CERTAIN CONGRUENCES ON ORTHODOX SEMIGROUPS

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Letting $\kappa$ be the minimum unitary-congruence on a regular semigroup $S$ and $\xi$ be the minimum congruence such that $S/\xi$ is a semilattice of groups, it is the purpose of this paper to characterize all regular semigroups for which $\kappa \cap \xi$ is the identity relation. That is, we describe all regular semigroups which are subdirect products of a unitary semigroup and a semilattice of groups. In the process of doing this, a description of $\xi$ is given for any orthodox semigroup.

In A. H. Clifford's paper on radicals in semigroups, [2], a diagram was given presenting the relationship between various classes of regular semigroups and certain minimum congruences. Two questions were left open. The first was to find all subdirect products of a band and a semilattice of groups, that is, all semigroups for which $\beta \cap \xi$ is the identity, where $\beta$ is the minimum band-congruence. This was solved by Schein in [14] and also by Petrich in Theorem 3.2 of [11]. The second question involves finding all subdirect products of a semilattice of groups and a regular semigroup whose set of idempotents is unitary. In this paper we find that any such semigroup can be described as a semilattice of unitary semigroups on which $\mathcal{H} \cap \sigma$ is a unitary-congruence, where $\sigma$ is the minimum group-congruence. A description will also be given in terms of restrictions on the structure homomorphisms. In order to accomplish this, we first give an explicit characterization of $\xi$, the minimum semilattice of groups-congruence, on any orthodox semigroup.

1. Preliminary results: For a regular semigroup $S$, $E_S$ denotes the set of idempotents of $S$. If $E_s$ is a subsemigroup then $S$ is said to be orthodox.

**Proposition 1.1.** [11; Proposition 2.5] On a regular semigroup $S$, the following are equivalent: for $s, t \in S$,

(i) $e, es \in E_S$ implies $s \in E_S$;
(ii) $e, se \in E_S$ implies $s \in E_S$;
(iii) $e, ese \in E_S$ implies $s \in E_S$;
(iv) $e, set \in E_S$ implies $st \in E_S$;
(v) $ese = e \in E_S$ implies $s \in E_S$.

If any one of these five conditions holds, then $E_S$ is said to be unitary. For brevity, we shall call $S$ unitary if $E_S$ is a unitary subset of $S$. It is easily seen that any unitary semigroup is an orthodox semigroup. For inverse semigroups, those whose idempotents satisfy...
condition (v) are called proper by McAlister in [8] and [9], and he has given a description of all such inverse semigroups in terms of partially ordered sets and groups [9]. Unitary semigroups are not closed under homomorphisms; in fact, as is shown in [8], every inverse semigroup is an idempotent-separating homomorphic image of a unitary inverse semigroup.

A congruence on a semigroup is an equivalence relation which is compatible with multiplication. For two congruences $\rho, \rho'$ on a semigroup $S$, $\rho \subseteq \rho'$ if $ab \in S$ implies $a \rho b$. The identity relation on $S$ will be denoted by $\iota$, or $\iota_S$, if emphasis is needed. For the basic properties of congruences, the reader is referred to [3; §1.4, §1.5].

For a class $\mathcal{C}$ of semigroups, a congruence $\rho$ is called a $\mathcal{C}$-congruence on $S$ if $S/\rho$ is in $\mathcal{C}$. For a regular semigroup, the following notation will be used:

- $\kappa = \text{the minimum unitary-congruence},$
- $\beta = \text{the minimum band-congruence},$
- $\eta = \text{the minimum semilattice-congruence},$
- $\mu = \text{the minimum inverse-congruence},$
- $\xi = \text{the minimum semilattice of groups-congruence},$
- $\sigma = \text{the minimum group-congruence},$
- $\mu = \text{the maximum idempotent-separating congruence}.$

The Green relations will be noted as usual, and for brevity, a semilattice of groups-congruence will be called a $SG$-congruence. That each of the above minimum congruences exists is explained in [5], and also noted there are some of the following relationships which will be useful here:

$$\mu \subseteq \kappa \subseteq \beta \subseteq \eta; \quad \kappa \subseteq \beta \cap \sigma; \quad \xi \subseteq \eta \cap \sigma.$$

The following result will be needed for later work.

**Lemma 1.2.** Let $\mathcal{B}$ be a class of regular semigroups and $\mathcal{C}$ be a subclass of $\mathcal{B}$ such that for any $S \in \mathcal{B}$, the minimum $\mathcal{C}$-congruence on $S$, $\rho$, exists. If $\tau$ is any congruence defined on all semigroups in $\mathcal{B}$ such that $\tau$ is the identity on any $\mathcal{C}$-semigroup, then $\tau \subseteq \rho$ in $\mathcal{B}$.

**Proof.** Let $\tau$ be such a congruence, $S \in \mathcal{B}$. Then $\tau \cup \rho$ is a congruence and $\tau \cup \rho = \iota$ on any $\mathcal{C}$-semigroup. Now $S/\rho$ is a $\mathcal{C}$-semigroup and $\rho \subseteq \tau \cup \rho$, so $(S/\rho)/(\tau \cup \rho)/\rho = S/\rho/\iota = S/\rho$. On the other hand, $(S/\rho)/(\tau \cup \rho)/\rho = S/(\tau \cup \rho)$. Hence $S/\rho = S/(\tau \cup \rho)$. Therefore $\rho = \tau \cup \rho$ and $\tau \subseteq \rho$.

Let $\{S_\alpha\}_{\alpha \in A}$ be a family of semigroups and $T$ be a subsemigroup of the direct product $\Pi_{\alpha \in A} S_\alpha$. For each $\alpha \in A$, $\pi_\alpha$ is the natural projection of $T$ into $S_\alpha$. A semigroup $S$ is a subdirect product of $S_\alpha$, $\alpha \in A$, if $S$ is
isomorphic to a subsemigroup $T$ of $\Pi_{\alpha \in A} S_\alpha$ such that $T \pi_\alpha = S_\alpha$ for all $\alpha$ in $A$. For the particular case we are interested in, the relationship between congruences and subdirect products is as follows (see \cite[II.1.4]{10}). For congruences $\lambda$, $\rho$ on a semigroup $S$, $S$ is a subdirect product of $S/\lambda$ and $S/\rho$ if and only if $\lambda \cap \rho = \iota_S$. The aim of this paper is to describe all subdirect products of a unitary semigroup and a semilattice of groups. It is evident that this is equivalent to finding all semigroups for which $\kappa \cap \xi$ is the identity congruence. It is the latter attack that we shall make. From now on, we will assume that all semigroups are regular.

2. The minimum semilattice of groups-congruence. It was shown in \cite{5} that $\eta \cap \sigma$ is the smallest congruence $\rho$ such that $S/\rho$ is a semilattice of groups and is unitary. Thus, in general, $\xi$ is strictly contained in $\eta \cap \sigma$. Recall \cite[Theorem 3.1]{7} that on an orthodox semigroup $S$,

$$ a \sigma b \iff eae = ebe \text{ for some } e \in E_S. $$

To find $\xi$ on any orthodox semigroup, we first describe $\xi$ on any inverse semigroup and extend it to an orthodox semigroup via the method developed in Theorem 3.1 of \cite{7}.

**Theorem 2.1.** Let $S$ be an inverse semigroup. The minimum $SG$-congruence $\xi$ on $S$ can be defined as follows:

$$ a \xi b \iff a \eta b \text{ and } ea = eb \text{ for some } e^2 = e \eta a. $$

**Proof.** Let $arb$ if and only if $a \eta b$ and $ea = eb$ for some $e^2 = e \eta a$. It is easily seen that $\tau$ is an equivalence relation on $S$. Let $arb$ and $x$ be in $S$. Then $a \eta b$ and $ea = eb$ for some $e^2 = e \eta a$. Since $\eta$ is a congruence, $ax \eta bx$. Let $f$ be any idempotent such that $f \eta x$. Then $ef \eta ax$ and

$$ (ef)(ax) = f(ea)x = f(eb)x = (fe)(bx) = (ef)(bx); $$

therefore, $ax \tau bx$. On the other hand, $xa \eta xb$ and $xe \eta xa$. Thus, since $\eta$ is a semilattice congruence, $xex^{-1} \eta xa$. In addition,

$$ (xex^{-1})(xa) = xe(x^{-1}x)a = xea = xeb = (xex^{-1})(xb); $$

that is, $xa \tau xb$.

To see that $S/\tau$ is a semilattice of groups, it is sufficient to show that $aa^{-1} \tau a^{-1}a$ for all $a$ in $S$. But this is clear by letting $e = (aa^{-1})(a^{-1}a)$. Therefore, $\xi \subseteq \tau$.

Now if $S$ is in fact a semilattice of groups then $\eta = \mathcal{R}$ and $\tau$ is clearly the identity on $S$. Hence by Lemma 1.2, $\tau \subseteq \xi$. Consequently $\tau = \xi$. 

On an orthodox semigroup, the minimum inverse-congruence $\mathcal{Y}$ has been described by Hall [4] and Schein [13] as follows:

$$a \mathcal{Y} b \iff V(a) = V(b),$$

where $V(x)$ is the set of all inverses of $x$. For any $a$, $a \mathcal{Y}$ will denote the $\mathcal{Y}$-class containing $a$.

**Theorem 2.2.** Let $S$ be an orthodox semigroup. Then $\xi$ can be defined on $S$ as follows:

$$a \xi b \iff a \eta b \quad \text{and} \quad eae = ebe \quad \text{for some} \quad e^2 = e\eta.$$

**Proof.** Since $\mathcal{Y}$ is the minimum inverse-congruence then $\mathcal{Y} \subseteq \eta$. Thus for $a, b \in S$, $a \mathcal{Y} \eta b \mathcal{Y}$ implies $a \eta b$.

Now, $S/\mathcal{Y}$ is the maximum inverse homomorphic image of $S$, and therefore, letting $\xi'$ be the minimum $SG$-congruence on $S/\mathcal{Y}$, we have, via Theorem 2.1,

$$(*) \quad a \xi b \leftrightarrow a \mathcal{Y} \xi' \mathcal{b} \leftrightarrow a \mathcal{Y} \eta b \mathcal{Y} \quad \text{and} \quad x \mathcal{Y} a \mathcal{Y} = x \mathcal{Y} b \mathcal{Y},$$

for some $(x \mathcal{Y})^2 = x \mathcal{Y}$ with $x \mathcal{Y} \eta a \mathcal{Y}$.

Since $(x \mathcal{Y})^2 = x \mathcal{Y}$, there exists an idempotent $f$ such that $f \mathcal{Y} x$, and thus $f \eta a$. Therefore, using the fact that $\mathcal{Y}$ is a congruence, $(*)$ is equivalent to

$$a \mathcal{Y} \eta b \mathcal{Y} \quad \text{and} \quad (fa) \mathcal{Y} = (fb) \mathcal{Y} \quad \text{for some} \quad f^2 = f \eta a.$$

By definition of $\mathcal{Y}$, this means

$$a \xi b \leftrightarrow a \eta b \quad \text{and} \quad V(fa) = V(fb) \quad \text{for some} \quad f^2 = f \eta a.$$

The rest of the proof that $\xi$ can be defined as in the statement of the theorem is very similar to that of Lemma 3.2 of [7], using the additional fact that $\eta$ is a semilattice-congruence.

**Corollary 2.3.** (See [14] or [11; Theorem 3.2].) Let $S$ be a regular semigroup. Then $\beta \cap \xi = \iota_s$ if and only if $S$ is an orthodox band of groups.

**Proof.** Let $\beta \cap \xi = \iota_s$. Now we know that $\mathcal{H} \subseteq \beta \subseteq \eta$. We will show that $\beta$ is idempotent-separating. Let $e$ and $f$ be idempotents with $e \beta f$. Then $e \eta f$ and $(ef)e(ef) = (ef)f(ef)$ with $ef \eta e$. That is, by Theorem 2.2, $e \xi f$. Since $\beta \cap \xi = \iota_s$, we have $e = f$. Therefore, $\beta \subseteq \mu \subseteq \mathcal{H}$. 
But $\mathcal{H} \subseteq \beta$ so $\beta = \mathcal{H}$ and $S$ is a band of groups. The converse follows easily from the fact that $\beta = \mathcal{H}$ and $\mathcal{H} \cap \xi = \iota$.

3. \(\kappa \cap \xi = \iota\). In this section we characterize those semigroups $S$ which are a subdirect product of a unitary semigroup and a semilattice of groups, that is, those semigroups $S$ for which $\kappa \cap \xi$ is the identity. Clearly, since a unitary semigroup and a semilattice of groups are both orthodox, then $S$ is again an orthodox semigroup.

Recall that a semigroup is $\eta$-simple if it has exactly one $\eta$-class.

**Lemma 3.1.** Let $S$ be an $\eta$-simple orthodox semigroup. Then $S$ is unitary if and only if $\kappa \cap \xi = \iota_5$.

**Proof.** If $S$ is unitary then $\kappa = \iota_5$ so $\kappa \cap \xi = \iota_5$. Conversely, let $\kappa \cap \xi = \iota_5$. Then, using Theorem 2.2 and the fact that $S$ is $\eta$-simple, we have

\[
a\xi b \leftrightarrow a\eta b \quad \text{and} \quad eae = ebe \quad \text{for some} \quad e^2 = e\eta a
\]

\[
\leftrightarrow eae = ebe \quad \text{for some} \quad e^2 = e \leftrightarrow a\sigma b.
\]

That is, $\xi = \sigma$ and $\kappa \cap \sigma = \iota_5$. But $\sigma$ is a unitary congruence so $\kappa \cap \sigma = \kappa$. Therefore $\kappa = \iota_5$ and $S$ is unitary.

**Lemma 3.2.** Let $S$ be a semigroup with $\kappa \cap \xi = \iota_5$. Then $S$ is a semilattice of $\eta$-simple unitary semigroups.

**Proof.** Since $\eta$ is a semilattice-congruence, we know that $S$ is a semilattice $Y$ of $\eta$-simple semigroups $S_\alpha$, $\alpha \in Y$. Now, on $S_\alpha$, $\kappa \mid S_\alpha$ is a unitary-congruence and $\xi \mid S_\alpha$ is a $SG$-congruence. Hence, on $S_\alpha$, $(\kappa \mid S_\alpha) \cap (\xi \mid S_\alpha) = \iota$, and thus the intersection of the minimum unitary-congruence on $S_\alpha$ and the minimum $SG$-congruence on $S_\alpha$ is also the identity. By Lemma 3.1, $S_\alpha$ is unitary.

**Lemma 3.3** \cite{5; Theorem 3.9}. If $S$ is a unitary semigroup then $\mathcal{H} \cap \sigma = \iota_5$.

**Lemma 3.4.** Let $S$ be a regular semigroup. Then $\mathcal{H} \cap \sigma \subseteq \kappa$.

**Proof.** Let $\mathcal{H}$ be the class of all unitary semigroups. Letting $\tau$ be the congruence generated by $\mathcal{H} \cap \sigma$, then $\tau = \iota_5$ for any $S \in \mathcal{H}$, by Lemma 3.3. Therefore, by Lemma 1.2, $\tau \subseteq \kappa$. That is, $\mathcal{H} \cap \sigma \subseteq \kappa$.

**Theorem 3.5.** Let $S$ be a regular semigroup. The following statements are equivalent.
(i) $\kappa \cap \xi = \iota$, where $\kappa$ is the minimum unitary-congruence and $\xi$ is the minimum SG-congruence.

(ii) $S$ is a semilattice of unitary semigroups and $\kappa = \mathcal{H} \cap \sigma = \mu \cap \sigma$.

(iii) $S$ is a semilattice of unitary semigroups and $\mathcal{H} \cap \sigma$ is a unitary congruence on $S$.

(iv) $S$ is a subdirect product of a unitary semigroup and a semilattice of groups.

Proof. (i) implies (ii). Let $\kappa \cap \xi = \iota$. By Lemma 3.2, $S$ is a semilattice of unitary semigroups. Since $\eta$ and $\sigma$ are both unitary-congruences, $\kappa$ is contained in both $\eta$ and $\sigma$. We shall show that $\kappa$ is idempotent-separating. For, let $ef$, with $e, f \in E_S$. Then $enf$ and $(ef)e(ef) = (ef)f(ef)$ with $ef\eta e$. That is, by Theorem 2.2, $e\xi f$. Since $\kappa \cap \xi = \iota$, then $e = f$. Hence $\kappa$ is idempotent-separating and $\kappa \subseteq \mu$. Consequently, using Lemma 3.4, we have $\mathcal{H} \cap \sigma \subseteq \kappa \subseteq \mu \cap \sigma$. But $\mu \subseteq \mathcal{H}$, so equality holds.

(ii) implies (iii). Clear.

(iii) implies (iv). Let $S$ be a semilattice of $\eta$-simple unitary semigroups $S_\alpha$, $\alpha \in Y$, with $\mathcal{H} \cap \sigma$ unitary. Then by Lemma 3.4, $\kappa = \mathcal{H} \cap \sigma$. Therefore

$$\kappa \cap \xi = (\mathcal{H} \cap \sigma) \cap \xi = \mathcal{H} \cap (\sigma \cap \xi) = \mathcal{H} \cap \xi.$$ 

Let $a \mathcal{H} \cap \xi b$. Then $a, b \in S_\alpha$ for some $\alpha$. Thus, since $S_\alpha$ is $\eta$-simple, in $S_\alpha$, $a \mathcal{H} \cap \sigma b$. But $S_\alpha$ is unitary, so by Lemma 3.3, on $S_\alpha$, $\mathcal{H} \cap \sigma = \iota$. Hence $a = b$. Consequently $\kappa \cap \xi = \iota$. Therefore $S$ is a subdirect product of $S/\kappa$ and $S/\xi$ (see II.1.4 of [10]).

(iv) implies (i). Let $S$ be a subdirect product of a unitary semigroup $U$ and a semilattice of groups $T$. Then the congruences induced on $S$ by the two projection maps are, respectively, a unitary-congruence $\lambda$, and a SG-congruence, $\rho$, and $\lambda \cap \rho = \iota$. Thus $\kappa \cap \xi \subseteq \lambda \cap \rho = \iota$. Hence $\kappa \cap \xi = \iota$.

It is not possible to eliminate either one of the two conditions:

1. $S$ is a semilattice of unitary semigroups,
2. $\mathcal{H} \cap \sigma$ is unitary.

For, any unitary semigroup (which is not a group) with a zero adjoined, satisfies (1), but for such a semigroup, $\kappa = \beta$ and $\beta \cap \xi \neq \iota$ by Corollary 2.3. On the other hand, let $S = B(G, \alpha)$ be any bisimple $\omega$-semigroup for which $\alpha$ is not one-to-one. Then $S$ is not unitary but $\kappa = \mathcal{H} \cap \sigma$ and $\kappa \cap \xi = \kappa \cap \sigma = \mathcal{H} \cap \sigma \neq \iota$.

**Corollary 3.6.** Let $S$ be a fundamental regular semigroup. Then $\kappa \cap \xi = \iota$ if and only if $S$ is unitary.
Proof. Since $S$ is fundamental, then $\mu = \iota$. Therefore, $\mu \cap \sigma = \iota$.

Every regular semigroup which is a semilattice $Y$ of semigroups $S_\alpha$, $\alpha \in Y$, can be constructed via certain homomorphisms $\phi_{\alpha, \beta}$ from $S_\alpha$ into $\Omega(S_\beta)$, the translational hull of $S_\beta$, for all $\alpha > \beta$; in this case, we shall denote $S$ by $(Y, S_\alpha, \phi_{\alpha, \beta})$. For a full description of this structure the reader is referred to [10; III.7.5]. In light of Theorem 3.5, it is of interest to know how the condition $\kappa = \mathcal{H} \cap \sigma$ can be expressed in terms of the structure homomorphisms $\phi_{\alpha, \beta}$. To do this we need to explore the translational hull $\Omega(T)$ of a unitary semigroup $T$. For the elementary properties of the translational hull, see Chapter V of [10]. Recall [10; III.7.5] that in $S = (Y, S_\alpha, \phi_{\alpha, \beta})$, if $s \in S_\alpha$, $t \in S_\beta$, with $\alpha > \beta$ then

$$st = \phi_{\alpha, \beta}^*t = \lambda^*t \quad \text{and} \quad ts = t\phi_{\alpha, \beta}^* = t\rho^*,$$

where $\phi_{\alpha, \beta}^* = (\lambda^*, \rho^*) \in \Omega(S_\beta)$.

**Lemma 3.7.** Let $S$ be unitary and $(\lambda, \rho) \in \Omega(S)$. If there exists an idempotent $e$ such that $\lambda e \in E_S$ or $ep \in E_S$ then $\lambda (E_S) \subseteq E_S$, $(E_S)\rho \subseteq E_S$.

**Proof.** Let $e$ and $\lambda e$ be in $E_S$. Then $(ep)e = e(\lambda e) \in E_S$, and since $S$ is unitary, by Proposition 1.1, $ep \in E_S$.

Let $f$ be in $E_S$. Then $e(\lambda f) = (ep)f \in E_S$, so again $\lambda f$ is in $E_S$. Thus $\lambda (E_S) \subseteq E_S$. Since $\lambda f \in E_S$ implies $fp \in E_S$, then also $(E_S)\rho \subseteq E_S$.

**Lemma 3.8.** Let $S$ be a unitary semigroup. Define

$$K(S) = \{(\lambda, \rho) \in \Omega(S) \mid \lambda (E_S) \subseteq E_S, (E_S)\rho \subseteq E_S\}.$$

Then $K(S)$ is a subsemigroup of $\Omega(S)$ which contains $E_{\Omega(S)}$.

**Proof.** That $K(S)$ is a semigroup is clear. Let $(\lambda, \rho)$ be in $E_{\Omega(S)}$. Then $\lambda^2 = \lambda$, $\rho^2 = \rho$. Let $a \in S$ and $\lambda a = b$. Let $b'$ be an inverse of $b$; then we have $\lambda(ab') = (\lambda a)b' = bb' \in E_S$. Hence there exists $x$ in $S$ such that $\lambda x = f \in E_S$. Moreover, $f = \lambda x = \lambda^2 x = \lambda (\lambda x) = \lambda f$. Therefore $\lambda f \in E_S$, and by Lemma 3.7, $\lambda (E_S) \subseteq E_S$. Similarly $(E_S)\rho \subseteq E_S$.

If $S$ is an inverse semigroup then $E_{\Omega(S)} = K(S)$, [1; Lemma 2.1]. However, in general, strict containment is possible. For, if $S$ is a rectangular group, we may assume $S = L \times G \times R$ where $L (R)$ is a left (right) zero semigroup and $G$ is a group. Then $\Omega(S) = T(L) \times G \times T'(R)$, where $T(L)$ ($T'(R)$) is the semigroup of all transformations of $L$ ($R$) written on the left (right) [10; V.3.12]. Under this isomorphism,
$$E_{0(S)} = \{(f, 1, f') | f, f' \text{ are retractions}\},$$

where a retraction is any mapping which is the identity on its range. On the other hand, $K(S) = T(L) \times 1 \times T(R)$ which is not equal to $E_{0(S)}$.

From [5] we recall that a congruence $\tau$ is unitary if $x^2 \tau x$ and $(sx)^2 \tau sx$ implies $s^2 \tau s$. For regular semigroups this is equivalent to:

$$\text{for } e, f \in E_s, \; se\tau f \text{ implies } s^2 \tau s.$$

We now explore the properties of $\phi_{a, \beta}$ which make $\kappa \cap \xi$ the identity. For a semigroup $S$, we denote $E_s$ by $E_s$ and $K(S)$ by $K_s$. For $S = (Y, S, \phi_{a, \beta})$ a semilattice of unitary semigroups $S$, let $\Gamma_{a, \beta} = K_{\beta} \phi_{a, \beta}^{-1}$ for all $\alpha > \beta$ and $\Gamma = \bigcup_{a > \beta} \Gamma_{a, \beta}$.

**Theorem 3.9.** Let $S$ be a regular semigroup. Then $\kappa \cap \xi = \iota_s$ if and only if $S = (Y, S_s, \phi_{a, \beta})$ is a semilattice of unitary semigroups satisfying the properties:

(i) $\Gamma$ is a band of groups,

(ii) for $s$ in $\Gamma \cap H_s$, $e \in E_s$ if $f < e$ with $f \in E_{\beta}$ then $\phi_{a, \beta}^*(f) \mathcal{H} f$, where $\phi_{a, \beta}^*(f)$ means both $\phi_{a, \beta}^{\lambda} f$ and $f \phi_{a, \beta}^{\lambda}$.

**Proof.** Let $\kappa \cap \xi = \iota_s$. Then $\mathcal{H} \cap \sigma = \mu \cap \sigma$ is unitary. Let $\alpha > \beta$ and $s$ be in $S_s$ with $\phi_{a, \beta} \in K_{\beta}$. Then $\phi_{a, \beta}^*(f) = (\lambda, \rho)$ and $\lambda (E_{\beta}) \subseteq E_{\beta}$. Let $g$ be in $E_{\beta}$. Then $\lambda g = f = f^2 \in E_{\beta}$, so by definition of multiplication in $S$, $sg = \phi_{a, \beta}^*(g) = \lambda g = f$. Hence $se\mu \cap \sigma f$, and $\mu \cap \sigma$ is unitary, so $s\mu \cap \sigma s^2$. Thus $s$ is contained in a group. Now since $\mu \cap \sigma$ is a congruence, the $\mu \cap \sigma$-classes which contain idempotents form a band of groups, $T$, and since $\phi_{a, \beta}$ is a homomorphism, then $K_{\beta} \phi_{a, \beta}^{-1} = \Gamma_{a, \beta}$ is a band of groups contained in $T$. Thus $\Gamma$ is a band of groups.

Now let $s$ be in $\Gamma_{a, \beta}$. Then $s$ is in a group so there exists an idempotent $h$ such that $ss^2 = s'h = h$ for some $s' \in V(s)$. Since $s\mu s^2$, then $s\mu h$ and for all idempotents $f$, $sf's = hfh$, $s'fs = hfh$, [6].

Let $f < h$, $f \in S_s$. Then $sf's = f$ and $fs'sf = fhf = f$. Thus if $\phi_{a, \gamma} = (\lambda', \rho')$ and $\phi_{a, \gamma} = (\lambda', \rho')$, $f = fs'sf = (f\rho') (\lambda' f)$. Therefore,

$$f\rho' = f(f\rho') = (f\rho')(\lambda' f)(f\rho'), \quad \lambda' f = (\lambda' f)(f\rho')(\lambda' f);$$

that is, $f\rho'$ is an inverse of $\lambda' f$. Now $sf's = f = fs'sf$ can be expressed by

$$(\lambda' f)(f\rho') = f = (f\rho')(\lambda' f).$$

Thus $\lambda' f \mathcal{H} f$. By considering $s'fs = fss'f = f$ we find $f\rho' \mathcal{H} f$. Thus $\phi_{a, \gamma}^*(f) \mathcal{H} f$.

Conversely, to show $\kappa \cap \xi = \iota_s$, using Theorem 3.5, we need only show that $\mu \cap \sigma$ is a unitary congruence. Let $s\mu \cap \sigma f$ with $e, f \in E_s$. 

Letting $s$ be in $S_\alpha$, $e$ in $E_\beta$, then $s \in E_{\alpha\beta}$. Since $se\sigma f$, there exists $g \in E_{\gamma}$, $\gamma \leq \alpha\beta$ such that $g(se)g = gfg \in E_{\gamma}$. This means that $(g\phi_{\alpha\gamma}^*)^*(\phi_{\beta\gamma}^*)g \in E_{\gamma}$. Since $e$ is idempotent so is $\phi_{\beta\gamma}^*$ and by Lemma 3.8, $\phi_{\beta\gamma}^*g$ is in $E_{\gamma}$. Thus since $S_\gamma$ is unitary, $g\phi_{\gamma\gamma}^*$ is idempotent by Proposition 1.1; by Lemma 3.7, $\phi_{\alpha\gamma}^*\phi_{\beta\gamma}^*$ is in $E_{\gamma}$. Consequently by property (i), $s$ is contained in a band $E_{\alpha\beta}$ of groups $G_b$, $b \in E_\alpha$. In particular, there exists $s' \in \mathcal{V}(s)$ such that $ss' = s's = h$ for some $h \in E_\alpha$. We need to show that $s\mu h$, and to do this it is sufficient to show that $ss' = f$, $s'fs = f$ for all $f \leq h$. Now if $f$ is in $E_\alpha$, $f \leq h$, then $s$ and $s'$ are in the group $G_h$ and

$$sfs' \in G_hG_fG_h \subseteq G_{hf} = G_f,$$

so that $sfs' = f$. Similarly $s'fs = f$. Now let $f$ be in $E_\alpha$, $\delta < \alpha$, with $f < h$. Let $\phi_{\delta\delta}^* = (\lambda^\delta, \rho^\delta)$, $\phi_{\alpha\delta}^* = (\lambda^\alpha, \rho^\alpha)$. By property (ii), $\lambda^\delta f \not\in \mathcal{H} f$, $fp^\delta \not\in \mathcal{H} f$. That is, $f(\lambda^\delta f) = \lambda^\delta f$, $(fp^\delta)f = fp^\delta$. Now since $f < h = ss'$, then $f = hf = \lambda^\alpha f = \lambda^\alpha \lambda^\delta f$, and thus

$$sfs' = (\lambda^\alpha f)(fp^\delta) = f(\lambda^\delta f)(fp^\delta) = (fp^\delta)f = (fp^\delta)(fp^\delta) = [(fp^\delta)f]p^\delta = (fp^\delta)p^\delta = fp^\delta f = f.$$

Similarly $s'fs = f$. Therefore $s\mu h$.

Since $\sigma$ is always unitary, $se\sigma f$ implies $s\sigma h$. Consequently, $s\mu \cap s\sigma h$, and $s\mu \cap s\sigma$ is a unitary congruence. By Theorem 3.5, $\kappa \cap \xi = \iota_s$.

A regular semigroup $S = (Y, S_\alpha, \phi_{\alpha\beta})$ is a strong semilattice of the semigroups $S_\alpha$, if $\phi_{\alpha\beta}$ maps $S_\alpha$ into $S_\beta$ for all $\alpha > \beta$. The conditions in Theorem 3.9 can be simplified considerably for strong semilattices of unitary semigroups.

**Corollary 3.10.** Let $S = (Y, S_\alpha, \phi_{\alpha\beta})$ be a strong semilattice of unitary semigroups $S_\alpha$. Then $\kappa \cap \xi = \iota_s$ if and only if $E_\beta \phi_{\alpha\beta}^{-1}$ is a band of groups for all $\alpha > \beta$.

**Proof.** It can be easily seen that $K_\beta \cap \Pi(S_\beta) = E_\beta$, where $\Pi(S_\beta)$ is the semigroup of inner bitranslations of $S_\beta$. Thus if $\phi_{\alpha\beta}$ maps $S_\alpha$ into $S_\beta$, then $K_\beta \phi_{\alpha\beta}^{-1} = E_\beta \phi_{\alpha\beta}^{-1}$. Property (ii) automatically holds since homomorphisms preserve $\mathcal{H}$-classes.

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**References**


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