Pacific Journal of Mathematics

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Vol. 64, No. 1 May 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 64, No. 1, 1976

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We determine the radius of convexity of functions $f(z)$ for which $\operatorname{Re}\{f'(z)/\phi'(z)\} > \beta$, where $\phi(z)$ is convex of order $\alpha (0 \leqq \alpha \leqq 1)$. We also find bounds for $|\arg f'(z)|$. All result are sharp.

1. Introduction. Let S be the class of normalized univalent functions analytic in the unit disk. Let $K(\alpha)$ denote the subclass of S consisting of functions $\phi(z)$ for which

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} \geqq \alpha \qquad (0 \leqq \alpha \leqq 1).$$

This class is called convex of order α. We say that an analytic function $f(z) = z + a_2 z^2 + \cdots$ is in the class $C(\alpha, \beta)$ if there exists a function $\phi(z) \in K(\alpha)$ such that

$$\operatorname{Re}\left\{\frac{f'(z)}{\phi'(z)}\right\} > \beta \qquad (0 \leqq \beta < 1, |z| < 1).$$

This class was defined by Libera **[5]**. Kaplan **[3]** showed that $C(0, 0)$, the class of close-to-convex functions, is univalent. Since $C(\alpha, \beta) \subset C(0, 0)$, we see that $C(\alpha, \beta)$ is a subclass of S.

Denote by P_β the functions $p(z)$ that are analytic in $|z| < 1$ and satisfy there the conditions

$$p(0) = 1 \quad \text{and} \quad \operatorname{Re} p(z) > \beta,$$

and set $P_0 = P$. It is well known that a function $q(z)$ is in P_β if and only if there exists a function $p(z) \in P$ such that

$$q(z) = (1-\beta)p(z) + \beta = \frac{p(z)+h}{1+h}, \quad \text{where}$$

$$h = \frac{\beta}{1-\beta}. \tag{1}$$

Thus if $f(z) \in K(\alpha, \beta)$, then we may write

$$f'(z) = \phi'(z)\frac{p(z)+h}{1+h}, \tag{2}$$

where $\phi(z) \in K(\alpha)$, $p(z) \in P$, and h is defined by (1). Taking logarithmic derivatives in (2), we find that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{z\phi''(z)}{\phi'(z)} + \frac{zp'(z)}{p(z)+h}. \tag{3}$$

It is our purpose in this paper to determine the radius of convexity for the class $C(\alpha, \beta)$. Note, for $|z| = r$, that (3) yields

$$\min_{f \in C(\alpha,\beta)} \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \min_{\phi \in K(\alpha)} \text{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} + \min_{p \in P} \text{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\}. \tag{4}$$

In [5], Libera found a disk $|z| < r$ in which $f(z) \in C(\alpha, \beta)$ is convex. His method essentially consisted of utilizing the inequaltiy

$$\min_{f \in C(\alpha,\beta)} \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqq \min_{\phi \in K(\alpha)} \text{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} - \max_{p \in P} \left|\frac{zp'(z)}{p(z)+h}\right|.$$

His result, however, was not sharp because for $|z| = r$,

$$\min_{p \in P} \text{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\} \geqq -\max_{p \in P} \left|\frac{zp'(z)}{p(z)+h}\right|,$$

with equality *only* when $h = 0$. The function that he claimed to be extremal need not be in $C(\alpha, \beta)$. See [9]. It is known [1] that

$$\min_{\substack{|z|=r \\ \phi \in K(\alpha)}} \text{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} = \frac{1-(1-2\alpha)r}{1+r},$$

with equality for functions of the form

$$\phi(z) = \begin{cases} \dfrac{1}{(1-2\alpha)\epsilon}\left[\dfrac{1}{(1-\epsilon z)^{1-2\alpha}} - 1\right] & (\alpha \neq \tfrac{1}{2}, |\epsilon| = 1) \\ -\bar{\epsilon}\log(1-\epsilon z) & (\alpha = \tfrac{1}{2}, |\epsilon| = 1). \end{cases}$$

Thus, taking into account (4), the radius of convexity of $C(\alpha, \beta)$ is seen to be the smallest positive r for which

$$\frac{1-(1-2\alpha)r}{1+r} + \min_{\substack{|z|=r \\ p \in P}} \text{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\} = 0. \tag{5}$$

In §2 we will use a theorem of V. A. Zmorovič to find $\min_{|z|=r, p\in P} \operatorname{Re}\{zp'(z)/(p(z)+h)\}$. In §3 we will determine the radius of convexity of $C(\alpha, \beta)$ and examine some of its consequences. Finally, in §4 we will find a sharp bound on $|\arg f'(z)|$ for $f(z) \in C(\alpha, \beta)$.

2. Consequences of Zmorovič's theorem.

The following theorem is due to V. A. Zmorovič **[11]**.

THEOREM A. *Let* $\Psi(w, W) = M(w) + N(w)W$, *where* $M(w)$ *and* $N(w)$ *are defined and are finite in the half plane* $\operatorname{Re}\{w\} > 0$. *Set*

$$w = \lambda_1 \frac{1+z_1^m}{1-z_1^m} + \lambda_2 \frac{1+z_2^m}{1-z_2^m},$$

$$W = \lambda_1 \frac{2mz_1^m}{(1-z_1^m)^2} + \lambda_2 \frac{2mz_2^m}{(1-z_2^m)^2},$$

where z_1 *and* z_2 *are any points on the circle* $|z| = r < 1$, m *is a positive integer,* $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, *and* $\lambda_1 + \lambda_2 = 1$. *Then the function* $\Psi(w, W)$ *can be put in the form*

$$\Psi(w, W) = M(w) + \frac{m}{2}(w^2 - 1)N(w) + \frac{m}{2}(\rho^2 - \rho_0^2)N(w)e^{2i\psi},$$

where

$$\frac{1+z_k^m}{1-z_k^m} = a + \rho e^{i\psi_k} \qquad (k = 1, 2),$$

$$w = a + \rho_0 e^{i\psi_0} \qquad (0 \leqq \rho_0 \leqq \rho),$$

$$a = \frac{1+r^{2m}}{1-r^{2m}}, \qquad \rho = \frac{2r^m}{1-r^{2m}}, \qquad e^{i\psi} = ie^{i(\psi_1+\psi_2)/2}.$$

Also,

$$\min \operatorname{Re}\{\Psi(w, W)\} \equiv \Psi_\rho(w) \tag{6}$$

$$= \operatorname{Re}\left\{M(w) + \frac{m}{2}(w^2 - 1)N(w)\right\} - \frac{m}{2}|N(w)|(\rho^2 - \rho_0^2).$$

This minimum is reached when

$$\exp[i(2\psi + \arg N(w))] = -1.$$

The importance of this formidable theorem lies in the fact that the minimum of $\operatorname{Re}\Psi(w, W)$ in the disk $|w - a| \leqq \rho$ depends only on the two variables $\operatorname{Re} w$ and $\operatorname{Im} w$, as can be seen by (6), and not on W, λ_1, or λ_2.

I would like to thank the referee for pointing out that the following theorem may be found in **[12]**. For completeness we include a more detailed proof of this useful result.

THEOREM 1. *Suppose $p(z)\in P$, h is defined by* (1), *and a is defined as in Theorem A. Then*

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\} \geqq \begin{cases} -\dfrac{2r}{(1+r)[(1+h-(1-h)r]} & (0\leqq r\leqq r_\beta) \\ 2\sqrt{h^2+ah}-a-2h & (r_\beta<r<1), \end{cases}$$

where r_β is the unique root of the equation $(1-2\beta)r^3-3(1-2\beta)r^2+3r-1=0$ *in the interval* $(0,1]$. *This result is sharp.*

Proof. Set $M(w)=0$, $N(w)=1/(w+h)$, $m=1$, and $w=p(z)=p$ in Theorem A, and note that $W=zp'(z)$. Thus $\Psi(w,W)=\Psi(p,zp')=zp'(z)/(p(z)+h)$ and, in view of (6),

$$\text{(7)}\qquad \operatorname{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\} \geqq \Psi_\rho(p)=\frac{1}{2}\operatorname{Re}\left[\frac{p^2-1}{p+h}-\frac{\rho^2-\rho_0^2}{|p+h|}\right].$$

Since $|p-a|=\rho_0\leqq\rho$, we may set $p=a+\xi+i\eta$, $\rho_0^2=\xi^2+\eta^2$, and $R=|p+h|$. Then

$$\begin{aligned}\text{(8)}\quad \operatorname{Re}\frac{p^2-1}{p+h} &= \frac{|p|^2(a+\xi)-(a+\xi+h)+h[(a+\xi)^2-\eta^2]}{R^2}\\ &= \frac{(a+\xi+h)[R^2-(h^2+2h(a+\xi)+1)]-2h\eta^2}{R^2}\\ &= \frac{(a+\xi+h)[R^2-2h(a+\xi+h)+(h^2-1)]-2h\eta^2}{R^2}\\ &= (a+\xi+h)-2h+\frac{(h^2-1)(a+\xi+h)}{R^2}.\end{aligned}$$

A substitution of (8) into (7) gives

$$\text{(9)}\qquad \Psi_\rho(p)=\frac{a+\xi+h}{2}-h+\frac{(h^2-1)(a+\xi+h)}{2R^2}-\frac{\rho^2-\xi^2-\eta^2}{2R}.$$

We now wish to minimize $\Psi_\rho(p)$ as a function of η. A differentiation shows that

$$\text{(10)}\qquad \frac{\partial\psi_\rho}{\partial\eta}=\frac{\eta}{2}\frac{S(\xi,\eta)}{R^4},$$

where

$$S(\xi,\eta)=[\xi^2+4(a+h)\xi+\rho^2+\eta^2+2(a+h)^2]R-2(h^2-1)(\xi+a+h)$$

$$\geqq[\xi^2+4(a+h)\xi+\rho^2+2(a+h)^2-2(h^2-1)](\xi+a+h). \tag{11}$$

But the last expression in (11) is an increasing function of ξ in the interval $[-\rho,\rho]$. Hence

$$S(\xi,\eta)\geqq S(-\rho,\eta)=2[(a-\rho)^2+2h(a-\rho)+1](a+h-\rho)>0.$$

We thus see from (10) that $\Psi_\rho(\xi,\eta)$ is minimized on every chord $\xi=\text{constant}$ of the circle $\xi^2+\eta^2=\rho_0^2$ at the point $\eta=0$. Therefore the minimum of $\Psi_\rho(\xi,\eta)$ in the disk $\xi^2+\eta^2\leqq\rho^2$ occurs somewhere on the diameter $\eta=0$. Setting $\eta=0$ in (9) and noting that $R=a+\xi+h$, we have

$$\Psi_\rho(p)\geqq\Psi_\rho(\xi,0)=l(R)=\frac{R}{2}-h+\frac{h^2+\xi^2-\rho^2-1}{2R}. \tag{12}$$

Using the identities $\xi=R-(a+h)$ and $\rho^2=a^2-1$ in (12), we get

$$l(R)=R+\frac{h^2+ah}{R}-(a+2h). \tag{13}$$

We must now determine the minimum of $l(R)$ for R in the interval $[a+h-\rho,\ a+h+\rho]$. A differentiation of (13) shows that $l(R)$ assumes its minimum at

$$R_0=\sqrt{h^2+ah} \tag{14}$$

as long as

$$a+h-\rho\leqq R_0\leqq a+h+\rho. \tag{15}$$

The right hand inequality in (15) is always true, but the left hand inequality will not hold when h (and consequently β) is small. In the latter case, $l(R)$ assumes its minimum at the point

$$R_1=a+h-\rho. \tag{16}$$

Substituting (14) and (16), respectively, into (13), we find

$$l(R_0)=2\sqrt{h^2+ah}-(a+2h) \tag{17}$$

$$l(R_1)=\frac{\rho^2-a\rho}{a+h-\rho}=-\frac{2r}{(1+r)[1+h-(1-h)r]}. \tag{18}$$

As β increases, the transition from $l(R_1)$ to $l(R_0)$ occurs at the point where $R_0=R_1$. But $R_0=R_1$ when $h^2+ah=(a-\rho+h)^2$, or in terms of r, when the polynomial equation

$$t(r)=(1-2\beta)r^3-3(1-2\beta)r^2+3r-1$$

has a root in the interval $(0,1]$. Note that

$$t'(r)=3[(1-2\beta)r^2-2(1-2\beta)r+1]>0 \qquad (0<r<1)$$

so that $t(r)$ is increasing. Further, $t(0)=-1$ and $t(1)=4\beta$ so that $t(r)$ has a unique root in the interval $(0,1]$. This completes the proof.

Equality holds in (18) for $p(z)=(1+z)/(1-z)$, and in (17) for

$$p(z)=\frac{1}{2}\left[\frac{1+ze^{-i\theta_0}}{1-ze^{-i\theta_0}}+\frac{1+ze^{i\theta_0}}{1-ze^{i\theta_0}}\right]=\frac{1-z^2}{1-2z\cos\theta_0+z^2},$$

where $\cos\theta_0$ is defined by the equation

$$h+(1-r_0^2)(1-2r_0\cos\theta_0+r_0^2)^{-1}=R_0 \qquad (r_0=l(R_0)). \tag{19}$$

3. Radius of convexity theorems. We may now use Theorem 1 to prove

THEOREM 2. *Suppose r_β is the unique root of*

$$t(r)=(1-2\beta)r^3-3(1-2\beta)r^2+3r-1$$

in the interval $(0,1]$. *Set*

$$r(\alpha,\beta)=\frac{1}{2-\alpha-2\beta+\sqrt{\alpha^2-2\alpha+4\beta^2-6\beta+3}}.$$

Then the radius of convexity of $C(\alpha,\beta)$ is $r(\alpha,\beta)$ when $0<r(\alpha,\beta)\leqq r_\beta$, and is otherwise the smallest root greater than r_β of the polynomial equation

$$v(r)=[\alpha^2-\beta(\alpha^2+2\alpha-1)]r^4-2(1-\alpha)(\beta+\alpha\beta-\alpha)r^3$$
$$+[(1-\alpha)^2(1-\beta)+2\alpha\beta]r^2+2\beta(1-\alpha)r-\beta.$$

This result is sharp for all α and β.

Proof. An application of Theorem 1 to (5) shows that the radius of convexity of $C(\alpha, \beta)$ is the smallest positive root of

$$(20)\qquad \begin{cases} \dfrac{1-(1-2\alpha)r}{1+r} - \dfrac{2r}{(1+r)[(1+h)-(1-h)r]} = 0 & (0 \leqq r \leqq r_\beta) \\ \dfrac{1-(1-2\alpha)r}{1+r} + 2\sqrt{h^2+ah} - a - 2h = 0 & (r_\beta < r < 1), \end{cases}$$

where a is defined in Theorem A and h is defined by (1). The first expression in (20) may be written as

$$\frac{(1-2\alpha)(1-2\beta)r^2 - 2(2-\alpha-2\beta)r + 1}{(1+r)[(1+h)-(1-h)r]} = 0,$$

whose roots are

$$\frac{(2-\alpha-2\beta) \mp \sqrt{(2-\alpha-2\beta)^2 - (1-2\alpha)(1-2\beta)}}{(1-2\alpha)(1-2\beta)} = \frac{1}{(2-\alpha-2\beta) \pm \sqrt{\alpha^2 - 2\alpha + 4\beta^2 - 6\beta + 3}}.$$

If both roots are positive, the minimum root is $r(\alpha, \beta)$. Similarly, a computation shows that r^* is a root of the second expression in (20) if and only if it is a root of $v(r)$. This completes the proof.

The extremal function is of the form

$$f(z) = \int_0^z \frac{1+(1-2\beta)t}{(1-t)^{3-2\alpha}}\, dt$$

when $0 < r(\alpha, \beta) \leqq r_\beta$, and is otherwise of the form

$$f(z) = \int_0^z \frac{1-2\beta\cos\theta_0 + (2\beta-1)t^2}{(1-2t\cos\theta_0 + t^2)(1-t)^{2(1-\alpha)}}\, dt,$$

where $\cos\theta_0$ is defined by (19).

COROLLARY. *If $0 \leqq \beta \leqq \frac{1}{10}$, then the radius of convexity of $C(\alpha, \beta)$ is $r(\alpha, \beta)$ for all α.*

Proof. We must show that $0 < r(\alpha, \beta) \leqq r_\beta$ for $0 \leqq \alpha \leqq 1$ and $0 \leqq \beta \leqq 1/10$. Note that $\partial t(r)/\partial\beta = 2r^2(3-r)$, so that $t(r)$ is an increasing function of β. This means that r_β is a decreasing function of β. Set $A = \sqrt{\alpha^2 - 2\alpha + 4\beta^2 - 6\beta + 3}$. Then

$$\frac{\partial}{\partial\alpha}\, r(\alpha,\beta) = \frac{A+1-\alpha}{S^3} \geqq 0 \qquad (0 \leqq \alpha \leqq 1)$$

and

$$\frac{\partial}{\partial\beta}\, r(\alpha,\beta) = \frac{A+3-4\beta}{A^3} \geqq 0 \qquad (0 \leqq \beta \leqq \tfrac{3}{4}).$$

Thus $r(\alpha,\beta) \leqq r(1,\frac{1}{10})$ for $0 \leqq \alpha \leqq 1$ and $0 \leqq \beta \leqq \frac{1}{10}$. The result follows upon observing that

$$r\left(1, \frac{1}{10}\right) = \frac{1}{2} \quad \text{and} \quad t\left(\frac{1}{2}\right) = \frac{8}{10}\left(\frac{1}{8}\right) - \frac{24}{10}\left(\frac{1}{4}\right) + \frac{3}{2} - 1 = 0.$$

REMARK. When $\beta = 0$, we see that

$$r(\alpha, 0) = \frac{1}{2-\alpha+\sqrt{\alpha^2-2\alpha+3}}\,.$$

In this case, Libera's result **[5]** is sharp.

We turn now to a distinguished subclass of $C(\alpha,\beta)$,and state the result as a separate theorem.

THEOREM 3. *If $f(z) \in S$ with* $\operatorname{Re} f'(z) > \beta$, *then $f(z)$ is convex in a disk of radius*

$$\begin{cases} \dfrac{1}{1-2\beta+\sqrt{4\beta^2-6\beta+2}} & (0 \leqq \beta \leqq \frac{1}{10}) \\[2ex] \left(1+\sqrt{\dfrac{1-\beta}{\beta}}\right)^{-\frac{1}{2}} & (\frac{1}{10} < \beta < 1). \end{cases}$$

This result is sharp.

Proof. Since $\phi(z) = z$ is the only function in $K(1)$, the class under consideration is $C(1,\beta)$ so that Theorem 2 may be applied. As we saw in the corollary to Theorem 2,

$$r(1,\beta) = \frac{1}{1-2\beta+\sqrt{4\beta^2-6\beta+2}} \leqq r_\beta \qquad (0 \leqq \beta \leqq \tfrac{1}{10}),$$

which gives the first part of the theorem.

Since $t(r_\beta) = 0$ when $\alpha = 1$ and $\beta = \frac{1}{10}$, the radius of convexity of $C(1,\beta)$ for $\beta > \frac{1}{10}$ is the only positive root of

$$(1-2\beta)r^4+2\beta r^2-\beta=0, \quad \text{or}$$

$$r^2=\frac{-\beta+\sqrt{\beta-\beta^2}}{1-2\beta}=\frac{1}{1+\sqrt{\dfrac{1-\beta}{\beta}}}.$$

This completes the proof.

REMARK. The cases $\beta=0$ and $\beta=\frac{1}{2}$ were proved, respectively, by MacGregor **[6]** and Hallenbeck **[2]**.

4. An argument theorem.

THEOREM 4. *If* $f(z)\in C(\alpha,\beta)$, *then*

$$|\arg f'(z)|\leqq 2(1-\alpha)\sin^{-1}r+\sin^{-1}\left[\frac{2(1-\beta)r}{1+(1-2\beta)r^2}\right].$$

This result is sharp.

Proof. We may write

$$f'(z)=\phi'(z)q(z), \quad \text{where} \quad \phi(z)\in K(\alpha) \quad \text{and} \quad q(z)\in P_\beta.$$

Hence

$$|\arg f'(z)|\leqq|\arg\phi'(z)|+|\arg q(z)|. \tag{21}$$

But by a result of Pinchuk **[8]**,

$$|\arg\phi'(z)|\leqq 2(1-\alpha)\sin^{-1}r \qquad (|z|\leqq r). \tag{22}$$

Since $\operatorname{Re} q(z)>\beta$, the function

$$\omega(z)=\frac{(q(z)-\beta)-(1-\beta)}{(q(z)-\beta)+(1-\beta)}=\frac{q(z)-1}{q(z)-(2\beta-1)}$$

is analytic with $\omega(0)=0$ and $|\omega(z)|<1$ in $|z|<1$.
Thus by Schwarz's lemma,

$$\left|\frac{q(z)-1}{q(z)-(2\beta-1)}\right|<|z| \quad \text{for} \quad |z|<1.$$

Hence the values of $q(z)$ are contained in the circle of Apollonius whose diameter is the line segment from $(1+(2\beta-1)r)/(1+r)$ to

$(1-(2\beta-1)r)/(1-r)$. The circle is centered at the point $(1+(1-2\beta)r^2)/(1-r^2)$ and has radius $(2(1-\beta)r)/(1-r^2)$. Thus $|\arg q(z)|$ attains its maximum at points where a ray from the origin is tangent to the circle, that is, when

$$\arg q(z) = \pm \sin^{-1} \frac{2(1-\beta)r}{1+(1-2\beta)r^2}. \tag{23}$$

Substituting (22) and (23) into (21), the result follows.

Equality holds for functions of the form

$$f(z) = \int_0^z \frac{1+(1-2\beta)\eta t}{(1-\epsilon t)^{2(1-\alpha)}(1-\eta t)}\, dt$$

with suitably chosen ϵ, η, where $|\epsilon| = |\eta| = 1$.

REMARK. For $\alpha = \beta = 0$, this reduces to

$$|\arg f'(z)| \leqq 2\sin^{-1} r + \sin^{-1} \frac{2r}{1+r^2} = 2(\sin^{-1} r + \tan^{-1} r),$$

a result of Krzyz **[4]**.

THEOREM 5. *Suppose $f(z)$, $g(z) \in C(\alpha, \beta)$. Then*

$$\lambda f(z) + (1-\lambda)g(z) \qquad (0 \leqq \lambda \leqq 1)$$

is univalent in a disk $|z| < r$, where r is the smallest positive root of the equation

$$2(1-\alpha)\sin^{-1} r + \sin^{-1}\left(\frac{2(1-\beta)r}{1+(1-2\beta)r^2}\right) = \frac{\pi}{2}.$$

This result is sharp.

Proof. In **[7]**, MacGregor showed that the exact radius of univalence of convex linear combinations of a rotation and conjugation invariant subclass of S is given by the supremum of the values of r for which $\operatorname{Re} f'(z) > 0$, $|z| < r$, where $f(z)$ varies over all functions in the class. Since $K(\alpha)$ is rotation and conjugation invariant, see **[10]**, so is $C(\alpha, \beta)$. That is, $f(z) \in C(\alpha, \beta)$ if and only if $\overline{f(\bar{z})}$ is in $C(\alpha, \beta)$. Since $\operatorname{Re} f'(z) > 0$ if and only if $|\arg f'(z)| < \pi/2$, the result follows from Theorem 4.

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Received September 12, 1975.

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Pacific Journal of Mathematics

Vol. 64, No. 1 May, 1976