnothing \neq A \subset X,\ A \text{ connected}\}, \tag{1}$$

$$\sigma_0(X) = \operatorname{Sup}\{\sigma_0^*(A) \colon \varnothing \neq A \subset X,\ A \text{ connected}\}. \tag{2}$$

It follows directly from the definitions that the following inequalities hold:

$$0 \leqq \sigma^*(X) \leqq \sigma(X) \leqq \sigma_0(X) \leqq \operatorname{diam} X, \tag{3}$$

$$0 \leqq \sigma^*(X) \leqq \sigma_0^*(X) \leqq \sigma_0(X) \leqq \operatorname{diam} X, \tag{4}$$

$$\sigma(A) \leqq \sigma(X), \qquad \sigma_0(A) \leqq \sigma_0(X) \qquad (A \subset X). \tag{5}$$

It is not difficult to check that the above definition of the span, formula (1), is equivalent to the definition given in **[5]**. Continua of surjective span zero were defined in **[10]**. For each arc, as well as for each arc-like continuum, all these four quantities are equal to zero (cf. **[8]**, Propositions 1.3 and 2.1). Nevertheless, they are quite useful in the theory of tree-like continua (see **[2]**, **[3]**, **[6]**, **[7]** and **[8]**). From this

view-point, the relationship between different kinds of span of a tree-like continuum seems to be of some importance. There is an easy example of a continuum X such that $\sigma(X)=\sigma_0(X)=1$ and $\sigma^*(X)=\sigma_0^*(X)=\frac{1}{2}$ (see **[8]**, Example 1.4). Its existence shows, among other things, that the analogues of inequalities (5) for surjective span and surjective semispan instead of span and semispan, respectively, do not hold. The example from **[8]**, however, is not a tree-like continuum: it contains a simple closed curve. It is the aim of the present paper to describe an example of a tree-like continuum with the same span properties. Moreover, our example belongs topologically to the simplest, besides arcs, class of tree-like continua, namely, that of simple triods. By a *simple triod* we understand the union of three arcs having a common end-point and mutually disjoint except at that point.

EXAMPLE. There exists a simple triod X in the 3-space such that $\sigma(X)=\sigma_0(X)=1$ and $\sigma^*(X)=\sigma_0^*(X)=\frac{1}{2}$.

Proof. The 3-space R^3 metrized by the ordinary Pythagorean distance will be used. Given two points $x, y \in R^3$, we denote by $\overline{xy}$ the straight-line segment with end-points x and y. Setting

$$(6)\quad q_i=\left(\frac{1}{2}\cos\frac{2\pi i}{3},\frac{1}{2}\sin\frac{2\pi i}{3},0\right),\quad r_i=\left(\cos\frac{2\pi(i+1)}{3},\sin\frac{2\pi(i+1)}{3},0\right)\qquad(i=1,2,3)$$

and

$$(7)\quad \begin{aligned} s_0&=\left(\frac{1}{4}\cos\frac{7\pi}{12},0,0\right), & s_1&=\left(\frac{1}{2}\cos\frac{7\pi}{12},-\frac{1}{2}\sin\frac{7\pi}{12},0\right),\\ s_2&=\left(0,0,\frac{1}{4}\right), & s_3&=\left(\frac{1}{2}\cos\frac{7\pi}{12},\frac{1}{2}\sin\frac{7\pi}{12},0\right), \end{aligned}$$

we get ten points $q_i, r_i, s_i \in R^3$. Let $o=(0,0,0)$ be the origin. We take the polygonal arcs:

$$S_0=\overline{or_2}\cup\overline{r_2s_2}\cup\overline{s_2s_0}\cup\overline{s_0q_1}\cup\overline{q_1q_2},$$

$$S_1=\overline{os_1}\cup\overline{s_1r_1},$$

$$S_2=\overline{os_3}\cup\overline{s_3r_3},$$

and we define X to be the union

$$X=S_0\cup S_1\cup S_2.$$

Clearly, X is a simple triod in R^3 (Fig. 1).

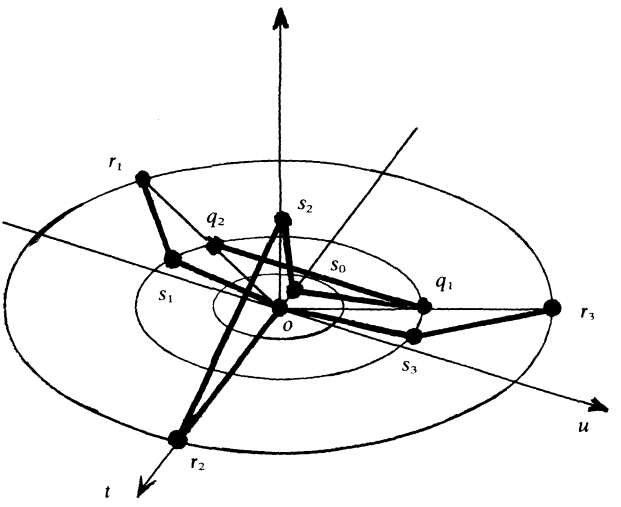

FIGURE 1

Let $A_0 = \overline{or_2}$ and $A_i = S_i$ for $i = 1, 2$. The set $T = A_0 \cup A_1 \cup A_2$ is also a simple triod. Its width (cf. [1] and [7]) is given by the formula[1]

$$w(T) = \operatorname{Min}\{\operatorname{Max}\{\rho(x, A_{i+1} \cup A_{i+2}): x \in A_i\}: i = 0, 1, 2\},$$

where the subscripts of A_j are taken mod 3 (see [7], Proposition 2.4).

Moreover, we have

$$\operatorname{Max}\{\rho(x, A_{i+1} \cup A_{i+2}): x \in A_i\} = 1 \qquad (i = 0, 1, 2),$$

whence $w(T) = 1$. It is known (see [9], p. 210) that $w(T) \leqq \sigma(T)$. Since $T \subset X$, we get $\sigma(T) \leqq \sigma(X)$, by (5). Thus $1 \leqq \sigma(X)$. We claim that $\sigma_0(X) \leqq 1$. If $\sigma_0(X) > 1$, there would exist, by (2), a connected nonempty set $A \subset X$ such that $\sigma_0^*(A) > 1$. Hence there would be a connected set $C_1 \subset X \times X$ such that $1 < \operatorname{dist}(x, x')$ for $(x, x') \in C_1$ and $p_1(C_1) = A$. By (6) and (7), each of the ten points q_i, r_i and s_i belongs to the sphere of radius 1 and center at the origin. Consequently, the set X is contained in this sphere, too, and therefore $\operatorname{dist}(o, x') \leqq 1$ for $x' \in X$. It follows that the origin o cannot be a point of $p_1(C_1)$, whence $o \notin A$, i.e., $A \subset X \setminus \{o\}$. The point o cuts the simple triod X into the three components $S_i \setminus \{o\}$ $(i = 0, 1, 2)$. Since A is a connected set, there has to be a

[1] Here $\rho(x, A)$ denotes the greatest lower bound of distances between the point x and all points of A.

subscript $j = 0, 1, 2$ such that $A \subset S_j \setminus \{o\}$. But then $A \subset S_j$ and $\sigma_0(S_j) \geqq \sigma_0^*(A) > 1$, by (2). This, however, is impossible because S_j is an arc (see **[8]**, Proposition 1.3). As a result, we obtain $\sigma_0(X) \leqq 1$, and $\sigma(X) = \sigma_0(X) = 1$, by (3).

The width of the simple triod X, i.e.,

$$w(X) = \operatorname{Min}\{\operatorname{Max}\{\rho(x, S_{i+1} \cup S_{i+2}): x \in S_i\}: i = 0, 1, 2\}$$

is less than the width of the simple triod T contained in X. Indeed, it is not difficult to check (cf. Fig. 1) that

$$\operatorname{Max}\{\rho(x, S_{i+1} \cup S_{i+2}): x \in S_i\} = \begin{cases} 1 & i = 0, \\ \frac{1}{2} & i = 1, \\ \frac{1}{2} & i = 2, \end{cases}$$

where, again, the subscripts are taken mod 3. Consequently, $w(X) = \frac{1}{2}$. We have $w(X) \leqq \sigma^*(X)$ (see **[9]**, p. 210)[2]. To complete the proof of all the properties of the example, it is now sufficient, by (4), to prove that $\sigma_0^*(X) \leqq \frac{1}{2}$.

Suppose, on the contrary, that $\sigma_0^*(X) > \frac{1}{2}$. Then there exist a number $\alpha_0 > \frac{1}{2}$ and a connected set $C_{\alpha_0} \subset X \times X$ such that $\alpha_0 \leqq \operatorname{dist}(x, x')$ for $(x, x') \in C_{\alpha_0}$ and $p_1(C_{\alpha_0}) = X$. The closure C of C_{α_0} in $X \times X$ is a continuum, and we also have $\alpha_0 \leqq \operatorname{dist}(x, x')$ for $(x, x') \in C$ and $p_1(C) = X$. The existence of such a continuum C will lead to a contradiction in each of a number of cases considered below. Before listing them, we need to establish some geometric properties of the polygonal arcs which form the simple triod X.

The arcs

$$B_0 = \overline{s_0 q_1} \cup \overline{q_1 q_2}, \qquad B_1 = \overline{s_0 s_2} \cup \overline{s_2 r_2},$$

and $B = B_0 \cup B_1$ are subarcs of the arc S_0. By (6) and (7), the points q_1, q_2 and s_0 belong to the sphere of radius $\frac{1}{2}$ and center at the origin. Thus

$$\operatorname{dist}(o, x) \leqq \tfrac{1}{2} \qquad (x \in B_0). \tag{8}$$

Let $f\colon X \to R^3$ be the projection of X onto the t-axis (Fig. 1). By (7), any plane containing a point of the segment $\overline{s_0 s_2}$ and perpendicular to the t-axis meets the simple triod X in a set contained in the sphere of radius $\frac{1}{4}$ and center at the origin (Fig. 2). Consequently, the diameter of the set $f^{-1}f(x)$ does not exceed $\frac{1}{2}$ for $x \in \overline{s_0 s_2}$. If $x \in \overline{s_2 r_2}$, the point $f(x)$ is a point

[2] The lemma in **[9]** states that $w(X) \leqq \sigma(X)$, but its proof actually provides an estimate of the surjective span rather than the span. Hence $w(X) \leqq \sigma^*(X)$ for each simple triod X.

of $\overline{or_2}$ having a distance not greater than $\frac{1}{4}$ from x. No other point of X projects onto $f(x)$ besides x and $f(x)$ itself. Hence $f^{-1}f(x) = \{x, f(x)\}$ for $x \in \overline{s_2r_2}$, and we get

$$\text{(9)} \qquad \operatorname{diam} f^{-1}f(x) \leqq \tfrac{1}{2} \qquad (x \in B_1).$$

Now, since the continuum C is mapped onto X by the projection p_1, the point q_2 belongs to $p_1(C)$. The set $p_1^{-1}(B) \cap C$ is a closed proper subset of C which meets $p_1^{-1}(q_2)$. There exists a component K of $p_1^{-1}(B) \cap C$ containing a point of $p_1^{-1}(q_2)$, so that $q_2 \in p_1(K)$. We observe that the arc B has a degenerate boundary in X, namely $\{r_2\}$. The continuum K meets the boundary of $p_1^{-1}(B) \cap C$ in C (see **[4]**, p. 172), whence $p_1(K)$ must contain a point of the boundary of B in X, i.e., $r_2 \in p_1(K)$. The continuum $p_1(K)$, however, is a subset of the arc B whose end-points are q_2 and r_2. Thus $B = p_1(K)$. We denote $g = p_1 | K$. Hence $g\colon K \to B$ is a continuous mapping of K onto B.

The point s_0 is a common end-point of the arcs B_0 and B_1 whose union is B. Let $\mathcal{K}_i$ denote the collection of all components of the set $g^{-1}(B_i)$ $(i = 0, 1)$. Each component of $g^{-1}(B_i)$ meets the boundary of $g^{-1}(B_i)$ in K (ibidem). Each point of the boundary of $g^{-1}(B_i)$ in K is sent under g into a point of the boundary of B_i in B. The latter boundary is degenerate, namely it is $\{s_0\}$. It follows that

$$\text{(10)} \qquad g^{-1}(s_0) \cap Z \neq \varnothing \qquad (Z \in \mathcal{K}_i\,;\ i = 0, 1).$$

The sets M_j $(j = 0, 1)$, defined by the formulae:

$$M_0 = \{y \in g^{-1}(s_0)\colon fp_1(y) \leqq fp_2(y)\},$$
$$M_1 = \{y \in g^{-1}(s_0)\colon fp_1(y) \geqq fp_2(y)\},$$

are closed subsets of K and their union is $g^{-1}(s_0)$. We denote by $\mathcal{K}_{ij}$ the subcollection of $\mathcal{K}_i$ consisting of all those elements of $\mathcal{K}_i$ which meet the set M_j $(i, j = 0, 1)$.

The remainder of the proof is divided into several cases, each of them ending with a contradiction. In this way, we shall prove that the surjective semispan of X is, indeed, less than or equal to $\frac{1}{2}$.

Case 1. $\mathcal{K}_{00} = \varnothing$. Since $q_2 \in B_0 \subset B = g(K)$, the point-inverse $g^{-1}(q_2)$ is a nonempty subset of $g^{-1}(B_0)$. Let K' be a component of $g^{-1}(B_0)$ which meets $g^{-1}(q_2)$. Then $K' \in \mathcal{K}_0$ and, by (10), the component K' also meets $g^{-1}(s_0) = M_0 \cup M_1$. Hence it must meet M_0 or M_1, that is, $K' \in \mathcal{K}_{00}$ or $K' \in \mathcal{K}_{01}$, respectively. But $\mathcal{K}_{00}$ being empty, we obtain $K' \in \mathcal{K}_{01}$. On the other hand, the continuum $g(K')$ is a subset of the arc

B_0 whose end-points are q_2 and s_0. Since both q_2 and s_0 belong to $g(K')$, we have $B_0 = g(K')$. Also $q_1 \in B_0$, so that $q_1 \in g(K')$. Let $y_k \in K'$ be points such that $g(y_k) = q_k$ $(k = 1, 2)$. It follows from the inclusions

$$K' \subset K \subset C \subset X \times X$$

that $g(y_k) = p_1(y_k)$ and

$$\tfrac{1}{2} < \alpha_0 \leqq \operatorname{dist}[p_1(y_k), p_2(y_k)] = \operatorname{dist}[q_k, p_2(y_k)] \qquad (k = 1, 2),$$

where $p_2(y_k)$ is a point of the simple triod X. Observe that each point of the arc S_2 has a distance less than or equal to $\frac{1}{2}$ from q_1. Similarly, each point of the arc S_1 has a distance less than or equal to $\frac{1}{2}$ from q_2. Thus $p_2(y_k) \in S_0 \cup S_k$ $(k = 1, 2)$. As a result, the continuum $p_2(K')$ meets both sets $S_0 \cup S_1$ and $S_0 \cup S_2$ whose union X contains $p_2(K')$. The continuum $p_2(K')$ must, therefore, intersect the common part of these two sets which is equal to S_0. We conclude Case 1 by proving that the conditions $K' \in \mathcal{K}_{01}$ and $p_2(K') \cap S_0 \neq \emptyset$ are incompatible. First, we note that $K' \in \mathcal{K}_{01}$ means $K' \cap M_1 \neq \emptyset$, i.e., there is a point $y_3 \in K'$ with $g(y_3) = s_0$ and $fp_2(y_3) \leqq fp_1(y_3)$. Again, $g(y_3) = p_1(y_3)$. The t-coordinate of $p_2(y_3)$ is less than or equal to the t-coordinate of s_0; this is what the last inequality says. The point $p_2(y_3)$ is, of course, a point of X, and we see (cf. Fig. 1) that it can be located only on B_0 or $S_1 \cup S_2$; the remaining points of S_0 have t-coordinates greater than the t-coordinate of s_0. However, we also have

$$\tfrac{1}{2} < \alpha_0 \leqq \operatorname{dist}[p_1(y_3), p_2(y_3)] = \operatorname{dist}[s_0, p_2(y_3)],$$

and each point of the arc B_0 has a distance less than $\frac{1}{2}$ from s_0 (cf. Fig. 2). Hence $p_2(y_3) \in S_1 \cup S_2$. If the continuum $p_2(K')$ intersected S_0, it would intersect both S_0 and $S_1 \cup S_2$, so that it would have to intersect the common part of S_0 and $S_1 \cup S_2$ which is the degenerate set $\{o\}$. In such a situation, there would exist a point $y_0 \in K'$ with $o = p_2(y_0)$, or, which is the same thing, a point $x_0 \in X$ such that $(x_0, o) \in K'$. Then $\frac{1}{2} < \alpha_0 \leqq \operatorname{dist}(x_0, o)$ and $x_0 \in p_1(K')$. Since $p_1(K') = g(K') \subset B_0$, we would get $x_0 \in B_0$, contradicting (8).

Case 2. $\mathcal{K}_{11} = \emptyset$. Since $r_2 \in B_1 \subset B = g(K)$, the point-inverse $g^{-1}(r_2)$ is a nonempty subset of $g^{-1}(B_1)$. Let K'' be a component of $g^{-1}(B_1)$ which meets $g^{-1}(r_2)$. Then $K'' \in \mathcal{K}_1$, and K'' also meets $g^{-1}(s_0) = M_0 \cup M_1$, by (10). Hence $K'' \in \mathcal{K}_{10}$ or $K'' \in \mathcal{K}_{11}$. But $\mathcal{K}_{11}$ being empty, we obtain $K'' \in \mathcal{K}_{10}$, which means that $K'' \cap M_0 \neq \emptyset$. Consequently, there is a point $z_0 \in K''$ with $fp_1(z_0) \leqq fp_2(z_0)$. On the other hand, there exists a point $z_2 \in K''$ such that $g(z_2) = r_2$. Thus $z_2 \in K$ and $g(z_2) = p_1(z_2)$. The

t-coordinate $f(r_2)$ of the point r_2 is equal to 1, and it is the maximum t-coordinate of points belonging to X (cf. Fig. 1). It follows that $fp_2(z_2) \leqq f(r_2) = fp_1(z_2)$. The continuum K'' then contains two points, z_0 and z_2, at which the real-valued functions fp_1 and fp_2 switch the order. As a result, there exists a point $z_1 \in K''$ with $fp_1(z_1) = fp_2(z_1)$. We denote $x_1 = p_1(z_1)$ and observe that $z_1 \in K \subset C$. Therefore we have

$$\tfrac{1}{2} < \alpha_0 \leqq \operatorname{dist}[p_1(z_1), p_2(z_1)] = \operatorname{dist}[x_1, p_2(z_1)],$$

where both points x_1 and $p_2(z_1)$ belong to the set $f^{-1}f(x_1)$. Then $\frac{1}{2} < \operatorname{diam} f^{-1}f(x_1)$. Since $p_1(K'') = g(K'') \subset B_1$, we get $x_1 \in B_1$, which contradicts (9).

Case 3. $\mathcal{K}_{00} \neq \emptyset \neq \mathcal{K}_{11}$. By (10), we see that $\mathcal{K}_i = \mathcal{K}_{i0} \cup \mathcal{K}_{i1}$ $(i = 0, 1)$. Consequently, the continuum K admits the decomposition[3]

$$K = g^{-1}(B) = g^{-1}(B_0 \cup B_1) = g^{-1}(B_0) \cup g^{-1}(B_1)$$

$$= |\mathcal{K}_0| \cup |\mathcal{K}_1| = (|\mathcal{K}_{00}| \cup |\mathcal{K}_{10}|) \cup (|\mathcal{K}_{01}| \cup |\mathcal{K}_{11}|),$$

where both sets $|\mathcal{K}_{00}| \cup |\mathcal{K}_{10}|$ and $|\mathcal{K}_{01}| \cup |\mathcal{K}_{11}|$ are nonempty. These two sets, however, are closed in K (see **[4]**, p. 182). Their common part must be nonempty, too, and Case 3 splits into four subcases.

Case 3(a). $|\mathcal{K}_{00}| \cap |\mathcal{K}_{01}| \neq \emptyset$. In this case, there exists an element $K^* \in \mathcal{K}_{00} \cap \mathcal{K}_{01}$. In particular, $K^* \in \mathcal{K}_{00}$ means $K^* \cap M_0 \neq \emptyset$, i.e., there is a point $y^* \in K^*$ with $g(y^*) = s_0$ and $fp_1(y^*) \leqq fp_2(y^*)$. Thus $y^* \in K$ and $g(y^*) = p_1(y^*)$, whence $f(s_0) \leqq fp_2(y^*)$. The t-coordinate of $p_2(y^*)$ is, therefore, greater than or equal to the t-coordinate of s_0. The only points of the arcs S_1 and S_2 whose t-coordinates satisfy this inequality are those points of the segments $\overline{os_1}$ and $\overline{os_3}$, respectively, which belong to the sphere of radius $\frac{1}{4}$ and center at the origin (cf. Fig. 2). The point s_0 also belongs to this sphere, so that its distance to each of those points is less than or equal to $\frac{1}{2}$. On the other hand, it follows from $y^* \in K \subset C$ that

$$\tfrac{1}{2} < \alpha_0 \leqq \operatorname{dist}[p_1(y^*), p_2(y^*)] = \operatorname{dist}[s_0, p_2(y^*)]$$

and $p_2(y^*) \in X$. We conclude that the point $p_2(y^*)$ cannot be located on S_1 or S_2. Hence $p_2(y^*) \in S_0$, and thus $p_2(K^*) \cap S_0 \neq \emptyset$. Also, $K^* \in \mathcal{K}_{01}$. It has been proved in Case 1, for K' instead of K^*, that the last two conditions are incompatible.

[3] Given a collection $\mathcal{K}$ of sets, we denote by $|\mathcal{K}|$ the union of all elements of $\mathcal{K}$.

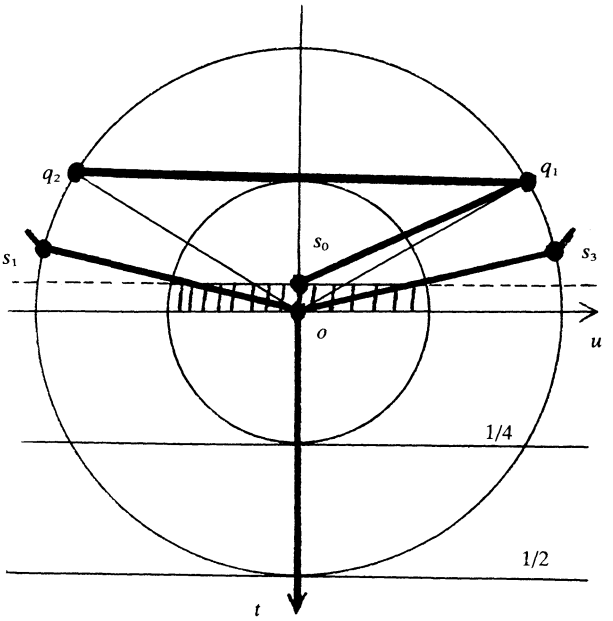

FIGURE 2

Case 3(b). $|\mathcal{K}_{10}| \cap |\mathcal{K}_{11}| \neq \emptyset$. In this case, there exists an element $K^{**} \in \mathcal{K}_{10} \cap \mathcal{K}_{11}$. In particular, we have $K^{**} \in \mathcal{K}_1$, whence $K^{**} \subset K$ and $p_1(K^{**}) = g(K^{**}) \subset B_1$. Being an element of $\mathcal{K}_{1j}$, the continuum K^{**} meets M_j $(j = 0, 1)$ and, by the definition of the sets M_j, it contains two points at which the functions fp_1 and fp_2 switch their order. We have shown in Case 2, for K'' in lieu of K^{**}, that this combination of properties leads to a contradiction.

Case 3(c). $|\mathcal{K}_{00}| \cap |\mathcal{K}_{11}| \neq \emptyset$. Let $K_i \in \mathcal{K}_{ii}$ $(i = 0, 1)$ be elements that intersect, and let $c \in K_0 \cap K_1$ be a point. Then $K_i \subset g^{-1}(B_i)$, whence

$$g(c) \in g(K_0) \cap g(K_1) \subset B_0 \cap B_1 = \{s_0\},$$

that is, $g(c) = s_0$. Thus $c \in g^{-1}(s_0) = M_0 \cup M_1$, and $c \in M_0$ or $c \in M_1$. If $c \in M_0$, then $K_1 \in \mathcal{K}_{10}$ and Case 3(c) reduces to Case 3(b). If $c \in M_1$, then $K_0 \in \mathcal{K}_{01}$ and Case 3(c) reduces to Case 3(a).

Case 3(d). $|\mathcal{K}_{10}| \cap |\mathcal{K}_{01}| \neq \emptyset$. Let $L_0 \in \mathcal{K}_{01}$ and $L_1 \in \mathcal{K}_{10}$ be elements that intersect, and let $d \in L_0 \cap L_1$ be a point. As in Case 3(c), we obtain $d \in M_0 \cup M_1$. If $d \in M_0$, then $L_0 \in \mathcal{K}_{00}$ and Case 3(d) reduces to Case 3(a). If $d \in M_1$, then $L_1 \in \mathcal{K}_{11}$ and Case 3(d) reduces to Case 3(b).

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