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Letting κ be the minimum unitary-congruence on a regular semigroup S and ξ be the minimum congruence such that S/ξ is a semilattice of groups, it is the purpose of this paper to characterize all regular semigroups for which $\kappa \cap \xi$ is the identity relation. That is, we describe all regular semigroups which are subdirect products of a unitary semigroup and a semilattice of groups. In the process of doing this, a description of ξ is given for any orthodox semigroup.

In A. H. Clifford's paper on radicals in semigroups, **[2]**, a diagram was given presenting the relationship between various classes of regular semigroups and certain minimum congruences. Two questions were left open. The first was to find all subdirect products of a band and a semilattice of groups, that is, all semigroups for which $\beta \cap \xi$ is the identity, where β is the minimum band-congruence. This was solved by Schein in **[14]** and also by Petrich in Theorem 3.2 of **[11]**. The second question involves finding all subdirect products of a semilattice of groups and a regular semigroup whose set of idempotents is unitary. In this paper we find that any such semigroup can be described as a semilattice of unitary semigroups on which $\mathcal{H} \cap \sigma$ is a unitary-congruence, where σ is the minimum group-congruence. A description will also be given in terms of restrictions on the structure homomorphisms. In order to accomplish this, we first give an explicit characterization of ξ, the minimum semilattice of groups-congruence, on any orthodox semigroup.

1. Preliminary results.

For a regular semigroup S, E_S denotes the set of idempotents of S. If E_s is a subsemigroup then S is said to be orthodox.

PROPOSITION 1.1. **[11**; Proposition 2.5] *On a regular semigroup S, the following are equivalent: for $s, t \in S$,*

(i) *$e, es \in E_S$ implies $s \in E_S$;*
(ii) *$e, se \in E_S$ implies $s \in E_S$;*
(iii) *$e, ese \in E_S$ implies $s \in E_S$;*
(iv) *$e, set \in E_S$ implies $st \in E_S$;*
(v) *$ese = e \in E_S$ implies $s \in E_S$.*

If any one of these five conditions holds, then E_S is said to be *unitary*. For brevity, we shall call S unitary if E_S is a unitary subset of S. It is easily seen that any unitary semigroup is an orthodox semigroup. For inverse semigroups, those whose idempotents satisfy condition (v) are called proper by McAlister in **[8]** and **[9]**, and he has given a description of all such inverse semigroups in terms of partially ordered sets and groups **[9]**. Unitary semigroups are not closed under homomorphisms; in fact, as is shown in **[8]**, every inverse semigroup is an idempotent-separating homomorphic image of a unitary inverse semigroup.

A congruence on a semigroup is an equivalence relation which is compatible with multiplication. For two congruences ρ, ρ' on a semigroup S, $\rho \subseteq \rho'$ if $a\rho b$ implies $a\rho' b$. The identity relation on S will be denoted by ι, or ι_S, if emphasis is needed. For the basic properties of congruences, the reader is referred to [**3**; §1.4, §1.5].

For a class $\mathscr{C}$ of semigroups, a congruence ρ is called a $\mathscr{C}$-congruence on S if S/ρ is in $\mathscr{C}$. For a regular semigroup, the following notation will be used:

κ = the minimum unitary-congruence,
β = the minimum band-congruence,
η = the minimum semilattice-congruence,
$\mathscr{Y}$ = the minimum inverse-congruence,
ξ = the minimum semilattice of groups-congruence,
σ = the minimum group-congruence,
μ = the maximum idempotent-separating congruence.

The Green relations will be noted as usual, and for brevity, a semilattice of groups-congruence will be called a SG-congruence. That each of the above minimum congruences exists is explained in **[5]**, and also noted there are some of the following relationships which will be useful here:

$$\mu \subseteq \mathscr{H} \subseteq \beta \subseteq \eta; \qquad \kappa \subseteq \beta \cap \sigma; \qquad \xi \subseteq \eta \cap \sigma.$$

The following result will be needed for later work.

LEMMA 1.2. *Let $\mathscr{B}$ be a class of regular semigroups and $\mathscr{C}$ be a subclass of $\mathscr{B}$ such that for any $S \in \mathscr{B}$, the minimum $\mathscr{C}$-congruence on S, ρ, exists. If τ is any congruence defined on all semigroups in $\mathscr{B}$ such that τ is the identity on any $\mathscr{C}$-semigroup, then $\tau \subseteq \rho$ in $\mathscr{B}$.*

Proof. Let τ be such a congruence, $S \in \mathscr{B}$. Then $\tau \vee \rho$ is a congruence and $\tau \vee \rho = \iota$ on any $\mathscr{C}$-semigroup. Now S/ρ is a $\mathscr{C}$-semigroup and $\rho \subseteq \tau \vee \rho$, so $(S/\rho)/(\tau \vee \rho)/\rho = S/\rho/\iota = S/\rho$. On the other hand, $(S/\rho)/(\tau \vee \rho)/\rho = S/(\tau \vee \rho)$. Hence $S/\rho \simeq S/(\tau \vee \rho)$. Therefore $\rho = \tau \vee \rho$ and $\tau \subseteq \rho$.

Let $\{S_\alpha\}_{\alpha \in A}$ be a family of semigroups and T be a subsemigroup of the direct product $\Pi_{\alpha \in A} S_\alpha$. For each $\alpha \in A$, π_α is the natural projection of T into S_α. A semigroup S is a subdirect product of S_α, $\alpha \in A$, if S is isomorphic to a subsemigroup T of $\Pi_{\alpha \in A} S_\alpha$ such that $T\pi_\alpha = S_\alpha$ for all α in A. For the particular case we are interested in, the relationship between congruences and subdirect products is as follows (see **[10**; II.1.4]). For congruences λ, ρ on a semigroup S, S is a subdirect product of S/λ and S/ρ if and only if $\lambda \cap \rho = \iota_S$. The aim of this paper is to describe all subdirect products of a unitary semigroup and a semilattice of groups. It is evident that this is equivalent to finding all semigroups for which $\kappa \cap \xi$ is the identity congruence. It is the latter attack that we shall make. From now on, we will assume that all semigroups are regular.

2. The minimum semilattice of groups-congruence. It was shown in **[5]** that $\eta \cap \sigma$ is the smallest congruence ρ such that S/ρ is a semilattice of groups and is unitary. Thus, in general, ξ is strictly contained in $\eta \cap \sigma$. Recall [**7**; Theorem 3.1] that on an orthodox semigroup S,

$$a\sigma b \leftrightarrow eae = ebe \text{ for some } e \text{ in } E_S.$$

To find ξ on any orthodox semigroup, we first describe ξ on any inverse semigroup and extend it to an orthodox semigroup via the method developed in Theorem 3.1 of **[7]**.

THEOREM 2.1. *Let S be an inverse semigroup. The minimum SG-congruence ξ on S can be defined as follows*:

$$a\xi b \leftrightarrow a\eta b \quad \textit{and} \quad ea = eb \quad \textit{for some} \quad e^2 = e\eta a.$$

Proof. Let $a\tau b$ if and only if $a\eta b$ and $ea = eb$ for some $e^2 = e\eta a$. It is easily seen that τ is an equivalence relation on S. Let $a\tau b$ and x be in S. Then $a\eta b$ and $ea = eb$ for some $e^2 = e\eta a$. Since η is a congruence, $ax\eta bx$. Let f be any idempotent such that $f\eta x$. Then $ef\eta ax$ and

$$(ef)(ax) = f(ea)x = f(eb)x = (fe)(bx) = (ef)(bx);$$

therefore, $ax\tau bx$. On the other hand, $xa\eta xb$ and $xe\eta xa$. Thus, since η is a semilattice congruence, $xex^{-1}\eta xa$. In addition,

$$(xex^{-1})(xa) = xe(x^{-1}x)a = xea = xeb = (xex^{-1})(xb);$$

that is, $xa\tau xb$.

To see that S/τ is a semilattice of groups, it is sufficient to show that $aa^{-1}\tau a^{-1}a$ for all a in S. But this is clear by letting $e = (aa^{-1})(a^{-1}a)$. Therefore, $\xi \subseteq \tau$.

Now if S is in fact a semilattice of groups then $\eta = \mathscr{H}$ and τ is clearly the identity on S. Hence by Lemma 1.2, $\tau \subseteq \xi$. Consequently $\tau = \xi$.

On an orthodox semigroup, the minimum inverse-congruence $\mathcal{Y}$ has been described by Hall **[4]** and Schein **[13]** as follows:

$$a\mathcal{Y}b \leftrightarrow V(a) = V(b),$$

where $V(x)$ is the set of all inverses of x. For any a, $a\mathcal{Y}$ will denote the $\mathcal{Y}$-class containing a.

THEOREM 2.2. *Let S be an orthodox semigroup. Then ξ can be defined on S as follows*:

$$a\xi b \leftrightarrow a\eta b \quad \text{and} \quad eae = ebe \quad \text{for some} \quad e^2 = e\eta a.$$

Proof. Since $\mathcal{Y}$ is the minimum inverse-congruence then $\mathcal{Y} \subseteq \eta$. Thus for $a, b \in S$, $a\mathcal{Y}\eta b\mathcal{Y}$ implies $a\eta b$.

Now, $S/\mathcal{Y}$ is the maximum inverse homomorphic image of S, and therefore, letting ξ' be the minimum SG-congruence on $S/\mathcal{Y}$, we have, via Theorem 2.1,

$$(*) \qquad a\xi b \leftrightarrow a\mathcal{Y}\xi' b\mathcal{Y} \leftrightarrow a\mathcal{Y}\eta b\mathcal{Y} \quad \text{and} \quad x\mathcal{Y}a\mathcal{Y} = x\mathcal{Y}b\mathcal{Y},$$

for some $(x\mathcal{Y})^2 = x\mathcal{Y}$ with $x\mathcal{Y}\eta a\mathcal{Y}$.

Since $(x\mathcal{Y})^2 = x\mathcal{Y}$, there exists an idempotent f such that $f\mathcal{Y}x$, and thus $f\eta a$. Therefore, using the fact that $\mathcal{Y}$ is a congruence, $(*)$ is equivalent to

$$a\mathcal{Y}\eta b\mathcal{Y} \quad \text{and} \quad (fa)\mathcal{Y} = (fb)\mathcal{Y} \quad \text{for some} \quad f^2 = f\eta a.$$

By definition of $\mathcal{Y}$, this means

$$a\xi b \leftrightarrow a\eta b \quad \text{and} \quad V(fa) = V(fb) \quad \text{for some} \quad f^2 = f\eta a.$$

The rest of the proof that ξ can be defined as in the statement of the theorem is very similar to that of Lemma 3.2 of **[7]**, using the additional fact that η is a semilattice-congruence.

COROLLARY 2.3. (See **[14]** or **[11**; Theorem 3.2].) *Let S be a regular semigroup. Then $\beta \cap \xi = \iota_S$ if and only if S is an orthodox band of groups.*

Proof. Let $\beta \cap \xi = \iota_S$. Now we know that $\mathcal{H} \subseteq \beta \subseteq \eta$. We will show that β is idempotent-separating. Let e and f be idempotents with $e\beta f$. Then $e\eta f$ and $(ef)e(ef) = (ef)f(ef)$ with $ef\eta e$. That is, by Theorem 2.2, $e\xi f$. Since $\beta \cap \xi = \iota_S$, we have $e = f$. Therefore, $\beta \subseteq \mu \subseteq \mathcal{H}$.

But $\mathscr{H} \subseteq \beta$ so $\beta = \mathscr{H}$ and S is a band of groups. The converse follows easily from the fact that $\beta = \mathscr{H}$ and $\mathscr{H} \cap \xi = \iota$.

3. $\kappa \cap \xi = \iota$. In this section we characterize those semigroups S which are a subdirect product of a unitary semigroup and a semilattice of groups, that is, those semigroups S for which $\kappa \cap \xi$ is the identity. Clearly, since a unitary semigroup and a semilattice of groups are both orthodox, then S is again an orthodox semigroup.

Recall that a semigroup is η-simple if it has exactly one η-class.

LEMMA 3.1. *Let S be an η-simple orthodox semigroup. Then S is unitary if and only if $\kappa \cap \xi = \iota_S$.*

Proof. If S is unitary then $\kappa = \iota_S$ so $\kappa \cap \xi = \iota_S$. Conversely, let $\kappa \cap \xi = \iota_S$. Then, using Theorem 2.2 and the fact that S is η-simple, we have

$$a \xi b \leftrightarrow a \eta b \quad \text{and} \quad eae = ebe \quad \text{for some} \quad e^2 = e \eta a$$
$$\leftrightarrow eae = ebe \quad \text{for some} \quad e^2 = e \leftrightarrow a \sigma b.$$

That is, $\xi = \sigma$ and $\kappa \cap \sigma = \iota_S$. But σ is a unitary congruence so $\kappa \cap \sigma = \kappa$. Therefore $\kappa = \iota_S$ and S is unitary.

LEMMA 3.2. *Let S be a semigroup with $\kappa \cap \xi = \iota_S$. Then S is a semilattice of η-simple unitary semigroups.*

Proof. Since η is a semilattice-congruence, we know that S is a semilattice Y of η-simple semigroups S_α, $\alpha \in Y$. Now, on S_α, $\kappa \mid S_\alpha$ is a unitary-congruence and $\xi \mid S_\alpha$ is a SG-congruence. Hence, on S_α, $(\kappa \mid S_\alpha) \cap (\xi \mid S_\alpha) = \iota$, and thus the intersection of the minimum unitary-congruence on S_α and the minimum SG-congruence on S_α is also the identity. By Lemma 3.1, S_α is unitary.

LEMMA 3.3 [**5**; Theorem 3.9]. *If S is a unitary semigroup then $\mathscr{H} \cap \sigma = \iota_S$.*

LEMMA 3.4. *Let S be a regular semigroup. Then $\mathscr{H} \cap \sigma \subseteq \kappa$.*

Proof. Let $\mathscr{K}$ be the class of all unitary semigroups. Letting τ be the congruence generated by $\mathscr{H} \cap \sigma$, then $\tau = \iota_S$ for any $S \in \mathscr{K}$, by Lemma 3.3. Therefore, by Lemma 1.2, $\tau \subseteq \kappa$. That is, $\mathscr{H} \cap \sigma \subseteq \kappa$.

THEOREM 3.5. *Let S be a regular semigroup. The following statements are equivalent.*

(i) $\kappa \cap \xi = \iota_S$, *where* κ *is the minimum unitary-congruence and* ξ *is the minimum SG-congruence.*

(ii) *S is a semilattice of unitary semigroups and* $\kappa = \mathcal{H} \cap \sigma = \mu \cap \sigma$.

(iii) *S is a semilattice of unitary semigroups and* $\mathcal{H} \cap \sigma$ *is a unitary congruence on S.*

(iv) *S is a subdirect product of a unitary semigroup and a semilattice of groups.*

Proof. (i) implies (ii). Let $\kappa \cap \xi = \iota$. By Lemma 3.2, S is a semilattice of unitary semigroups. Since η and σ are both unitary-congruences, κ is contained in both η and σ. We shall show that κ is idempotent-separating. For, let $e\kappa f$, with $e, f \in E_S$. Then $e\eta f$ and $(ef)e(ef) = (ef)f(ef)$ with $ef\eta e$. That is, by Theorem 2.2, $e\xi f$. Since $\kappa \cap \xi = \iota$, then $e = f$. Hence κ is idempotent-separating and $\kappa \subseteq \mu$. Consequently, using Lemma 3.4, we have $\mathcal{H} \cap \sigma \subseteq \kappa \subseteq \mu \cap \sigma$. But $\mu \subseteq \mathcal{H}$, so equality holds.

(ii) implies (iii). Clear.

(iii) implies (iv). Let S be a semilattice of η-simple unitary semigroups S_α, $\alpha \in Y$, with $\mathcal{H} \cap \sigma$ unitary. Then by Lemma 3.4, $\kappa = \mathcal{H} \cap \sigma$. Therefore

$$\kappa \cap \xi = (\mathcal{H} \cap \sigma) \cap \xi = \mathcal{H} \cap (\sigma \cap \xi) = \mathcal{H} \cap \xi.$$

Let $a\,\mathcal{H} \cap \xi\, b$. Then $a, b \in S_\alpha$ for some α. Thus, since S_α is η-simple, in S_α, $a\,\mathcal{H} \cap \sigma\, b$. But S_α is unitary, so by Lemma 3.3, on S_α, $\mathcal{H} \cap \sigma = \iota$. Hence $a = b$. Consequently $\kappa \cap \xi = \iota_S$. Therefore S is a subdirect product of S/κ and S/ξ (see II.1.4 of **[10]**).

(iv) implies (i). Let S be a subdirect product of a unitary semigroup U and a semilattice of groups T. Then the congruences induced on S by the two projection maps are, respectively, a unitary-congruence λ, and a SG-congruence, ρ, and $\lambda \cap \rho = \iota_S$. Thus $\kappa \cap \xi \subseteq \lambda \cap \rho = \iota_S$.

It is not possible to eliminate either one of the two conditions:

(1) S is a semilattice of unitary semigroups,

(2) $\mathcal{H} \cap \sigma$ is unitary.

For, any unitary semigroup (which is not a group) with a zero adjoined, satisfies (1), but for such a semigroup, $\kappa = \beta$ and $\beta \cap \xi \neq \iota$ by Corollary 2.3. On the other hand, let $S = B(G, \alpha)$ be any bisimple ω-semigroup for which α is not one-to-one. Then S is not unitary but $\kappa = \mathcal{H} \cap \sigma$ and $\kappa \cap \xi = \kappa \cap \sigma = \mathcal{H} \cap \sigma \neq \iota$.

COROLLARY 3.6. *Let S be a fundamental regular semigroup. Then* $\kappa \cap \xi = \iota$ *if and only if S is unitary.*

Proof. Since S is fundamental, then $\mu = \iota$. Therefore, $\mu \cap \sigma = \iota$.

Every regular semigroup which is a semilattice Y of semigroups S_α, $\alpha \in Y$, can be constructed via certain homomorphisms $\phi_{\alpha,\beta}$ from S_α into $\Omega(S_\beta)$, the translational hull of S_β, for all $\alpha > \beta$; in this case, we shall denote S by $(Y, S_\alpha, \phi_{\alpha,\beta})$. For a full description of this structure the reader is referred to [**10**; III.7.5]. In light of Theorem 3.5, it is of interest to know how the condition $\kappa = \mathcal{H} \cap \sigma$ can be expressed in terms of the structure homomorphisms $\phi_{\alpha,\beta}$. To do this we need to explore the translational hull $\Omega(T)$ of a unitary semigroup T. For the elementary properties of the translational hull, see Chapter V of [**10**]. Recall [**10**; III.7.5] that in $S = (Y, S_\alpha, \phi_{\alpha,\beta})$, if $s \in S_\alpha$, $t \in S_\beta$, with $\alpha > \beta$ then

$$st = \phi^s_{\alpha,\beta} t = \lambda^s t \quad \text{and} \quad ts = t\phi^s_{\alpha,\beta} = t\rho^s,$$

where $\phi^s_{\alpha,\beta} = (\lambda^s, \rho^s) \in \Omega(S_\beta)$.

LEMMA 3.7. *Let S be unitary and $(\lambda, \rho) \in \Omega(S)$. If there exists an idempotent e such that $\lambda e \in E_S$ or $e\rho \in E_S$ then $\lambda(E_S) \subseteq E_S$, $(E_S)\rho \subseteq E_S$.*

Proof. Let e and λe be in E_S. Then $(e\rho)e = e(\lambda e) \in E_S$, and since S is unitary, by Proposition 1.1, $e\rho \in E_S$.

Let f be in E_S. Then $e(\lambda f) = (e\rho)f \in E_S$, so again λf is in E_S. Thus $\lambda(E_S) \subseteq E_S$. Since $\lambda f \in E_S$ implies $f\rho \in E_S$, then also $(E_S)\rho \subseteq E_S$.

LEMMA 3.8. *Let S be a unitary semigroup. Define*

$$K(S) = \{(\lambda, \rho) \in \Omega(S) \mid \lambda(E_S) \subseteq E_S,\ (E_S)\rho \subseteq E_S\}.$$

Then $K(S)$ is a subsemigroup of $\Omega(S)$ which contains $E_{\Omega(S)}$.

Proof. That $K(S)$ is a semigroup is clear. Let (λ, ρ) be in $E_{\Omega(S)}$. Then $\lambda^2 = \lambda$, $\rho^2 = \rho$. Let $a \in S$ and $\lambda a = b$. Let b' be an inverse of b; then we have $\lambda(ab') = (\lambda a)b' = bb' \in E_S$. Hence there exists x in S such that $\lambda x = f \in E_S$. Moreover, $f = \lambda x = \lambda^2 x = \lambda(\lambda x) = \lambda f$. Therefore $\lambda f \in E_S$, and by Lemma 3.7, $\lambda(E_S) \subseteq E_S$. Similarly $(E_S)\rho \subseteq E_S$.

If S is an inverse semigroup then $E_{\Omega(S)} = K(S)$, [**1**; Lemma 2.1]. However, in general, strict containment is possible. For, if S is a rectangular group, we may assume $S = L \times G \times R$ where L (R) is a left (right) zero semigroup and G is a group. Then $\Omega(S) = T(L) \times G \times T'(R)$, where $T(L)$ ($T'(R)$) is the semigroup of all transformations of L (R) written on the left (right) [**10**; V.3.12]. Under this isomorphism,

$$E_{\Omega(S)} = \{(f, 1, f') \mid f, f' \text{ are retractions}\},$$

where a retraction is any mapping which is the identiy on its range. On the other hand, $K(S) = T(L) \times 1 \times T'(R)$ which is not equal to $E_{\Omega(S)}$.

From **[5]** we recall that a congruence τ is unitary if $x^2 \tau x$ and $(sx)^2 \tau sx$ implies $s^2 \tau s$. For regular semigroups this is equivalent to:

$$\text{for } e, f \in E_S,\ se\tau f \text{ implies } s^2 \tau s.$$

We now explore the properties of $\phi_{\alpha,\beta}$ which make $\kappa \cap \xi$ the identity. For a semigroup S_α, we denote E_{S_α} by E_α, and $K(S_\alpha)$ by K_α. For $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ a semilattice of unitary semigroups S_α, let $\Gamma_{\alpha,\beta} = K_\beta \phi^{-1}_{\alpha,\beta}$ for all $\alpha > \beta$ and $\Gamma = \bigcup_{\alpha>\beta} \Gamma_{\alpha,\beta}$.

THEOREM 3.9. *Let S be a regular semigroup. Then $\kappa \cap \xi = \iota_S$ if and only if $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ is a semilattice of unitary semigroups satisfying the properties*:

(i) *Γ is a band of groups,*

(ii) *for s in $\Gamma \cap H_e$, $e \in E_\alpha$, if $f < e$ with $f \in E_\beta$ then $\phi^s_{\alpha,\beta}(f)\,\mathscr{H} f$, where $\phi^s_{\alpha,\beta}(f)$ means both $\phi^s_{\alpha,\beta} f$ and $f \phi^s_{\alpha,\beta}$.*

Proof. Let $\kappa \cap \xi = \iota_S$. Then $\mathscr{H} \cap \sigma = \mu \cap \sigma$ is unitary. Let $\alpha > \beta$ and s be in S_α with $\phi^s_{\alpha,\beta} \in K_\beta$. Then $\phi^s_{\alpha,\beta} = (\lambda, \rho)$ and $\lambda(E_\beta) \subseteq E_\beta$. Let g be in E_β. Then $\lambda g = f = f^2 \in E_\beta$, so by definition of multiplication in S, $sg = \phi^s_{\alpha,\beta} g = \lambda g = f$. Hence $se\mu \cap \sigma f$, and $\mu \cap \sigma$ is unitary, so $s\mu \cap \sigma s^2$. Thus s is contained in a group. Now since $\mu \cap \sigma$ is a congruence, the $\mu \cap \sigma$-classes which contain idempotents form a band of groups, T, and since $\phi_{\alpha,\beta}$ is a homomorphism, then $K_\beta \phi^{-1}_{\alpha,\beta} = \Gamma_{\alpha,\beta}$ is a band of groups contained in T. Thus Γ is a band of groups.

Now let s be in $\Gamma_{\alpha,\beta}$. Then s is in a group so there exists an idempotent h such that $ss' = s's = h$ for some $s' \in V(s)$. Since $s\mu s^2$, then $s\mu h$ and for all idempotents f, $sfs' = hfh$, $s'fs = hfh$, **[6]**.

Let $f < h$, $f \in S_\gamma$. Then $sfs' = f$ and $fs'sf = fhf = f$. Thus if $\phi^s_{\alpha,\gamma} = (\lambda^s, \rho^s)$ and $\phi^s_{\alpha,\gamma} = (\lambda^{s'}, \rho^{s'})$, $f = fs'sf = (f\rho^{s'})(\lambda^s f)$. Therefore,

$$f\rho^{s'} = f(f\rho^{s'}) = (f\rho^{s'})(\lambda^s f)(f\rho^{s'}), \qquad \lambda^s f = (\lambda^s f)f = (\lambda^s f)(f\rho^{s'})(\lambda^s f);$$

that is, $f\rho^{s'}$ is an inverse of $\lambda^s f$. Now $sfs' = f = fs'sf$ can be expressed by

$$(\lambda^s f)(f\rho^{s'}) = f = (f\rho^{s'})(\lambda^s f).$$

Thus $\lambda^s f \mathscr{H} f$. By considering $s'fs = fss'f = f$ we find $f\rho^s \mathscr{H} f$. Thus $\phi^s_{\alpha,\gamma}(f)\mathscr{H} f$.

Conversely, to show $\kappa \cap \xi = \iota_S$, using Theorem 3.5, we need only show that $\mu \cap \sigma$ is a unitary congruence. Let $se\mu \cap \sigma f$ with $e, f \in E_S$.

Letting s be in S_α, e in E_β, then $f \in E_{\alpha\beta}$. Since $se\sigma f$, there exists $g \in E_\gamma$, $\gamma \leqq \alpha\beta$ such that $g(se)g = gfg \in E_\gamma$. This means that $(g\phi^s_{\alpha,\gamma})(\phi^e_{\beta,\gamma}g) \in E_\gamma$. Since e is idempotent so is $\phi^e_{\beta,\gamma}$ and by Lemma 3.8, $\phi^e_{\beta,\gamma}g$ is in E_γ. Thus since S_γ is unitary, $g\phi^s_{\alpha,\gamma}$ is idempotent by Proposition 1.1; by Lemma 3.7, $\phi^s_{\alpha,\gamma}$ is in K_γ. Consequently by property (i), s is contained in a band E_α of groups G_b, $b \in E_\alpha$. In particular, there exists $s' \in V(s)$ such that $ss' = s's = h$ for some $h \in E_\alpha$. We need to show that $s\mu h$, and to do this it is sufficient to show that $sfs' = f$, $s'fs = f$ for all $f \leqq h$. Now if f is in E_α, $f \leqq h$, then s and s' are in the group G_h and

$$sfs' \in G_hG_fG_h \subseteq G_{hfh} = G_f,$$

so that $sfs' = f$. Similarly $s'fs = f$. Now let f be in E_δ, $\delta < \alpha$, with $f < h$. Let $\phi^s_{\alpha,\delta} = (\lambda^s, \rho^s)$, $\phi^{s'}_{\alpha,\delta} = (\lambda^{s'}, \rho^{s'})$. By property (ii), $\lambda^s f \mathscr{H} f$, $f\rho^s \mathscr{H} f$. That is, $f(\lambda^s f) = \lambda^s f$, $(f\rho^s)f = f\rho^s$. Now since $f < h = ss'$, then $f = hf = \lambda^{ss'}f = \lambda^s\lambda^{s'}f$, and thus

$$\begin{aligned} sfs' &= (\lambda^s f)(f\rho^{s'}) = f(\lambda^s f)(f\rho^{s'}) = (f\rho^s)f(f\rho^{s'}) = (f\rho^s)(f\rho^{s'}) \\ &= [(f\rho^s)f]\rho^{s'} = (f\rho^s)\rho^{s'} = f\rho^{ss'} = f. \end{aligned}$$

Similarly $s'fs = f$. Therefore $s\mu h$.

Since σ is always unitary, $se\sigma f$ implies $s\sigma h$. Consequently, $s\mu \cap \sigma h$, and $\mu \cap \sigma$ is a unitary congruence. By Theorem 3.5, $\kappa \cap \xi = \iota_S$.

A regular semigroup $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ is a strong semilattice of the semigroups S_α, if $\phi_{\alpha,\beta}$ maps S_α into S_β for all $\alpha > \beta$. The conditions in Theorem 3.9 can be simplified considerably for strong semilattices of unitary semigroups.

COROLLARY 3.10. *Let $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of unitary semigroups S_α. Then $\kappa \cap \xi = \iota_S$ if and only if $E_\beta\phi^{-1}_{\alpha,\beta}$ is a band of groups for all $\alpha > \beta$.*

Proof. It can be easily seen that $K_\beta \cap \Pi(S_\beta) \simeq E_\beta$ where $\Pi(S_\beta)$ is the semigroup of inner bitranslations of S_β. Thus if $\phi_{\alpha,\beta}$ maps S_α into $S_\beta \simeq \Pi(S_\beta)$, then $K_\beta\phi^{-1}_{\alpha,\beta} = E_\beta\phi^{-1}_{\alpha,\beta}$. Property (ii) automatically holds since homomorphisms preserve $\mathscr{H}$-classes.

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