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Let X be a locally compact closed convex subset of a locally convex Hausdorff topological linear space E. Then every exposed point of X is strongly exposed. The definitions of denting (strongly extreme) ray and strongly exposed ray are given for convex subsets of E. If X does not contain a line, then every extreme ray is strongly extreme and every exposed ray is strongly exposed. An example is given to show that the hypothesis that X be locally compact is necessary in both cases.

By a locally convex space we mean a real Hausdorff locally convex topological linear space E. E^* will denote the topological dual of E. The set of extreme points of X will be denoted by ext X. The closed line segment between the points x and y in E will be denoted $[x, y]$. The following definition was given by M. Rieffel [**6**, p. 75] for subsets of a Banach space. I. Namioka also studied these points in [**4**].

DEFINITION 1. If X is a subset of a locally convex space, then $x \in X$ is called a denting (strongly extreme) point of X if for any nbhd U of x, $x \notin \text{cl-conv}(X \setminus U)$. The set of all denting points of X will be denoted by dent X.

Clearly, every denting point is an extreme point. It follows from the separation theorem for convex sets that x_0 is a denting point of X iff for each nbhd U of x_0 there exist $f \in E^*$ and $\alpha \in R$ such that $x_0 \in \{x : f(x) < \alpha\} \cap X \subseteq X \cap U$. An example is given in [**6**, p. 75] to show that not every extreme point is a denting point. However, this is not the case in a locally compact set. For completeness we state the following theorem due to J. Reif and V. Zizler [**5**, p. 64].

THEOREM 1. *Assume X is a locally compact closed convex set in a locally convex space E. Then any extreme point of X is a strongly extreme point of X with respect to the relative topology from E.*

A point p of a set X in a locally convex space E is an exposed point of X if there exists an $f \in E^*$ such that $f(x) > f(p)$ for each $x \in X \setminus \{p\}$. The following definition was given by J. Lindenstrauss [**3**, p. 140] for subsets of a Banach space.

DEFINITION 2. A point $x \in X$, where $X \subseteq E$, is called a strongly exposed point of X whenever (i) there exists an $f \in E^*$ such that $f(y) > f(x)$ for each $y \in X \backslash \{x\}$, and (ii) for any net $\{x_\alpha\} \subseteq X$, $f(x_\alpha) \to f(x)$ in R implies that $x_\alpha \to x$ in E. The set of all strongly exposed points of X is denoted by strexp X.

It is easy to see from the definition that every strongly exposed point is an exposed point. J. Lindenstrauss in [**3**, p. 145] gave an example of a set which has an exposed point that is not strongly exposed. However, this is not the case if the set is locally compact.

THEOREM 2. *Let X be a locally compact closed convex subset of a locally convex space E, then every exposed point of X is a strongly exposed point of X.*

Proof. Let U be a closed convex nbhd of x such that $U \cap X$ is compact and assume $f \in E^*$ such that $f(x) < f(y)$, for all $y \in X \backslash \{x\}$. Since x is an exposed point of X, x is an extreme point of X. By Theorem 1, x is a denting point of X. Thus, there exist $g \in E^*$ and $\alpha \in R$ such that $\{x: g(x) < \alpha\} \cap X \subseteq (\text{int } U) \cap X$.

If $\{x: g(x) \geqq \alpha\} \cap (X \cap U) = \emptyset$, then it follows immediately that $U \cap X \subseteq \{x: g(x) < \alpha\} \cap X \subseteq (\text{int } U) \cap X$. Therefore $U \cap X$ is a nonempty open and closed set in the connected set X. Hence, $U \cap X = X$ which implies X is compact. Let $\{x_\alpha\}$ be a net in X such that $f(x_\alpha) \to f(x)$ in R. Since X is compact, there is a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ and a vector $y \in X$ such that $x_\beta \to y$. Thus, $f(x_\beta) \to f(y) = f(x)$ in R and so $y = x$. For any subnet $\{x_\gamma\} \subseteq \{x_\alpha\}$ there is similarly a subnet which converges to x, which proves that $x_\alpha \to x$ in E.

On the other hand, if $W = \{x: g(x) \geqq \alpha\} \cap (X \cap U) \neq \emptyset$, then W is a nonempty compact convex subset of X which does not contain x. Hence, there is a $w \in W$ such that $f(x) < f(w) = \inf f(W)$. Let $y \in X \backslash U$, then $[x, y] \subseteq X$. U is a closed convex nbhd of x; hence, there exists a $z \in \text{Bdry } U$ such that $z \in [x, y]$. Since $z \in \text{Bdry } U$, then $z \not\in \text{int } U$ and $z \not\in \{x: g(x) < \alpha\}$. Therefore, $z \in \{x: g(x) \geqq \alpha\} \cap (X \cap U)$ so $f(z) \geqq f(w)$. But $y - x = \lambda(z - x)$ where $\lambda > 1$. Hence, $f(y - x) = \lambda f(z - x) > f(z - x)$ which implies $f(y) > f(z) \geqq f(w)$. Let $\{y_\alpha\}$ be a net in X such that $f(y_\alpha) \to f(x)$ in R. Since $\{y_\alpha\} \subseteq X$ and $f(y) \geqq f(w) > f(x)$ for each $y \in X \backslash U$, we may assume that $\{y, y_a\} \subseteq U \cap X$. Since $U \cap X$ is compact, it follows from the previous argument that $y_\alpha \to x$ in E.

As V. Klee has shown in [**1**] and [**2**], it is possible to extend the Krein–Milman theorem to certain noncompact convex sets with the aid of the notion of extreme ray. An extreme ray of a closed convex set X

is a closed half-line $\rho \subseteq X$ such that whenever x, $y \in X$ and $\lambda x + (1-\lambda)y \in \rho$ for some λ with $0 < \lambda < 1$, x, $y \in \rho$.

DEFINITION 3. A ray $\rho = \{x + \lambda z : \lambda \geqq 0, z \neq 0\}$ of a convex set X in a topological linear space E is a denting (strongly extreme) ray of X if for any nbhd U of 0, $\rho' \cap \text{cl-conv}[X' \backslash (x + \langle z \rangle + U)] = \emptyset$, where X' is any bounded convex subset of X, $\rho' = \rho \cap X'$ and $\langle z \rangle$ denotes the one-dimensional linear subspace generated by z. Denote the union of all denting rays of X by rdent X.

It is easy to show that every denting ray of a convex set X is an extreme ray of X. The following theorem and example show that extreme rays and denting rays coincide in some instances and are distinct in others.

THEOREM 3. *Let X be a locally compact closed convex subset of a locally convex space E, then every extreme ray of X is a denting ray of X.*

Proof. Let ρ be an extreme ray of X. We may assume without loss of generality that $\rho = \{\lambda x_0 : \lambda \geqq 0\}$, $x_0 \neq 0$. Let X' be a bounded convex subset of X and let f_0 be in E^* such that f_0 is positive on $K \backslash \{0\}$, where K is the union of all rays in X which emanate from 0, and $X \cap \{x : f_0(x) \leqq t\}$ is compact, for each $t \in R$. Such a functional exists by Theorem 3.2 in **[1]**. Since X' is bounded and convex, $\text{cl}(X')$ is bounded and convex. According to a result of Klee [**1**, p. 236], $\text{cl}(X')$ is compact which implies $\sup f_0(\text{cl}(X')) < \infty$. Then we may assume $X' \subseteq \{x : f_0(x) \leqq 1\} \cap X = X''$. Let $W = \{x : f_0(x) = 1\} \cap X$ and assume $f_0(x_0) = 1$. Then $x_0 \in \text{ext}(W)$ and W is compact, since X'' is compact. By Theorem 1, x_0 is a denting point of W. Let U be a nbhd of zero and let $g \in E^*$ and $\alpha > 0$ such that $x_0 \in \{x : g(x) < \alpha\} \cap W \subseteq (x_0 + U) \cap W$. Let $T = \{x : g(x) = \alpha\} \cap W$. Then T is compact, convex and $T \cap \langle x_0 \rangle = \emptyset$. Let $f \in E^*$ and $\beta > 0$ such that $f(\langle x_0 \rangle) < \beta < \inf f(T)$. Since $0 \in \langle x_0 \rangle$, we have $0 = f(\langle x_0 \rangle) < \beta < \inf f(T)$.

If $y \in W$ such that $f(y) < \beta$, then $f_0(y) = 1$ and $[x_0, y] \cap T = \emptyset$, since $f(x_0) < \beta$. It follows that $g(y) < \alpha$ and hence, $y \in (x_0 + U) \cap W \subseteq \langle x_0 \rangle + U$.

On the other hand, if $y \in X$ such that $f_0(y) < 1$ and $f(y) < \beta$, then there is a unique $\lambda > 0$ such that $f_0(y + \lambda x_0) = 1$. Again from Klee [**1**, p. 235] we have $y + \lambda x_0 \in X$. Hence, $y + \lambda x_0 \in W$ and $f(y + \lambda x_0) = f(y) < \beta$. By the previous argument, it follows that $y + \lambda x_0 \in x_0 + U$ and so $y \in (1 - \lambda)x_0 + U \subseteq \langle x_0 \rangle + U$.

In both cases we have $y \in \{x : f(x) < \beta\} \cap X''$ implies $y \in \langle x_0 \rangle + U$. Hence, $X'' \backslash (\langle x_0 \rangle + U) \subseteq X'' \backslash \{x : f(x) < \beta\} \subseteq \{x : f(x) \geqq \beta\}$. Thus, $\text{cl-conv}[X' \backslash (\langle x_0 \rangle + U)] \subseteq \{x : f(x) \geqq \beta\}$. Now $f(\rho') < \beta$, since

$f(\langle x_0\rangle) < \beta$ and $\rho' = (X' \cap \rho)$, so $\rho' \cap \text{cl-conv}[X' \backslash (\langle x_0\rangle + U)] = \emptyset$. Therefore, ρ is a denting ray of X.

EXAMPLE 1. Let the space be ℓ_2 with the canonical basis $\{e_n\}$, and $X = \text{cl-conv}(\{e_n;\, n = 2, 3, \cdots\})$. Then $0 \in X$ and e_1 is in $\ell_2 \backslash X$. Let C be the cone generated by X with vertex e_1, then C is a closed convex subset of ℓ_2. Let ρ be the ray of the cone through 0. Clearly, ρ is an extreme ray of C. Let $S_{\frac{1}{2}}(0)$ be the open ball of radius 1/2 centered on 0. Clearly, $e_n \notin S_{\frac{1}{2}}(0)$ so $e_n \notin \langle e_1\rangle + S_{\frac{1}{2}}(0)$ and it follows that $e_n \in \text{cl-conv}[X \backslash (\langle e_1\rangle + S_{\frac{1}{2}}(0))]$ for $n \geqq 2$. However, $\{e_n\}$ converges weakly to 0 and $\text{cl-conv}[X \backslash (\langle e_1\rangle + S_{\frac{1}{2}}(0))]$ is weakly closed so $0 \in \text{cl-conv}[X \backslash (\langle e_1\rangle + S_{\frac{1}{2}}(0))]$. Hence ρ is not a denting ray of C.

A ray ρ in X, where $X \subseteq E$, is an exposed ray of X if there exist $f \in E^*$ and $\alpha \in R$ such that $\rho = \{x : f(x) = \alpha\} \cap X$ and $f(X \backslash \rho) > \alpha$. The next definition was given by V. Zizler in [**7**, p. 55] for subsets of a Banach space.

DEFINITION 4. Let X be a convex set in a locally convex space E and ρ a closed ray in X. Then ρ is a strongly exposed ray of X if (i) there exist $f \in E^*$ and $r \in R$ such that $f(x) = r$ for $x \in \rho$ and $f(x) > r$ for $x \in X \backslash \rho$, and (ii) $\{x_\alpha\}$ is eventually in $\rho + U$, whenever U is a nbhd of 0 and $\{x_\alpha\}$ is a bounded net in X such that $f(x_\alpha) \to r$. The set of all strongly exposed rays will be denoted by $\text{rstrexp}\, X$.

Clearly every strongly exposed ray is an exposed ray. The following proposition, theorem, and examples show the relationships among denting ray, exposed ray and strongly exposed ray.

PROPOSITION 1. *Let ρ be a strongly exposed ray of a convex set X in a locally convex space E. Then ρ is a denting ray of X.*

Proof. We may assume $\rho = \{\lambda x_0 : \lambda \geqq 0\}$, $x_0 \neq 0$. Let $f \in E^*$ such that $\rho = \{x : f(x) = 0\} \cap X$ and $f(x) > 0$ for each $x \in X \backslash \rho$. Let U be a nbhd of zero and X' a bounded convex subset of X. Assume for each positive integer n there is an $x_n \in \{x : f(x) < (1/n)\} \cap X'$ such that $x_n \notin \langle x_0\rangle + U$. Clearly $\{x_n\}$ is bounded and $f(x_n) \to 0$. Hence, there exists a positive integer N such that $x_n \in \rho + U$ for $n \geqq N$. This is a contradiction; so there is a positive integer N' such that $\{x : f(x) < (1/N')\} \cap X' \subseteq (\langle x_0\rangle + U) \cap X'$. Thus, $\text{cl-conv}[X' \backslash (\langle x_0\rangle + U)] \subseteq \{x : f(x) \geqq (1/N')\}$ which implies $(\rho \cap X') \cap \text{cl-conv}[X' \backslash (\langle x_0\rangle + U)] = \emptyset$; so ρ is a denting ray of X.

THEOREM 4. *Let X be a locally compact closed convex subset of a locally convex space E, then every exposed ray of X is a strongly exposed ray of X.*

Proof. Let ρ be an exposed ray of X. We may assume that ρ emanates from the origin. Let $f \in E^*$ such that $\rho = X \cap \{x: f(x) = 0\}$ and $f(x) > 0$ for $x \in X \backslash \rho$. Let $\{x_\alpha\}$ be a bounded net in X such that $f(x_\alpha) \to 0$ in R and let U be a nbhd of 0. There exists a nbhd V of 0 such that V is closed, balanced and convex, $V \subseteq U$ and $V \cap X$ is compact. Let $\{x_\beta\}$ denote the set of all vectors in the net $\{x_\alpha\}$ which lie in $X \backslash U$. If $\{x_\beta\}$ is not a subnet of $\{x_\alpha\}$, then $\{x_\alpha\}$ is eventually in $U = 0 + U \subseteq \rho + U$ and the conclusion follows.

If $\{x_\beta\}$ is a subnet of $\{x_\alpha\}$, then it suffices to show that $\{x_\beta\}$ is eventually in $\rho + U$. By Theorem 1, 0 is a denting point of X, since 0 is an extreme point of X. Let $g \in E^*$ and $a > 0$ such that $\{x: g(x) < a\} \cap X \subseteq V \cap X$. Since $x_\beta \notin V$, then $g(x_\beta) \geqq a$, for each β. The net $\{x_\alpha\}$ is bounded, so there exists a number $b > 0$ such that $g(x_\beta) \leqq b$, for each β. Hence, $0 < a \leqq g(x_\beta) \leqq b$, for each β. If $y_\beta = [a/g(x_\beta)]x_\beta$, then $y_\beta \in \{x: g(x) = a\} \cap X$. Since $\{x: g(x) < a\} \cap X \subseteq V \cap X$ and $V \cap X$ is compact, then $\{x: g(x) = a\} \cap X$ is compact; so there is a subnet $\{y_\gamma\} \subseteq \{y_\beta\}$ and a point $y \in \{x: g(x) = a\} \cap X$ such that $y_\gamma \to y$ in E. Since $g(x_\beta)$ is bounded and $f(x_\beta) \to 0$ in R, we have $y \in \{x: f(x) = 0\} \cap X$ and thus, $y \in \rho$. Hence, $y \in \{x: g(x) = a\} \cap \rho$. It follows immediately that $\{y\} = \{x: g(x) = a\} \cap \rho$. Let $W = \{x: g(x) = a\} \cap X$ and $z \in W \backslash \{y\}$. Then $z \in W \backslash \rho$ which implies $f(z) > 0$. Thus, y is exposed by f on W. Since $f(y_\beta) \to 0 = f(y)$, by Theorem 2 we have $y_\beta \to y$ in E. Hence, there is a λ_0 such that $y_\beta \in y + (a/b)V$, for $\beta \geqq \lambda_0$. If $z_\beta = [g(x_\beta)/a]y$, then $z_\beta \in \rho$, for each β. But $y_\beta = [a/g(x_\beta)]x_\beta$, so $x_\beta \in [g(x_\beta)/a]y + [g(x_\beta)/a](a/b)V \subseteq \rho + V \subseteq \rho + U$, for all $\beta \geqq \lambda_0$. Therefore, the net $\{x_\beta\}$ is eventually in $\rho + U$ and it follows that ρ is a strongly exposed ray of X.

EXAMPLE 2. The ray ρ defined in Example 1 is exposed by $f = (0, \frac{1}{2}, \frac{1}{3}, \cdots, 1/n, \cdots)$ on C. Therefore ρ is an exposed ray of C that is not a denting ray of C so by Proposition 1 ρ is not a strongly exposed ray of C.

EXAMPLE 3. Let the space be R^3 and

$$X = \operatorname{conv}[\{(x, y, z): x^2 + y^2 \leqq 1,\ -1 \leqq y \leqq 0 \text{ and } z = 1\} \cup (1,1,1)].$$

Let C be the cone generated by X with vertex $(0,0,0)$. Then C is a closed convex subset of R^3. Let ρ be the ray of the cone through the point $(1,0,1)$. It is easy to see ρ is not an exposed ray of C, but ρ is a denting ray of C.

From the preceding work we can restate two of Klee's theorems ([**2,** Th. 2.3, p. 91], [**1,** Th. 3.4, p. 237]) as follows:

THEOREM 5. *Suppose X is a locally compact closed convex subset of a normed linear space, and X contains no line. Then* $\operatorname{ext} X \subseteq \operatorname{cl}(\operatorname{strexp} X)$ *and* $X = \text{cl-conv}(\operatorname{strexp} X \cup \operatorname{rstrexp} X)$.

THEOREM 6. *If X is a locally compact closed convex subset of a locally convex space, and X contains no line, then* $X = \text{cl-conv}(\operatorname{dent} X \cup \operatorname{rdent} U)$.

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