

Pacific Journal of Mathematics

CONVEXITY THEOREMS FOR SUBCLASSES OF UNIVALENT FUNCTIONS

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We determine the radius of convexity of functions $f(z)$ for which $\operatorname{Re}\{f'(z)/\phi'(z)\} > \beta$, where $\phi(z)$ is convex of order α ($0 \leq \alpha \leq 1$). We also find bounds for $|\arg f'(z)|$. All results are sharp.

1. Introduction. Let S be the class of normalized univalent functions analytic in the unit disk. Let $K(\alpha)$ denote the subclass of S consisting of functions $\phi(z)$ for which

$$\operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} \geq \alpha \quad (0 \leq \alpha \leq 1).$$

This class is called convex of order α . We say that an analytic function $f(z) = z + a_2z^2 + \dots$ is in the class $C(\alpha, \beta)$ if there exists a function $\phi(z) \in K(\alpha)$ such that

$$\operatorname{Re} \left\{ \frac{f'(z)}{\phi'(z)} \right\} > \beta \quad (0 \leq \beta < 1, |z| < 1).$$

This class was defined by Libera [5]. Kaplan [3] showed that $C(0, 0)$, the class of close-to-convex functions, is univalent. Since $C(\alpha, \beta) \subset C(0, 0)$, we see that $C(\alpha, \beta)$ is a subclass of S .

Denote by P_β the functions $p(z)$ that are analytic in $|z| < 1$ and satisfy there the conditions

$$p(0) = 1 \quad \text{and} \quad \operatorname{Re} p(z) > \beta,$$

and set $P_0 = P$. It is well known that a function $q(z)$ is in P_β if and only if there exists a function $p(z) \in P$ such that

$$q(z) = (1 - \beta)p(z) + \beta = \frac{p(z) + h}{1 + h}, \quad \text{where}$$

$$(1) \quad h = \frac{\beta}{1 - \beta}.$$

Thus if $f(z) \in C(\alpha, \beta)$, then we may write

$$(2) \quad f'(z) = \phi'(z) \frac{p(z) + h}{1 + h},$$

where $\phi(z) \in K(\alpha)$, $p(z) \in P$, and h is defined by (1). Taking logarithmic derivatives in (2), we find that

$$(3) \quad 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{z\phi''(z)}{\phi'(z)} + \frac{zp'(z)}{p(z)+h}.$$

It is our purpose in this paper to determine the radius of convexity for the class $C(\alpha, \beta)$. Note, for $|z| = r$, that (3) yields

$$(4) \quad \min_{f \in C(\alpha, \beta)} \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \min_{\phi \in K(\alpha)} \operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} \\ + \min_{p \in P} \operatorname{Re} \left\{ \frac{zp'(z)}{p(z)+h} \right\}.$$

In [5], Libera found a disk $|z| < r$ in which $f(z) \in C(\alpha, \beta)$ is convex. His method essentially consisted of utilizing the inequality

$$\min_{f \in C(\alpha, \beta)} \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \min_{\phi \in K(\alpha)} \operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} - \max_{p \in P} \left| \frac{zp'(z)}{p(z)+h} \right|.$$

His result, however, was not sharp because for $|z| = r$,

$$\min_{p \in P} \operatorname{Re} \left\{ \frac{zp'(z)}{p(z)+h} \right\} \geq - \max_{p \in P} \left| \frac{zp'(z)}{p(z)+h} \right|,$$

with equality *only* when $h = 0$. The function that he claimed to be extremal need not be in $C(\alpha, \beta)$. See [9]. It is known [1] that

$$\min_{\substack{|z|=r \\ \phi \in K(\alpha)}} \operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} = \frac{1 - (1 - 2\alpha)r}{1 + r},$$

with equality for functions of the form

$$\phi(z) = \begin{cases} \frac{1}{(1-2\alpha)\epsilon} \left[\frac{1}{(1-\epsilon z)^{1-2\alpha}} - 1 \right] & (\alpha \neq \frac{1}{2}, |\epsilon| = 1) \\ -\bar{\epsilon} \log(1 - \epsilon z) & (\alpha = \frac{1}{2}, |\epsilon| = 1). \end{cases}$$

Thus, taking into account (4), the radius of convexity of $C(\alpha, \beta)$ is seen to be the smallest positive r for which

$$(5) \quad \frac{1 - (1 - 2\alpha)r}{1 + r} + \min_{\substack{|z|=r \\ p \in P}} \operatorname{Re} \left\{ \frac{zp'(z)}{p(z)+h} \right\} = 0.$$

In §2 we will use a theorem of V. A. Zmorovič to find $\min_{|z|=r, p \in P} \operatorname{Re}\{zp'(z)/(p(z) + h)\}$. In §3 we will determine the radius of convexity of $C(\alpha, \beta)$ and examine some of its consequences. Finally, in §4 we will find a sharp bound on $|\arg f'(z)|$ for $f(z) \in C(\alpha, \beta)$.

2. Consequences of Zmorovič's theorem. The following theorem is due to V. A. Zmorovič [11].

THEOREM A. *Let $\Psi(w, W) = M(w) + N(w)W$, where $M(w)$ and $N(w)$ are defined and are finite in the half plane $\operatorname{Re}\{w\} > 0$. Set*

$$w = \lambda_1 \frac{1 + z_1^m}{1 - z_1^m} + \lambda_2 \frac{1 + z_2^m}{1 - z_2^m},$$

$$W = \lambda_1 \frac{2mz_1^m}{(1 - z_1^m)^2} + \lambda_2 \frac{2mz_2^m}{(1 - z_2^m)^2},$$

where z_1 and z_2 are any points on the circle $|z| = r < 1$, m is a positive integer, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\lambda_1 + \lambda_2 = 1$. Then the function $\Psi(w, W)$ can be put in the form

$$\Psi(w, W) = M(w) + \frac{m}{2} (w^2 - 1)N(w) + \frac{m}{2} (\rho^2 - \rho_0^2)N(w)e^{2i\psi},$$

where

$$\frac{1 + z_k^m}{1 - z_k^m} = a + \rho e^{i\psi_k} \quad (k = 1, 2),$$

$$w = a + \rho_0 e^{i\psi_0} \quad (0 \leq \rho_0 \leq \rho),$$

$$a = \frac{1 + r^{2m}}{1 - r^{2m}}, \quad \rho = \frac{2r^m}{1 - r^{2m}}, \quad e^{i\psi} = ie^{i(\psi_1 + \psi_2)/2}.$$

Also,

$$(6) \quad \min \operatorname{Re}\{\Psi(w, W)\} \equiv \Psi_\rho(w)$$

$$= \operatorname{Re}\left\{M(w) + \frac{m}{2} (w^2 - 1)N(w)\right\} - \frac{m}{2} |N(w)|(\rho^2 - \rho_0^2).$$

This minimum is reached when

$$\exp[i(2\psi + \arg N(w))] = -1.$$

The importance of this formidable theorem lies in the fact that the minimum of $\operatorname{Re} \Psi(w, W)$ in the disk $|w - a| \leq \rho$ depends only on the two variables $\operatorname{Re} w$ and $\operatorname{Im} w$, as can be seen by (6), and not on W, λ_1 , or λ_2 .

I would like to thank the referee for pointing out that the following theorem may be found in [12]. For completeness we include a more detailed proof of this useful result.

THEOREM 1. *Suppose $p(z) \in P$, h is defined by (1), and a is defined as in Theorem A. Then*

$$\operatorname{Re} \left\{ \frac{zp'(z)}{p(z)+h} \right\} \cong \begin{cases} -\frac{2r}{(1+r)[(1+h-(1-h)r]} & (0 \leq r \leq r_\beta) \\ \frac{2\sqrt{h^2+ah}-a-2h}{2\sqrt{h^2+ah}-a-2h} & (r_\beta < r < 1), \end{cases}$$

where r_β is the unique root of the equation $(1-2\beta)r^3 - 3(1-2\beta)r^2 + 3r - 1 = 0$ in the interval $(0, 1]$. This result is sharp.

Proof. Set $M(w) = 0$, $N(w) = 1/(w+h)$, $m = 1$, and $w = p(z) = p$ in Theorem A, and note that $W = zp'(z)$. Thus $\Psi(w, W) = \Psi(p, zp') = zp'(z)/(p(z)+h)$ and, in view of (6),

$$(7) \quad \operatorname{Re} \left\{ \frac{zp'(z)}{p(z)+h} \right\} \cong \Psi_\rho(p) = \frac{1}{2} \operatorname{Re} \left[\frac{p^2-1}{p+h} - \frac{\rho^2-\rho_0^2}{|p+h|} \right].$$

Since $|p-a| = \rho_0 \leq \rho$, we may set $p = a + \xi + i\eta$, $\rho_0^2 = \xi^2 + \eta^2$, and $R = |p+h|$. Then

$$\begin{aligned} (8) \quad \operatorname{Re} \frac{p^2-1}{p+h} &= \frac{|p|^2(a+\xi) - (a+\xi+h) + h[(a+\xi)^2 - \eta^2]}{R^2} \\ &= \frac{(a+\xi+h)[R^2 - (h^2 + 2h(a+\xi) + 1)] - 2h\eta^2}{R^2} \\ &= \frac{(a+\xi+h)[R^2 - 2h(a+\xi+h) + (h^2-1)] - 2h\eta^2}{R^2} \\ &= (a+\xi+h) - 2h + \frac{(h^2-1)(a+\xi+h)}{R^2}. \end{aligned}$$

A substitution of (8) into (7) gives

$$(9) \quad \Psi_\rho(p) = \frac{a+\xi+h}{2} - h + \frac{(h^2-1)(a+\xi+h)}{2R^2} - \frac{\rho^2 - \xi^2 - \eta^2}{2R}.$$

We now wish to minimize $\Psi_\rho(p)$ as a function of η . A differentiation shows that

$$(10) \quad \frac{\partial \psi_\rho}{\partial \eta} = \frac{\eta}{2} \frac{S(\xi, \eta)}{R^4},$$

where

$$(11) \quad \begin{aligned} S(\xi, \eta) &= [\xi^2 + 4(a+h)\xi + \rho^2 + \eta^2 + 2(a+h)^2]R - 2(h^2 - 1)(\xi + a + h) \\ &\cong [\xi^2 + 4(a+h)\xi + \rho^2 + 2(a+h)^2 - 2(h^2 - 1)](\xi + a + h). \end{aligned}$$

But the last expression in (11) is an increasing function of ξ in the interval $[-\rho, \rho]$. Hence

$$S(\xi, \eta) \cong S(-\rho, \eta) = 2[(a - \rho)^2 + 2h(a - \rho) + 1](a + h - \rho) > 0.$$

We thus see from (10) that $\Psi_\rho(\xi, \eta)$ is minimized on every chord $\xi = \text{constant}$ of the circle $\xi^2 + \eta^2 = \rho_0^2$ at the point $\eta = 0$. Therefore the minimum of $\Psi_\rho(\xi, \eta)$ in the disk $\xi^2 + \eta^2 \leq \rho^2$ occurs somewhere on the diameter $\eta = 0$. Setting $\eta = 0$ in (9) and noting that $R = a + \xi + h$, we have

$$(12) \quad \Psi_\rho(p) \cong \Psi_\rho(\xi, 0) = l(R) = \frac{R}{2} - h + \frac{h^2 + \xi^2 - \rho^2 - 1}{2R}.$$

Using the identities $\xi = R - (a + h)$ and $\rho^2 = a^2 - 1$ in (12), we get

$$(13) \quad l(R) = R + \frac{h^2 + ah}{R} - (a + 2h).$$

We must now determine the minimum of $l(R)$ for R in the interval $[a + h - \rho, a + h + \rho]$. A differentiation of (13) shows that $l(R)$ assumes its minimum at

$$(14) \quad R_0 = \sqrt{h^2 + ah}$$

as long as

$$(15) \quad a + h - \rho \leq R_0 \leq a + h + \rho.$$

The right hand inequality in (15) is always true, but the left hand inequality will not hold when h (and consequently β) is small. In the latter case, $l(R)$ assumes its minimum at the point

$$(16) \quad R_1 = a + h - \rho.$$

Substituting (14) and (16), respectively, into (13), we find

$$(17) \quad l(R_0) = 2\sqrt{h^2 + ah} - (a + 2h)$$

$$(18) \quad l(R_1) = \frac{\rho^2 - a\rho}{a + h - \rho} = - \frac{2r}{(1+r)[1+h-(1-h)r]}.$$

As β increases, the transition from $l(R_1)$ to $l(R_0)$ occurs at the point where $R_0 = R_1$. But $R_0 = R_1$ when $h^2 + ah = (a - \rho + h)^2$, or in terms of r , when the polynomial equation

$$t(r) = (1 - 2\beta)r^3 - 3(1 - 2\beta)r^2 + 3r - 1$$

has a root in the interval $(0, 1]$. Note that

$$t'(r) = 3[(1 - 2\beta)r^2 - 2(1 - 2\beta)r + 1] > 0 \quad (0 < r < 1)$$

so that $t(r)$ is increasing. Further, $t(0) = -1$ and $t(1) = 4\beta$ so that $t(r)$ has a unique root in the interval $(0, 1]$. This completes the proof.

Equality holds in (18) for $p(z) = (1+z)/(1-z)$, and in (17) for

$$p(z) = \frac{1}{2} \left[\frac{1 + ze^{-i\theta_0}}{1 - ze^{-i\theta_0}} + \frac{1 + ze^{i\theta_0}}{1 - ze^{i\theta_0}} \right] = \frac{1 - z^2}{1 - 2z \cos \theta_0 + z^2},$$

where $\cos \theta_0$ is defined by the equation

$$(19) \quad h + (1 - r_0^2)(1 - 2r_0 \cos \theta_0 + r_0^2)^{-1} = R_0 \quad (r_0 = l(R_0)).$$

3. Radius of convexity theorems. We may now use Theorem 1 to prove

THEOREM 2. *Suppose r_β is the unique root of*

$$t(r) = (1 - 2\beta)r^3 - 3(1 - 2\beta)r^2 + 3r - 1$$

in the interval $(0, 1]$. Set

$$r(\alpha, \beta) = \frac{1}{2 - \alpha - 2\beta + \sqrt{\alpha^2 - 2\alpha + 4\beta^2 - 6\beta + 3}}.$$

Then the radius of convexity of $C(\alpha, \beta)$ is $r(\alpha, \beta)$ when $0 < r(\alpha, \beta) \leq r_\beta$, and is otherwise the smallest root greater than r_β of the polynomial equation

$$v(r) = [\alpha^2 - \beta(\alpha^2 + 2\alpha - 1)]r^4 - 2(1 - \alpha)(\beta + \alpha\beta - \alpha)r^3 + [(1 - \alpha)^2(1 - \beta) + 2\alpha\beta]r^2 + 2\beta(1 - \alpha)r - \beta.$$

This result is sharp for all α and β .

Proof. An application of Theorem 1 to (5) shows that the radius of convexity of $C(\alpha, \beta)$ is the smallest positive root of

$$(20) \quad \begin{cases} \frac{1 - (1 - 2\alpha)r}{1 + r} - \frac{2r}{(1 + r)[(1 + h) - (1 - h)r]} = 0 & (0 \leq r \leq r_\beta) \\ \frac{1 - (1 - 2\alpha)r}{1 + r} + 2\sqrt{h^2 + ah} - a - 2h = 0 & (r_\beta < r < 1), \end{cases}$$

where a is defined in Theorem A and h is defined by (1). The first expression in (20) may be written as

$$\frac{(1 - 2\alpha)(1 - 2\beta)r^2 - 2(2 - \alpha - 2\beta)r + 1}{(1 + r)[(1 + h) - (1 - h)r]} = 0,$$

whose roots are

$$\begin{aligned} \frac{(2 - \alpha - 2\beta) \mp \sqrt{(2 - \alpha - 2\beta)^2 - (1 - 2\alpha)(1 - 2\beta)}}{(1 - 2\alpha)(1 - 2\beta)} \\ = \frac{1}{(2 - \alpha - 2\beta) \pm \sqrt{\alpha^2 - 2\alpha + 4\beta^2 - 6\beta + 3}}. \end{aligned}$$

If both roots are positive, the minimum root is $r(\alpha, \beta)$. Similarly, a computation shows that r^* is a root of the second expression in (20) if and only if it is a root of $v(r)$. This completes the proof.

The extremal function is of the form

$$f(z) = \int_0^z \frac{1 + (1 - 2\beta)t}{(1 - t)^{3-2\alpha}} dt$$

when $0 < r(\alpha, \beta) \leq r_\beta$, and is otherwise of the form

$$f(z) = \int_0^z \frac{1 - 2\beta \cos \theta_0 + (2\beta - 1)t^2}{(1 - 2t \cos \theta_0 + t^2)(1 - t)^{2(1-\alpha)}} dt,$$

where $\cos \theta_0$ is defined by (19).

COROLLARY. *If $0 \leq \beta \leq \frac{1}{10}$, then the radius of convexity of $C(\alpha, \beta)$ is $r(\alpha, \beta)$ for all α .*

Proof. We must show that $0 < r(\alpha, \beta) \leq r_\beta$ for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1/10$. Note that $\partial t(r)/\partial \beta = 2r^2(3 - r)$, so that $t(r)$ is an increasing function of β . This means that r_β is a decreasing function of β . Set $A = \sqrt{\alpha^2 - 2\alpha + 4\beta^2 - 6\beta + 3}$. Then

$$\frac{\partial}{\partial \alpha} r(\alpha, \beta) = \frac{A + 1 - \alpha}{S^3} \geq 0 \quad (0 \leq \alpha \leq 1)$$

and

$$\frac{\partial}{\partial \beta} r(\alpha, \beta) = \frac{A + 3 - 4\beta}{A^3} \geq 0 \quad (0 \leq \beta \leq \frac{3}{4}).$$

Thus $r(\alpha, \beta) \leq r(1, \frac{1}{10})$ for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq \frac{1}{10}$. The result follows upon observing that

$$r\left(1, \frac{1}{10}\right) = \frac{1}{2} \quad \text{and} \quad t\left(\frac{1}{2}\right) = \frac{8}{10} \left(\frac{1}{8}\right) - \frac{24}{10} \left(\frac{1}{4}\right) + \frac{3}{2} - 1 = 0.$$

REMARK. When $\beta = 0$, we see that

$$r(\alpha, 0) = \frac{1}{2 - \alpha + \sqrt{\alpha^2 - 2\alpha + 3}}.$$

In this case, Libera's result [5] is sharp.

We turn now to a distinguished subclass of $C(\alpha, \beta)$, and state the result as a separate theorem.

THEOREM 3. *If $f(z) \in S$ with $\operatorname{Re} f'(z) > \beta$, then $f(z)$ is convex in a disk of radius*

$$\begin{cases} \frac{1}{1 - 2\beta + \sqrt{4\beta^2 - 6\beta + 2}} & (0 \leq \beta \leq \frac{1}{10}) \\ \left(1 + \sqrt{\frac{1-\beta}{\beta}}\right)^{-\frac{1}{2}} & (\frac{1}{10} < \beta < 1). \end{cases}$$

This result is sharp.

Proof. Since $\phi(z) = z$ is the only function in $K(1)$, the class under consideration is $C(1, \beta)$ so that Theorem 2 may be applied. As we saw in the corollary to Theorem 2,

$$r(1, \beta) = \frac{1}{1 - 2\beta + \sqrt{4\beta^2 - 6\beta + 2}} \leq r_\beta \quad (0 \leq \beta \leq \frac{1}{10}),$$

which gives the first part of the theorem.

Since $t(r_\beta) = 0$ when $\alpha = 1$ and $\beta = \frac{1}{10}$, the radius of convexity of $C(1, \beta)$ for $\beta > \frac{1}{10}$ is the only positive root of

$$(1 - 2\beta)r^4 + 2\beta r^2 - \beta = 0, \quad \text{or}$$

$$r^2 = \frac{-\beta + \sqrt{\beta - \beta^2}}{1 - 2\beta} = \frac{1}{1 + \sqrt{\frac{1-\beta}{\beta}}}.$$

This completes the proof.

REMARK. The cases $\beta = 0$ and $\beta = \frac{1}{2}$ were proved, respectively, by MacGregor [6] and Hallenbeck [2].

4. An argument theorem.

THEOREM 4. If $f(z) \in C(\alpha, \beta)$, then

$$|\arg f'(z)| \leq 2(1 - \alpha)\sin^{-1} r + \sin^{-1} \left[\frac{2(1 - \beta)r}{1 + (1 - 2\beta)r^2} \right].$$

This result is sharp.

Proof. We may write

$$f'(z) = \phi'(z)q(z), \quad \text{where } \phi(z) \in K(\alpha) \quad \text{and} \quad q(z) \in P_\beta.$$

Hence

$$(21) \quad |\arg f'(z)| \leq |\arg \phi'(z)| + |\arg q(z)|.$$

But by a result of Pinchuk [8],

$$(22) \quad |\arg \phi'(z)| \leq 2(1 - \alpha)\sin^{-1} r \quad (|z| \leq r).$$

Since $\operatorname{Re} q(z) > \beta$, the function

$$\omega(z) = \frac{(q(z) - \beta) - (1 - \beta)}{(q(z) - \beta) + (1 - \beta)} = \frac{q(z) - 1}{q(z) - (2\beta - 1)}$$

is analytic with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $|z| < 1$.

Thus by Schwarz's lemma,

$$\left| \frac{q(z) - 1}{q(z) - (2\beta - 1)} \right| < |z| \quad \text{for } |z| < 1.$$

Hence the values of $q(z)$ are contained in the circle of Apollonius whose diameter is the line segment from $(1 + (2\beta - 1)r)/(1 + r)$ to

$(1 - (2\beta - 1)r)/(1 - r)$. The circle is centered at the point $(1 + (1 - 2\beta)r^2)/(1 - r^2)$ and has radius $(2(1 - \beta)r)/(1 - r^2)$. Thus $|\arg q(z)|$ attains its maximum at points where a ray from the origin is tangent to the circle, that is, when

$$(23) \quad \arg q(z) = \pm \sin^{-1} \frac{2(1 - \beta)r}{1 + (1 - 2\beta)r^2}.$$

Substituting (22) and (23) into (21), the result follows.

Equality holds for functions of the form

$$f(z) = \int_0^z \frac{1 + (1 - 2\beta)\eta t}{(1 - \epsilon t)^{2(1-\alpha)}(1 - \eta t)} dt$$

with suitably chosen ϵ, η , where $|\epsilon| = |\eta| = 1$.

REMARK. For $\alpha = \beta = 0$, this reduces to

$$|\arg f'(z)| \leq 2 \sin^{-1} r + \sin^{-1} \frac{2r}{1 + r^2} = 2(\sin^{-1} r + \tan^{-1} r),$$

a result of Krzyz [4].

THEOREM 5. Suppose $f(z), g(z) \in C(\alpha, \beta)$. Then

$$\lambda f(z) + (1 - \lambda)g(z) \quad (0 \leq \lambda \leq 1)$$

is univalent in a disk $|z| < r$, where r is the smallest positive root of the equation

$$2(1 - \alpha)\sin^{-1} r + \sin^{-1} \left(\frac{2(1 - \beta)r}{1 + (1 - 2\beta)r^2} \right) = \frac{\pi}{2}.$$

This result is sharp.

Proof. In [7], MacGregor showed that the exact radius of univalence of convex linear combinations of a rotation and conjugation invariant subclass of S is given by the supremum of the values of r for which $\operatorname{Re} f'(z) > 0$, $|z| < r$, where $f(z)$ varies over all functions in the class. Since $K(\alpha)$ is rotation and conjugation invariant, see [10], so is $C(\alpha, \beta)$. That is, $f(z) \in C(\alpha, \beta)$ if and only if $f(\bar{z})$ is in $C(\alpha, \beta)$. Since $\operatorname{Re} f'(z) > 0$ if and only if $|\arg f'(z)| < \pi/2$, the result follows from Theorem 4.

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Received September 12, 1975.

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Walter Allegretto, <i>Nonoscillation theory of elliptic equations of order $2n$</i>	1
Bruce Allen Anderson, <i>Sequencings and starters</i>	17
Friedrich-Wilhelm Bauer, <i>A shape theory with singular homology</i>	25
John Kelly Beem, <i>Characterizing Finsler spaces which are pseudo-Riemannian of constant curvature</i>	67
Dennis K. Burke and Ernest A. Michael, <i>On certain point-countable covers</i>	79
Robert Chen, <i>A generalization of a theorem of Chacon</i>	93
Francis H. Clarke, <i>On the inverse function theorem</i>	97
James Bryan Collier, <i>The dual of a space with the Radon-Nikodým property</i>	103
John E. Cruthirds, <i>Infinite Galois theory for commutative rings</i>	107
Artatrana Dash, <i>Joint essential spectra</i>	119
Robert M. DeVos, <i>Subsequences and rearrangements of sequences in FK spaces</i>	129
Geoffrey Fox and Pedro Morales, <i>Non-Hausdorff multifunction generalization of the Kelley-Morse Ascoli theorem</i>	137
Richard Joseph Fleming, Jerome A. Goldstein and James E. Jamison, <i>One parameter groups of isometries on certain Banach spaces</i>	145
Robert David Gulliver, II, <i>Finiteness of the ramified set for branched immersions of surfaces</i>	153
Kenneth Hardy and István Juhász, <i>Normality and the weak cb property</i>	167
C. A. Hayes, <i>Derivation of the integrals of $L^{(q)}$-functions</i>	173
Frederic Timothy Howard, <i>Roots of the Euler polynomials</i>	181
Robert Edward Jamison, II, Richard O'Brien and Peter Drummond Taylor, <i>On embedding a compact convex set into a locally convex topological vector space</i>	193
Andrew Lelek, <i>An example of a simple triod with surjective span smaller than span</i>	207
Janet E. Mills, <i>Certain congruences on orthodox semigroups</i>	217
Donald J. Newman and A. R. Reddy, <i>Rational approximation of e^{-x} on the positive real axis</i>	227
John Robert Quine, Jr., <i>Homotopies and intersection sequences</i>	233
Nambury Sitarama Raju, <i>Periodic Jacobi-Perron algorithms and fundamental units</i>	241
Herbert Silverman, <i>Convexity theorems for subclasses of univalent functions</i>	253
Charles Frederick Wells, <i>Centralizers of transitive semigroup actions and endomorphisms of trees</i>	265
Volker Wrobel, <i>Spectral approximation theorems in locally convex spaces</i>	273
Hidenobu Yoshida, <i>On value distribution of functions meromorphic in the whole plane</i>	283