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**RATIONAL APPROXIMATION OF  $e^{-x}$  ON THE POSITIVE  
REAL AXIS**

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## RATIONAL APPROXIMATION OF $e^{-x}$ ON THE POSITIVE REAL AXIS

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**In this paper we obtain error bounds to approximations of  $e^{-x}$  on  $[0; \infty)$  by rational functions having zeros and poles only on the negative real axis.**

Our main concern in this paper is the question of approximating  $e^{-x}$  on the positive real axis by reciprocals of polynomials and by rational functions, especially by those which have all their zeros and poles on the negative real axis.

NOTATION. Let  $\pi_n$  represent the set of all polynomials of degree  $\leq n$ . Let  $\pi_n^*$  represent the set of all polynomials in  $\pi_n$  all of whose zeros are in the left half plane and  $\pi_n^{**}$  represent the set of all polynomials in  $\pi_n^*$  all of whose zeros are real and negative. Similarly let  $\rho_n, \rho_n^*, \rho_n^{**}$  represent the sets of rational functions of total degree  $n$  whose numerators and denominators are in  $\pi_n, \pi_n^*, \pi_n^{**}$  respectively. Let  $\| \cdot \|$  denote  $\| \cdot \|_{L^\infty[0, \infty)}$ . Then we define

$$\begin{aligned} \lambda_{0,n}(f) &= \inf_{p \in \pi_n} \left\| f - \frac{1}{p} \right\|, \\ \lambda_{0,n}^*(f) &= \inf_{p \in \pi_n^*} \left\| f - \frac{1}{p} \right\|, \\ \lambda_{0,n}^{**}(f) &= \inf_{p \in \pi_n^{**}} \left\| f - \frac{1}{p} \right\|, \\ \lambda_n(f) &= \inf_{r \in \rho_n} \|f - r\|, \\ \lambda_n^*(f) &= \inf_{r \in \rho_n^*} \|f - r\|, \\ \lambda_n^{**}(f) &= \inf_{r \in \rho_n^{**}} \|f - r\|. \end{aligned}$$

LEMMA (Newman [1], Theorem 2). *Let  $p \in \pi_n^{**}$  where  $n \geq 2$ , then*

$$\|e^x - p\|_{L^\infty[0,1]} \geq (16n + 1)^{-1}.$$

We obtain the following results.

(Theorems 1, 2):  $(17e^2n)^{-1} \leq \lambda_{0,n}^{**}(e^{-x}) \leq (en)^{-1}$ ,  $n \geq 2$ .

- (Theorem 3):  $\lambda_{0,2n}^*(e^{-x}) \leq 2(ne)^{-2}$ ,  $n \geq 1$ .
- (Theorems 4, 5):  $e^{-6\sqrt{n}} \leq \lambda_n^{**}(e^{-x}) \leq n^{-c \log n}$ ,  $n \geq 2$ .
- (Theorem 6):  $e^{-5n^{2/3}} \leq \lambda_n^*(e^{-x})$ ,  $n \geq 2$ .

THEOREM 1. For all  $n \geq 1$ ,

$$(1) \quad \left\| e^{-x} - \left(1 + \frac{x}{n}\right)^{-n} \right\| \leq \frac{1}{ne}.$$

*Proof.* For all  $x \geq 0$  and  $n \geq 1$  we have

$$0 \leq \left(1 + \frac{x}{n}\right)^n \leq e^x.$$

Hence

$$0 \leq \left(1 + \frac{x}{n}\right)^{-n} - e^{-x} \leq \left(1 + \frac{x}{n}\right)^{-n} - \left(1 + \frac{x}{n}\right)^{-n-1} \leq \frac{1}{ne} \quad \text{for all } x \geq 0,$$

because,  $(1 + (x/n))^{-n} - e^{-x}$  attains its maximum when  $e^x = (1 + (x/n))^{n+1}$ . Hence (1) follows.

THEOREM 2. For all  $n \geq 2$  we have

$$(2) \quad \lambda_{0,n}^{**}(e^{-x}) \geq (17e^2n)^{-1}.$$

*Proof.* Set

$$(3) \quad \left\| e^{-x} - \frac{1}{p_n(x)} \right\| = \epsilon.$$

Then

$$\| e^x - p_n(x) \|_{L^\infty[0,1]} \leq \epsilon e p_n(1),$$

since  $p_n(x)$  has only nonnegative coefficients. From (3), we get

$$(4) \quad [p_n(1)]^{-1} \geq e^{-1} - \epsilon = \frac{1 - \epsilon e}{e}.$$

From (3) and (4), we have

$$(5) \quad \| e^x - p_n(x) \|_{L^\infty[0,1]} \leq \frac{\epsilon e^2}{1 - \epsilon e}.$$

On the other hand we have from the lemma that

$$(6) \quad \|e^x - p_n(x)\|_{L_\infty[0,1]} \geq (16n + 1)^{-1}.$$

From (5) and (6), we get

$$\epsilon e^2(16n + 1) \geq 1 - \epsilon e.$$

Hence (2) follows.

**THEOREM 3.** *For all even  $n$*

$$\left\| e^{-x} - \left( 1 + \frac{2x}{n} + \frac{2x^2}{n^2} \right)^{-n/2} \right\| \leq 8(ne)^{-2}.$$

*Proof.* For all  $x \geq 0, n \geq 1$ , we have

$$\exp\left(\frac{2x}{n}\right) \geq \left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right).$$

We also know that

$1 + x + x^2/2!$  has zeros only in the left half plane.

The function

$$\left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right)^{-n/2} - e^{-x}$$

attains its maximum when

$$e^x = \left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right)^{n/2+1} \left(1 + \frac{2x}{n}\right)^{-1}.$$

Therefore

$$0 \leq \left(1 + \frac{2x}{n} + \frac{2x^2}{n^2}\right)^{-n/2} - e^{-x} \leq \frac{2x^2}{n^2 e^x} \leq \frac{8}{n^2 e^2}.$$

Hence the theorem is proved.

**THEOREM 4.** *There is a constant  $c > 0$  so that for all  $n \geq 2$ , we have*

$$(7) \quad \lambda_n^{**}(e^{-x}) \leq n^{-c \log n}.$$

*Proof.* We use the following formula.

$$(8) \quad \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{1}{k+s} = \frac{m!}{s(s+1)(s+2)\cdots(s+m)}.$$

Set  $N = \text{l.c.m}[1, 2, \dots, m]$ ,  $t = N/s \geq 0$  and  $\epsilon = (m!)^2 N^{-m}$ . Then using the fact that  $t^m \leq m! e^t$ , we get

$$(9) \quad 1 + \sum_{k=1}^m (-1)^k \frac{N}{k} \binom{m}{k} \frac{1}{t + \frac{N}{k}} = \frac{m! t^m}{(N+t)(N+2t)\cdots(N+mt)} \\ \leq \frac{m! t^m}{N^m + m! t^m} \\ \leq \frac{(m!)^2 e^t}{N^m + (m!)^2 e^t} \leq \frac{\epsilon e^t}{1 + \epsilon e^t}.$$

By integrating (9) with respect to  $t$  from 0 to  $x$  we get

$$(10) \quad 0 \leq x + \log R(x) \leq \log(1 + \epsilon e^x),$$

where  $R(x) = \prod_{k=1}^m (1 + (xk/N))^{(-1)^k (N/k)^{\binom{m}{k}}}$ . From (10), we get

$$0 \leq e^x R(x) - 1 \leq \epsilon e^x.$$

That is

$$0 \leq R(x) - e^{-x} \leq \epsilon.$$

From prime number theory we know that there exist positive constants  $\alpha, \beta$  so that  $e^{\alpha m} < N < e^{\beta m}$  for all  $m \geq 1$ . Hence  $\deg R(x) \leq N 2^m \leq n$  if we choose  $\gamma \log n < m < \delta \log n$ , where  $\gamma, \delta$  are positive constants. From this choice of  $m$ , we obtain (7). That is,

$$\epsilon \leq n^{-c \log n} \quad \text{as required.}$$

**THEOREM 5.** *For all  $n \geq 2$  we have*

$$(11) \quad \lambda_n^{**}(e^{-x}) \geq e^{-6\sqrt{n}}.$$

*Proof.* In (8) set  $s = m(1+t)$  and integrate, then we get

$$(12) \quad \sum_{k=0}^m (-1)^k \binom{m}{k} \log \left( 1 + \frac{mUA}{m+k} \right) \\ = \frac{1}{\binom{2m}{m}} \int_0^{UA} \frac{dt}{(1+t) \left( 1 + \frac{tm}{m+1} \right) \cdots \left( 1 + \frac{tm}{2m} \right)},$$

and observe that for  $U, A > 0$  the right side of (12) is bounded by

$$(13) \quad \frac{1}{\binom{2m}{m}} \int_0^\infty \frac{dt}{\left(1 + \frac{t}{2}\right)^{m+1}} = \frac{2}{m \binom{2m}{m}}.$$

Again (8) with  $s = m$  give us

$$(14) \quad \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{mA}{m+k} = \frac{A}{\binom{2m}{m}}.$$

Assume there is a rational function of total degree  $n$ , set

$$r(x) = e^{-c} \prod_{k=1}^n (1 + xu_k)^{\epsilon_k}, \quad \epsilon_k = \pm 1, \quad u_k \geq 0,$$

such that

$$\|e^{-x} - r(x)\| = \epsilon,$$

thus

$$(15) \quad \|e^x r(x) - 1\|_{L^\infty[0, A]} \leq \epsilon e^A.$$

From (15), we obtain

$$c - \sum_{i=1}^n \epsilon_i \log(1 + xu_i) + x \leq \log(1 + \epsilon e^A) < \epsilon e^A, \quad \text{for } 0 \leq x \leq A.$$

Now set  $x = mA/(m+k)$  to get

$$(16) \quad c - \sum_{i=1}^n \epsilon_i \log\left(1 + \frac{mAu_i}{m+k}\right) + \frac{mA}{m+k} < \epsilon e^A, \quad k = 0, 1, 2, \dots, m.$$

Applying the difference operator  $m$  times on both sides of (16). We get in view of (13) and (14),

$$(17) \quad -\frac{2n}{m} + A \leq \binom{2m}{m} 2^m \epsilon e^A.$$

Now choose  $m = \lceil \sqrt{n} \rceil$ ,  $A = 3\sqrt{n}$  then

$$\epsilon \geq \sqrt{n}(2e)^{-3\sqrt{n}} > e^{-6\sqrt{n}}, \quad \text{as required.}$$

**THEOREM 6.** *For all  $n \geq 2$ , we have*

$$\lambda_n^*(e^{-x}) \geq e^{-5n^{2/3}}.$$

*Proof.* The proof of this theorem is not very different from the proof of Theorem 5, except that we use  $t = ve^{i\theta}$ ,  $|\theta| \leq \pi/2$ , and obtain

$$\begin{aligned} (18) \quad & \sum_{k=0}^m (-1)^k \binom{m}{k} \log \left( 1 + \frac{muA}{m+k} \right) \\ &= \frac{1}{\binom{2m}{m}} \int_0^{UA} \frac{dt}{(1+t) \left(1 + \frac{tm}{m+1}\right) \cdots \left(1 + \frac{tm}{2m}\right)} \\ &\leq \frac{1}{\binom{2m}{m}} \int_0^\infty \frac{dv}{\left(1 + \frac{v^2}{2}\right)^{m/2}} \leq \frac{1}{\binom{2m}{m} \sqrt{m}}. \end{aligned}$$

Now by using (18) instead of (12) and (13), we obtain as in the case of Theorem 5,

$$-\frac{n}{\sqrt{m}} + A \leq 2^m e^A \binom{2m}{m} \epsilon.$$

Choose  $m = [n^{2/3}]$ ,  $A = 2n^{2/3}$  then we get

$$\epsilon \geq n^{2/3} 8^{-n^{2/3}} e^{-2n^{2/3}} > e^{-5n^{2/3}} \quad \text{as required.}$$

#### REFERENCE

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Walter Allegretto, <i>Nonoscillation theory of elliptic equations of order <math>2n</math></i> . . . . .	1
Bruce Allen Anderson, <i>Sequencings and starters</i> . . . . .	17
Friedrich-Wilhelm Bauer, <i>A shape theory with singular homology</i> . . . . .	25
John Kelly Beem, <i>Characterizing Finsler spaces which are pseudo-Riemannian of constant curvature</i> . . . . .	67
Dennis K. Burke and Ernest A. Michael, <i>On certain point-countable covers</i> . . . . .	79
Robert Chen, <i>A generalization of a theorem of Chacon</i> . . . . .	93
Francis H. Clarke, <i>On the inverse function theorem</i> . . . . .	97
James Bryan Collier, <i>The dual of a space with the Radon-Nikodým property</i> . . . . .	103
John E. Cruthirds, <i>Infinite Galois theory for commutative rings</i> . . . . .	107
Artatrana Dash, <i>Joint essential spectra</i> . . . . .	119
Robert M. DeVos, <i>Subsequences and rearrangements of sequences in FK spaces</i> . . . . .	129
Geoffrey Fox and Pedro Morales, <i>Non-Hausdorff multifunction generalization of the Kelley-Morse Ascoli theorem</i> . . . . .	137
Richard Joseph Fleming, Jerome A. Goldstein and James E. Jamison, <i>One parameter groups of isometries on certain Banach spaces</i> . . . . .	145
Robert David Gulliver, II, <i>Finiteness of the ramified set for branched immersions of surfaces</i> . . . . .	153
Kenneth Hardy and István Juhász, <i>Normality and the weak cb property</i> . . . . .	167
C. A. Hayes, <i>Derivation of the integrals of <math>L^{(q)}</math>-functions</i> . . . . .	173
Frederic Timothy Howard, <i>Roots of the Euler polynomials</i> . . . . .	181
Robert Edward Jamison, II, Richard O'Brien and Peter Drummond Taylor, <i>On embedding a compact convex set into a locally convex topological vector space</i> . . . . .	193
Andrew Lelek, <i>An example of a simple triod with surjective span smaller than span</i> . . . . .	207
Janet E. Mills, <i>Certain congruences on orthodox semigroups</i> . . . . .	217
Donald J. Newman and A. R. Reddy, <i>Rational approximation of <math>e^{-x}</math> on the positive real axis</i> . . . . .	227
John Robert Quine, Jr., <i>Homotopies and intersection sequences</i> . . . . .	233
Nambury Sitarama Raju, <i>Periodic Jacobi-Perron algorithms and fundamental units</i> . . . . .	241
Herbert Silverman, <i>Convexity theorems for subclasses of univalent functions</i> . . . . .	253
Charles Frederick Wells, <i>Centralizers of transitive semigroup actions and endomorphisms of trees</i> . . . . .	265
Volker Wrobel, <i>Spectral approximation theorems in locally convex spaces</i> . . . . .	273
Hidenobu Yoshida, <i>On value distribution of functions meromorphic in the whole plane</i> . . . . .	283