THE SOLUTION OF A STIELTJES-VOLterra INTEGRAL EQUATION FOR RINGS

BURRELL WASHINGTON HELTON
THE SOLUTION OF A STIELTJES-VOLTERRA INTEGRAL EQUATION FOR RINGS

BURRELL W. HELTON

For a triple \((h, K, g)\) of functions and an interval \([a, x]\), the author defines a subdivision-refinement-type limit \(V(a, x; h, K, dg)\) of the set \(\{V(D, h, K, Ag)\}\) of determinants, where each subdivision \(D = \{x_i\}_i^n\) of \([a, x]\) defines an \(n \times n\) determinant of the set and each determinant has the form

\[
\begin{vmatrix}
  h_1 + h_0 K_{10} \Delta g_1 & -1 & 0 & 0 \\
  h_1 + h_0 K_{20} \Delta g_1 & K_2 \Delta g_2 & -1 & 0 \\
  h_1 + h_0 K_{30} \Delta g_1 & K_3 \Delta g_2 & K_3 \Delta g_3 & -1 \\
  h_1 + h_0 K_{40} \Delta g_1 & K_4 \Delta g_2 & K_4 \Delta g_3 & K_4 \Delta g_4
\end{vmatrix}
\]

The following theorem is proved. If \(f, g, h\) and \(K\) are functions to a ring and \(g\) has bounded variation on \([a, b]\), then \((f, K, g) \in OA^*\) and \(f(x) = h(x) + (L) \int_a^x f(t)K(x, t)dt\) on \([a, b]\) iff \((h, K, g) \in OM^*\) and \(f(x) = V(a, x; h, K, dg)\) on \([a, b]\). The \(OA^*\) and \(OM^*\) sets are defined and sufficient conditions are proved for \((f, K, g) \in OA^*\) and \((h, K, g) \in OM^*\), and for the existence of the limit \(V(a, x; h, K, dg)\), and for \(V(a, x; h, K, dg) = h(x) - (L) \int_a^x h(t)\Delta V(t, x; 1, K, dg)\).

Although the Volterra equation \(f(x) = h(x) + \int_a^x f(t)K(x, t)dt\) has been studied in depth by many persons, it seems that only Hinton [3], Reneke [4] [5] and Bitzer [1] [2] have published papers on the Volterra integral equation in which the integral is a subdivision-refinement-type Stieltjes integral. In this paper the solution of the Volterra equation and the development of the related properties do not depend on a Picard expansion or on the above quoted references. So far as the author has been able to determine, this subdivision-refinement definition of the solution \(V(a, x; h, K, dg)\) of the Volterra equation has not been published previously.

Definitions and notations. The symbol \(R\) denotes the set of real numbers and \(N\) is a ring which has a multiplicative identity element 1 and a norm \(\|\cdot\|\) with respect to which \(N\) is complete and \(\|1\| = 1; f, g\) and \(h\) are functions from \(R\) to \(N\) and \(K\) is a function from \(RXR\) to \(N\). Also, \(dg \in OB^0\) on \([a, b]\) means \(g\) has bounded variation on \([a, b]\). All
integrals are of the subdivision-refinement-type limits; the approximating
sum for \( (L) \int E(t)dg(t) \) is \( \Sigma E(t_i)[g(t_i) - g(t_{i-1})] \) and for
\( (R) \int E(t)dg(t) \) is \( \Sigma E(t_i)[g(t_i) - g(t_{i-1})] \). If no misunderstanding is likely,
the symbols \( K_{ij}, f_i \) and \( \Delta g_i \) will be used for \( K(x_i, x_i), f(x_i) \) and
\( g(x_i) - g(x_{i-1}) \), respectively.

If \( \{a_{ij}\}_{i,j=1} \) is a sequence of elements of \( N \) and \( p \) and \( q \) are integers
such that \( 1 \leq p \leq q \leq n \), then the symbol \( |a_{ij}|_p^q \) denotes the determinant
\[
\begin{vmatrix}
    a_{pp} & \cdots & a_{pq} \\
    \vdots & \ddots & \vdots \\
    a_{qp} & \cdots & a_{qq}
\end{vmatrix}
\]
and is defined by the sum of the \((q - p + 1)!\) products obtained as follows:
1. each term of the sum is a product, or the negative of a product, which
   contains one and only one element from each row and each column of
   \( |a_{ij}|_p^q \); 2. the factors of each term are ordered so that the second
   subscripts appear in the order \( p, p + 1, \ldots, q \); and 3. the product
   or the negative of a product is used as a term according as the number
   of inversions of the first subscripts is even or odd. Note that the usual
   theorems pertaining to determinants will hold, except where multiplica-
   tive commutativity is needed in the proofs. Also, if \( A = |a_{ij}|_p^q \), then \( |A| \)
   denotes the norm of \( A \) and, if \( 1 \leq p \leq q \leq n \), \( A_p = |a_{ij}|_p^q \), \( A_p = |a_{ij}|_p^q \), \( A_0 = 1 \), \( A_{n+1} = 1 \), and if \( 1 \leq k \leq p \), then \( *A_{pk} \) is the determinant obtained by replacing
   the first column of \( |a_{ij}|_p^q \) with the column \( \{a_{ik}\}_{i=1}^p \) of elements of \( \{a_{ij}\}_{i,j=1}^n \).

\( A = |a_{ij}|_p^q \) is a Volterra determinant means \( \{a_{ij}\}_{i,j=1}^n \) is a sequence
such that \( a_{ii} = -1 \) for \( j = i + 1 \) and \( a_{ii} = 0 \) for \( j > i + 1 \). \( A = |a_{ij}|_p^q \) is a delta
   determinant defined by the sequences \( \{c_{ij}\}_{i,j=1}^n \) and \( \{d_{ij}\}_{i=0}^n \) means \( A \) is a
Volterra determinant and \( a_{ij} = c_{ii}(d_{ij} - d_{i-1}) \) for \( 1 \leq j \leq i \leq n \).

If \( D = \{x_i\}_0^n \) is a subdivision of a number interval \([a, b]\), then
\( V(D, h, K, \Delta g) \) denotes the \( n \times n \) Volterra determinant \( |a_{ij}|_p^q \) such that
\( a_{ij} = h(x_i) + h(x_{i-1})K(x_i, x_{i-1})(g(x_i) - g(x_{i-1})) \) for \( i = 1, 2, \ldots, n \) and
\( a_{ij} = K(x_i, x_{i-1})(g(x_i) - g(x_{i-1})) \) for \( 1 \leq j \leq i \leq n \). If no misunderstanding is
likely, \( V(D) \) will be used to denote \( V(D, h, K, \Delta g) \).

The limit \( V(a, b; h, K, \Delta g) \) exists means there is an element \( J \) of \( N \)
such that if \( \epsilon > 0 \) then there is a subdivision \( D \) of \([a, b]\) such that if \( D' \) is a
refinement of \( D \) then \( |J - V(D', h, K, \Delta g)| < \epsilon \). The symbol
\( V(a, b; h, K, \Delta g) \) will be used to denote this limit \( J \).

If \( m > 1 \), the number \( M \) is an \( m \)-bound for \( V(\ldots, \Delta g) \) on \([a, b]\)
means \( M \geq m \) and, if \( |h| < m \) on \([a, b]\) and \( |K| < m \) on \([a, b] \times [a, b] \)
and \( D \) is a subdivision of a subinterval of \([a, b]\) and \( A = |a_{ij}|_p^q = V(D, h, K, \Delta g) \),
then \( |A| < M \) and each of \( |A_p|, |A_p| \) and \( |*A_p| \) is less
than \( M \) for \( 1 \leq p \leq n \) and \( 1 \leq j \leq p \).
The triple \((f, K, g) \in OA^*\) on \([a, b]\) means that
\[
(L) \int_a^b f(t)K(x, t)dg(t) \text{ exists for } x \in [a, b] \text{ and if } \epsilon > 0 \text{ then there is a subdivision } D \text{ of } [a, b] \text{ such that, if } \{t_i\}^n_0 \text{ is a refinement of } D \text{ and } 0 < p \leq n \text{ and } x = t_p, \text{ then}
\]
\[
\left| (L) \int_a^b f(t)K(x, t)dg(t) - \sum^p_{i=0} f(t_{i-1})K(x, t_{i-1})[g(t_i) - g(t_{i-1})] \right| < \epsilon.
\]

The triple \((h, K, g) \in OM^*\) means \(V(a, x; h, K, dg)\) exists for \(x \in [a, b]\) and if \(\epsilon > 0\) then there is a subdivision \(D\) of \([a, b]\) such that, if \(\{x_i\}^n_0\) is a refinement of \(D\) and \(0 < p \leq n\) and \(H = \{x_i\}^p_0\), then
\[
|V(a, x_p; h, K, dg) - V(H, h, K, \Delta g)| < \epsilon.
\]

The triple \((l, K, g) \in OM^{**}\) on \([a, b]\) means \(V(x, b; 1, K, dg)\) exists for \(x \in [a, b]\) and if \(\epsilon > 0\) then there is a subdivision \(D\) of \([a, b]\) such that, if \(\{x_i\}^n_0\) is a refinement of \(D\) and \(0 \leq p < n\) and \(H = \{x_i\}^p_0\), then
\[
|V(x_p, b; 1, K, dg) - V(H, 1, K, \Delta g)| < \epsilon,
\]
where 1 denotes the identity function.

In the following three definitions, \(G(x, y) = \int_x^y |dg|\).

\[
\int_a^b \int_a^b |dK| |dg| |dg| = 0 \text{ means if } \epsilon > 0 \text{ then there is a subdivision } D \text{ of } [a, b] \text{ such that, if } \{x_i\}^n_0 \text{ is a refinement of } D, \text{ then}
\]
\[
\sum^n_{i=1} \sum^n_{j=1} M_{ij} G(x_{i-1}, x_i) G(x_{j-1}, x_j) < \epsilon,
\]
where, for each \(i\) and \(j\), \(M_{ij}\) is the lub of \(|K(x_{i-1}, x_{j-1}) - K(x, y)|\) for \(x_{i-1} \leq x < x_i\) and \(x_{j-1} \leq y < x_j\).

\[
\int_a^b |dK(p, x)||dg(x)| = 0 \text{ uniformly on } [a, b] \text{ means if } \epsilon > 0 \text{ then there is a subdivision } D \text{ of } [a, b] \text{ such that, if } \{x_i\}^n_0 \text{ is a refinement of } D, \text{ then}
\]
\[
\sum^n_{i=1} M_i G(x_{i-1}, x_i) < \epsilon, \text{ where, for each } i, M_i = \text{ the lub of } |K(p, x_{i-1}) - K(p, x_i)| \text{ for } x_{i-1} \leq x < x_i.
\]

\[
\int_a^b |dK(p, x)||dg(x)| = 0 \text{ uniformly on } [a, b] \text{ means if } \epsilon > 0 \text{ then there is a subdivision } D \text{ of } [a, b] \text{ such that, if } \{x_i\}^n_0 \text{ is a refinement of } D \text{ and } a \leq p \leq b, \text{ then}
\]
\[
\sum^n_{i=1} M_i G(x_{i-1}, x_i) < \epsilon, \text{ where, for each } i, M_i = \text{ the lub of } |K(p, x_{i-1}) - K(p, x_i)| \text{ for } x_{i-1} \leq x < x_i.
\]

The set \(S\) of functions is bounded uniformly on \([a, b]\) means there is a number \(M\) such that, if \(f \in S\) and \(x \in [a, b]\), then \(|f(x)| < M\). The set \(S\)
of functions is quasicontinuous uniformly on \([a, b]\) means \(S\) is bounded uniformly on \([a, b]\) and if \(\epsilon > 0\) then there is a subdivision \(D = \{x_i\}_0^n\) of \([a, b]\) such that, if \(f \in S\) and \(0 < i \leq n\) and \(x_{i-1} < r < t < x_i\) then \(|f(r) - f(t)| < \epsilon\).

**Theorems.** In Theorems 1–5 we develop properties of the Volterra determinant. Theorem 6 gives the solution to the Stieltjes-Volterra integral equation.

**Theorem 1.** If \(A = |a_{ij}|_n^p\) is a Volterra determinant and \(0 < p \leq n\), then

1. \(A = a_{11} \ast A_2 + \ast A_{21};\)
2. \(A = \sum_{i=1}^n a_{ii} \ast A_{i+1} = \sum_{j=1}^n A_{j-1} a_{nj};\)
3. If \(0 < j \leq p\), then \(A_{pj} = a_{pq} \ast A_{p+1} + \ast A_{p+1,j} = \sum_{i=p} a_{ij} \ast A_{i+1};\)
4. \(A = \sum_{j=1}^p A_{j-1} a_{pj} \ast A_{p+1};\)
5. If \(p < n\) and \(B = |b_{ij}|_n^p\) and \(b_{p+1,p+1} = 0\) and \(b_{ij} = a_{ij}\) otherwise, then \(B = \sum_{j=1}^p A_{j-1} a_{pj} \ast A_{p+1};\)
6. \(A = \sum_{i=1}^n A_{i+1} a_{i1} \ast A_{i+1}.\)

Each item in Theorem 1 can be proved using the definition of a determinant or by mathematical induction. Note that \(A_0 = 1\) and \(\ast A_{n+1} = 1.\)

**Theorem 2.** If \(A\) is a delta determinant defined by the sequences \(\{c_{ij}\}_n^p\) and \(\{g_{ij}\}_n^n\) and \(c_{ij} \leq m\) for \(i, j = 1, 2, \cdots, n\), then

\[|A| \leq m |g_i - g_0| II^*_2(1 + m |g_i - g_{i-1}|).\]

**Proof.** (by induction) If \(A\) is a \(2 \times 2\) delta determinant defined by \(\{c_{ij}\}_n^p\) and \(\{g_{ij}\}_n^n\), then

\[|A| = |c_{11}\Delta g_1 c_{22}\Delta g_2 + c_{21}\Delta g_1| \leq |\Delta g_1||c_{11}|c_{22}|\Delta g_2| + |c_{21}| \leq m |\Delta g_1|(1 + m |\Delta g_2|).\]

Suppose that the theorem is true for \(n = p\) and \(A\) is a \((p + 1) \times (p + 1)\) delta determinant defined by \(\{c_{ij}\}_p^{p+1}\) and \(\{g_{ij}\}_p^{p+1}\); then \(A = c_{11}\Delta g_1 \ast A_2 + \ast A_{21}\) (Th. 1a). Since \(\ast A_2\) and \(\ast A_{21}\) are \(p \times p\) delta determinants which have \(\Delta g_2\) and \(\Delta g_1\) as factors of each element of the first columns, respectively, then

\[|A| \leq |c_{11}| |\Delta g_1||\ast A_2| + |\ast A_{21}| \leq |c_{11}| |\Delta g_1|[m |\Delta g_2| II^{p+1}_2(1 + m |\Delta g_1|)] + m |\Delta g_1| II^{p+1}_2(1 + m |\Delta g_1|) \leq m |\Delta g_1| II^{p+1}_2(1 + m |\Delta g_1|).\]
THEOREM 3. \(\text{If } m > 1 \text{ and } dg \in OB^0 \text{ on } [a, b], \text{ then there is a number } M \text{ such that } M \text{ is an } m\text{-bound for } V(, , , \Delta g) \text{ on } [a, b].\)

Proof. Suppose that \(m > 1\) and \(k\) is a bound for \(|g|\) on \([a, b]\). Let \(M = P + Q\), where \(P = m^2(1 + 2k)\) and \(Q = \int_a^b \int_a^b m \, dg \, \exp \int_a^b m \, dg \).

Let \(D = \{x_j\}_{j=0}^n\) be a subdivision of a subinterval of \([a, b]\) and let \(K\) and \(h\) be functions such that \(m\) bounds \(|K|\) on \([a, b] \times [a, b]\) and \(|h|\) on \([a, b]\).

Let \(A = |\alpha_i| = V(D, h, K, \Delta g)\); then \(*A_{i+1}\) is a delta determinant for \(i = 1, 2, \ldots, n - 1\). Hence, for \(i = 1, 2, \ldots, n\) and \(1 < j \leq i\), \(P = m^2 + 2m^2k > |K_{i,j-1}| |\Delta g_i| \geq |\alpha_i|\) and \(P = m^2 + 2m^2k > |h_i| + |h_{i0}| |K_{i0}| |\Delta g_i| \geq |\alpha_i|\); hence, if \(0 < i \leq n\) and \(0 < j \leq n\), then \(|\alpha_{ij}| < P\). Therefore,

\[
|A| \leq \sum_{i=1}^n |\alpha_i| |*A_{i+1}| + |a_n| \quad \text{(Th. 1b)}
\]

\[
< \sum_{i=1}^n P |*A_{i+1}| + P
\]

\[
\leq \sum_{i=1}^n Pm |\Delta g_{i+1}| \left| \sum_{i=2}^n 1 + m |\Delta g_i| \right| + P \quad \text{(Th. 2)}
\]

\[
\leq Pm \int_a^b \int_a^b m \, dg \, \exp \int_a^b m \, dg + P = Q + P = M.
\]

Similarly, \(|A_p| < M\), \(|*A_p| < M\) and \(|*A_{pj}| < M\) for \(1 \leq p \leq n\) and \(j = 1, 2, \ldots, p\).

THEOREM 4. \(\text{If } \{f_i\}^n_1 \text{ and } \{a_i\}^n_1 \text{ are sequences of elements of } N \text{ and } A = |a_i| = \text{an } n \times n \text{ Volterra determinant}, \text{ then the following statements are equivalent.} \)

1. \(f_i = a_i\) \(1 \leq i \leq n; \) and
2. \(f_i = A_i \text{ for } 0 < i \leq n.\)

Proof. If \(0 < i \leq n\), it follows from Theorem 1b that \(A_i = a_i + \sum_{i=2}^n A_{i-1}a_{ij}\); therefore, 1 \(\rightarrow 2\) by induction and 2 \(\rightarrow 1\) by induction.

THEOREM 5. \(\text{If } g \text{ is a function and } m \text{ is a number such that } m > 1 \text{ and } dg \in OB^0 \text{ on } [a, b], \text{ then there is a number } Q \text{ such that, if } m \text{ bounds the functions } H \text{ and } K \text{ on } [a, b] \times [a, b] \text{ and } h \text{ and } k \text{ on } [a, b] \text{ and } D = \{x_i\}_0^n \text{ is a subdivision of a subinterval of } [a, b], \text{ then} \)

\[
|A - B| \leq Q \sum_{i=1}^n |a_{pj} - b_{pj}| \left| g(x_{p+1}) - g(x_p) \right|,
\]

where \(A = |a_i| = V(D, h, H, \Delta g), \quad B = |b_i| = V(D, k, K, \Delta g)\) and \(|\Delta g_{n+1}| = 1.\)

Proof. Let \(g\) be a function and \(m\) be a number such that \(dg \in OB^0\) on \([a, b]\) and \(m > 1\). It follows from Theorem 3 that there is a number \(M\)
which is an \( m \)-bound for \( V(\cdot, \cdot, \cdot, \Delta g) \) on \([a, b]\). Let \( Q = Mm \exp\left( m \int_{a}^{b} |dg| \right) \). Let \( H, K, h \) and \( k \) be functions which are bounded by \( m \) on \([a, b]\).

First we will consider a special case. Suppose that \( D = \{x_{i}\}_{0}^{n} \) is a subdivision of a subinterval of \([a, b]\), \( A = |a_{ij}|^{*} = V(D, h, H, \Delta g) \), \( B = |b_{ij}|^{*} = V(D, k, K, \Delta g) \), \( 1 \leq p \leq n \), and \( a_{ij} = b_{ij} \) for \( i \neq p \); then \( A - B = |a_{ij}|^{*} - |b_{ij}|^{*} \) is an \( n \times n \) determinant \( C = |c_{ij}|^{*} \) such that \( c_{ij} = a_{ij} \) for \( i \neq p \), \( c_{pi} = a_{pi} - b_{pi} \) for \( 1 \leq j \leq p \) and \( c_{p,p+1} = 0 \) for \( p < n \).

If \( p = n \), then

\[
|A - B| = |C| = |\Sigma_{j=1}^{n} C_{j-1} c_{nj}|
\]

(Th. 1b)

\[
\leq \Sigma_{j=1}^{n} M |a_{nj} - b_{nj}| \leq Q \Sigma_{j=1}^{n} |a_{nj} - b_{nj}| |\Delta g_{n+1}|.
\]

If \( 1 \leq p < n \), then

\[
|A - B| = |C| = |\Sigma_{j=1}^{p} C_{j-1} c_{pj}^{*} C_{p+1}|
\]

(Th. 1e)

\[
\leq \Sigma_{j=1}^{p} M |a_{pj} - b_{pj}| |C_{p+1}|
\]

\[
\leq \Sigma_{j=1}^{p} M |a_{pj} - b_{pj}| m |\Delta g_{p+1}| \exp \int_{a}^{b} m |dg| \quad \text{(Th. 2)}
\]

\[
\leq Q \Sigma_{j=1}^{p} |a_{pj} - b_{pj}| |\Delta g_{p+1}|.
\]

We will now prove the general case. Suppose that \( D = \{x_{i}\}_{0}^{n} \) is a subdivision of \([a, b]\) and that \( A \) and \( B \) are the determinants \( A = |a_{ij}|^{*} = V(D, h, H, \Delta g) \) and \( B = |b_{ij}|^{*} = V(D, k, K, \Delta g) \). There exists a sequence \( \{R_{p}\}_{0}^{n} \) of \( n \times n \) determinants such that \( A = R_{n}, B = R_{n}, \) and \( A - B = \Sigma_{p=1}^{n} (R_{p-1} - R_{p}) \) and such that, if \( a < p \leq n \) and \( R_{p-1} = |u_{ij}|^{*} \) and \( R_{p} = |v_{ij}|^{*} \), then \( u_{ij} = v_{ij} \) for \( i \neq p \). For each integer \( p \), \( 0 < p \leq n \), \( R_{p-1} - R_{p} \) is the difference of two determinants as defined in the special case above; therefore,

\[
|A - B| = |\Sigma_{p=1}^{n} (R_{p-1} - R_{p})| \leq \Sigma_{p=1}^{n} |R_{p-1} - R_{p}|
\]

\[
\leq \Sigma_{p=1}^{n} Q \Sigma_{j=1}^{p} |a_{pj} - b_{pj}| |\Delta g_{p+1}|.
\]

**Theorem 6.** Given \( K \) is a bounded function from \( R \times R \) to \( \mathbb{N} \) and \( f, h \) and \( g \) are functions from \( R \) to \( \mathbb{N} \) and \( dg \in OB^{0} \) on \([a, b]\). Conclusion. The following statements are equivalent:

1. \( (f, K, g) \in OA^{*} \) on \([a, b]\) and, if \( x \in [a, b] \), then

\[
f(x) = h(x) + (L) \int_{a}^{x} f(t)K(x, t)dg(t);
\]
Proof of $1 \rightarrow 2$. Suppose that $m$ is a bound for $K$ and $\epsilon > 0$. Since $(f, K, g) \in OM^*$ on $[a, b]$, there exists a subdivision $H$ of $[a, b]$ such that, if $H' = \{x_i\}_{i=0}^n$ is a refinement of $H$ and $0 < i \leq n$ and $x = x_i$, then

\[
\left| (L) \int_a^x f(t)K(x, t)dg(t) - \sum_{j=1}^{i-1} f(x_{j-1})K(x, x_{j-1})\Delta g_i \right| < \epsilon/M,
\]

where $M = 4 \left[ m \int_a^b |dg| \exp \int_a^b m |dg| + 1 \right]$. Let $x \in (a, b]$ and let $H' = \{x_i\}_{i=0}^n$ be any refinement of $H$ such that $x \in H'$; let $x = x_p$ and $D = \{x_i\}_{i=0}^p$ where $0 < p \leq n$. For each integer $i$ such that $0 < i \leq p$ there exists an element $e_i \in N$ such that

\[
f(x_i) = h(x_i) + (L) \int_a^{x_i} f(t)K(x, t)dg
\]

\[
= h(x_i) + \sum_{j=1}^{i-1} f(x_{j-1})K(x, x_{j-1})[g(x_j) - g(x_{j-1})] + \epsilon_i
\]

\[
= (h_i + \epsilon_i + f_oK_o\Delta g_i) + \sum_{j=2}^{i-1} f_{j-1}K_{j-1}\Delta g_{j-1},
\]

where $\sum_{j=2}^{i-1} (\cdot) = 0$. Let $\epsilon$ be a function such that $\epsilon(a) = 0$ and $\epsilon(x_i) = \epsilon_i$ for $i = 1, 2, \cdots, p$. Let $V(D, h + \epsilon, K, \Delta g) = |v_i|^p$; then $|v_i|^p$ is a Volterra determinant such that $v_{ij} = h_i + \epsilon_i + h_oK_o\Delta g_i$ for $i = 1, 2, \cdots, p$, and $v_i = K_{i-1}\Delta g_{i-1}$ for $1 < j \leq i \leq p$. Hence, $f_i = v_{11}$ and $f_i = v_{11} + \sum_{j=2}^{i-1} f_{j-1}v_{i-1}$ for $1 < i \leq p$. Therefore,

\[
f(x) = f(x_p) = V(D, h + \epsilon, K, \Delta g)
\]

\[
= V(D, h, K, \Delta g) + V(D, \epsilon, K, \Delta g).
\]

Let $A = |a_o|^p = V(D, \epsilon, K, \Delta g)$, then $A_{i+1}$ is a delta determinant for $i = 1, 2, \cdots, p - 1$, and

\[
|A| = \left| \sum_{i=0}^p \epsilon_iA_{i+1} \right| \leq \sum_{i=0}^{p-1} |\epsilon_i| |A_{i+1}| \exp \int_a^b m |dg| + |\epsilon_p| < \epsilon,
\]

(Th. 2).

Therefore, $|f(x) - V(D, h, K, \Delta g)| = |A| < \epsilon$. Since $x$ is an arbitrary element of $(a, b]$ and $H'$ is an arbitrary refinement of $H$ containing $x$, it follows that $V(a, x; h, K, dg) = f(x)$ for $x \in [a, b]$ and that $(h, K, g) \in OM^*$ on $[a, b]$.

Proof of $2 \rightarrow 1$. Suppose that $\epsilon > 0$. Since $(h, K, g) \in OM^*$ on
There exists a subdivision $H$ of $[a, b]$ such that if $H' = \{x_i\}_0^n$ is a refinement of $H$ and $0 < i \leq n$, then

$$|f(x_i) - V(D_n, h, K, \Delta g)| < \varepsilon/2 \left(1 + m \int_a^b |dg|\right),$$

where $m$ is a bound for $K$ and $D_i = \{x_i\}_0^n$. Let $x \in (a, b]$ and let $H' = \{x_i\}_0^n$ be a refinement of $H$ such that $x \in H'$. Let $x = x_p$ and $D = \{x_i\}_0^p$, where $0 < p \leq n$. Then there is a sequence $\{\varepsilon_i\}_1^p$ such that $f(x_i) - \varepsilon_i = V(D_i, h, K, \Delta g)$ for $0 < i \leq p$, where $D_i = \{x_i\}_0^i$. Let $A = |a_0|_i^p = V(D, h, K, \Delta g)$; then $A_i = V(D_i, h, K, \Delta g)$ for $j = 1, 2, \cdots, p$, and

$$f(x) = f(x_p) = V(a, x; h, K, dg) = V(D, h, K, \Delta g) + \varepsilon_p$$

$$= a_p + \sum_{j=2}^p A_{j-1}a_{p_j} + \varepsilon_p$$

$$= (h_p + h_0K_p\Delta g_1) + \sum_{j=2}^p V(D_{j-1}, h, K, \Delta g)K_{p,j-1}\Delta g_j + \varepsilon_p$$

$$= (h_p + h_0K_p\Delta g_1) + \sum_{j=2}^p (f_{j-1} - \varepsilon_{j-1})K_{p,j-1}\Delta g_j + \varepsilon_p.$$

Since $h_p = h(x)$ and $h_0 = f_0 = f(a)$, then

$$|f(x) - h(x) - \sum_{j=1}^p f_{j-1}K_{n,j-1}\Delta g_j| \leq |\varepsilon_p| + \sum_{j=1}^p |f_{j-1}K_{n,j-1}\Delta g_j|$$

$$\leq |\varepsilon_p| + \left[\varepsilon/2/\left(1 + m \int_a^b |dg|\right)\right] < \varepsilon.$$

Since $x$ is an arbitrary element of $(a, b]$ and $H'$ is an arbitrary refinement of $H$ containing $x$, then $f(x) - h(x) = (L)\int_a^x f(t)K(x, t)dg(t)$ for $x \in [a, b]$ and $(f, K, g) \in OA^*$ on $[a, b]$.

In the next three theorems, we prove a set of sufficient conditions for a function triple $(h, K, g)$ to belong to each of $OA^*$, $OM^*$ and $OM^{**}$ and show that, with appropriate restrictions,

$$V(a, b; h, K, dg) = h(b) - (L)\int_a^b h(t)dV(t, b; 1, K, dg).$$

The following lemma is used in the proofs of these theorems.

**Lemma.** Given $f$ is a function from $R$ to $N$ and if $\varepsilon > 0$ then there is a subdivision $D = \{x_i\}_0^n$ of $[a, b]$ such that, if $0 < i \leq n$ and $x_{i-1} < x < y < x_n$ then $|f(x) - f(y)| < \varepsilon$. Conclusion. The function $f$ is quasicontinuous on $[a, b]$.

**Theorem 7.** Given. (1) The functions $f$ and $K$ are bounded and
THE SOLUTION OF A STIELTJES-VOLterra INTEGRAL EQUATION

\[ dg \in \Omega \delta \text{ on } [a, b] \text{ and } F(x) = (L) \int_a^x f(t)K(x, t)dg(t) \text{ exists for } a \leq x \leq b; \text{ and (2) if } \epsilon > 0 \text{ then there is a subdivision } D = \{x_i\}^n_0 \text{ of } [a, b] \text{ such that, if } 0 < i \leq n \text{ and } x_{i-1} < x < y < x, \text{ and } \{t_j\}^n_m \text{ is a refinement of } D \text{ such that } t_i \leq \{t_j\}^n_m \text{ and } y = t_n, \text{ then}

\[ |\sum_i f(t_{i-1})[K(x, t_{i-1}) - K(y, t_{i-1})][g(t_i) - g(t_{i-1})]| < \epsilon. \]

Conclusion. (1) The function \( F \) is quasicontinuous on \([a, b]\); and (2) \((f, K, g) \in OA^* \text{ on } [a, b]\).

Proof of Conclusion 1. Suppose that \( \epsilon > 0 \) and \( M \) is a bound for \(|f||K|\). There is a subdivision \( D = \{x_i\}^n_0 \text{ of } [a, b] \text{ such that, if } 0 < p \leq n \) and \( x \) and \( y \in (x_{p-1}, x_p) \), then \( \int_y^x |dg| < \epsilon/4M \) and, if \( \{t_j\}^n_m \) is any refinement of \( D \) and \( y = t_n \), then

\[ |\sum_i f(t_{i-1})[K(x, t_{i-1}) - K(y, t_{i-1})][g(t_i) - g(t_{i-1})]| < \epsilon/4. \]

Let \( p \) be an integer and \( x \) and \( y \) be numbers such that \( 0 < p \leq n \) and \( x \) and \( y \in (x_{p-1}, x_p) \). There is a refinement \( D' = \{t_j\}^n_0 \) of \( D \) and integers \( r \) and \( s \) such that \( x = t_r \), \( y = t_s \), and such that \( |A| < \epsilon/4 \) and \( |B| < \epsilon/4 \), where

\[ A = F(x) - \sum_i f(t_{i-1})K(x, t_{i-1})[g(t_i) - g(t_{i-1})], \quad \text{and} \]
\[ B = F(y) - \sum_i f(t_{i-1})K(y, t_{i-1})[g(t_i) - g(t_{i-1})]. \]

Hence,

\[ |F(y) - F(x)| \leq |A| + |B| + |C| + |E| < \epsilon, \]

where

\[ C = \sum_i f(t_{i-1})[K(y, t_{i-1}) - K(x, t_{i-1})][g(t_i) - g(t_{i-1})], \quad \text{and} \]
\[ E = \sum_{i-1} f(t_{i-1})K(y, t_{i-1})[g(t_i) - g(t_{i-1})], \]

and \(|C| < \epsilon/4 \) and \(|E| < \epsilon/4 \). Therefore, \( F \) is quasicontinuous on \([a, b]\).

Proof of Conclusion 2. Let \( \epsilon > 0 \) and \( M \) be a bound for \(|f||K|\). Since \( F \) is quasicontinuous on \([a, b]\), then there is a subdivision \( H_1 = \{z_i\}^n_0 \text{ of } [a, b] \) which is a refinement of the subdivision \( D \) defined above and such that, if \( 0 < p \leq m \) and \( z_{p-1} < x < y < z_p \), then \(|F(x) - F(y)| < \epsilon/4 \). Let \( H_2 \) be an interpolating sequence for \( H_1 \) and let \( H \) be a refinement of \( H_1 \cup H_2 \) such that, if \( H' = \{y_i\}^n_0 \) is a refinement of \( H \) and \( y_q \in H_1 \cup H_2 \), then
We now show that this subdivision $H$ satisfies the definition for $OA^*$. Let $H' = \{t_i\}_{i=0}^p$ be a refinement of $H$ and let $x = t_i \in H'$. If $x \notin H_1 \cup H_2$, then the $OA^*$ inequality $| \cdot | < \epsilon$ is satisfied. Suppose that $x \notin H_1 \cup H_2$; then there exist $y = t \in H_2$ and $z_{i-1}, z_i \in H_1$ such that $x$ and $y \in (z_{i-1}, z_i)$. For convenience we will assume that $x < y$. Hence,

$$|F(x) - \Sigma_i f(t_{i-1})K(x, t_{i-1})[g(t_i) - g(t_{i-1})]| < \epsilon/4,$$

where $A = F(x) - F(y)$ and $B, C$ and $E$ are defined as in Conclusion 1 of this proof. If $x > y$, the steps would be similar. Therefore, $(f, K, g) \in OA^*$ on $[a, b]$.

**Theorem 8.** Given. The function $K$ is bounded on $[a, b] \times [a, b]$ and on $[a, b]$ the functions $h$ and $g$ have bounded variation, the set

$$\left\{ F_q \mid q \in [a, b], F_q = \int_a^x |dK(t, q)| \right\}$$

of functions is quasicontinuous uniformly and $F(x) = V(a, h; K, dg)$ exists. Conclusion. (1) $F$ is quasicontinuous on $[a, b]$; and (2) $(h, K, g) \in OM^*$ on $[a, b]$.

Proof of Conclusion 1. Suppose that $0 < \epsilon < 1$ and $m > 1$ is a bound for $h$ and $K$; then there is a number $M$ which is an $m$-bound for $V(\cdot, \cdot, \Delta g)$ on $[a, b]$ and a number $Q > 1$ which has the properties stated in Theorem 5. There is a subdivision $D = \{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $0 < i \leq n$ and $x_{i-1} < x < y < x_i$, then

$$\int_x^y |dK(t, q)| < \epsilon/8mM \left( \int_a^b |dg| + 1 \right) \quad \text{for} \quad q \in [a, b],$$

$$\int_x^y |dg| < \epsilon/8QM, \quad \text{and} \quad \int_x^y |dh| < \epsilon/8M.$$

Suppose that $0 < i \leq n$ and that $x_{i-1} < x < y < x_i$; then there is a refinement $\{z_i\}_{i=0}^p$ of $D$ such that $x = z_p$ and $y = z_q$ and such that $|F(x) - V(P, h, K, \Delta g)| < \epsilon/8$ and $|F(y) - V(R, h, K, \Delta g)| < \epsilon/8$, where $P = \{z_i\}_{i=0}^p$ and $R = \{z_i\}_{i=0}^q$. Let $A = a_{ij} = V(R, h, K, \Delta g)$; then $V(P, h, K, \Delta g) = a_{ij} A_{ij}$. Let $B = |b_{ij}|$ be the $q \times q$ determinant such that $b_{ij} = 0$ for $p < j \leq i \leq q$ and $b_{ij} = a_{ij}$ otherwise. It follows from Theorem 5 that
THE SOLUTION OF A STIELTJES-VOLTERRA INTEGRAL EQUATION

\[ |A - B| \leq Q(\sum_{i=1}^{q} \sum_{j=1}^{\Gamma_i} a_{ij} - b_{ij}||g(z_{i+1}) - g(z_i)|| + \sum_{i=1}^{q} \sum_{j=1}^{\Gamma_i} a_{ij}||\Delta g_{i+1}|| + \sum_{i=1}^{q} \sum_{j=1}^{\Gamma_i} a_{ij}||\Delta g_{i+1}||)
\]

\[ = Q(\sum_{i=1}^{q} \sum_{j=1}^{\Gamma_i} a_{ij}||\Delta g_{i+1}|| + \sum_{i=1}^{q} \sum_{j=1}^{\Gamma_i} K_{i,j-1}||\Delta g_{i+1}||)
\]

\[ \leq Q \left[ M \left( \int_{\alpha}^{\beta} |dg| \right)^2 + M \int_{\alpha}^{\beta} |dg| \right] < \epsilon/4. \]

Also, \( B_{j-1} = A_{j-1} \) and \( *B_{pj} = a_{qj} \) (Th. 1f) for \( j = 1, 2, \ldots, p \); therefore,

\[ |B - A_p| = |\sum_{i=1}^{\rho} B_{i-1}*B_{pj} - \sum_{i=1}^{\rho} A_{j-1}a_{pj}| \leq \sum_{i=1}^{\rho} |A_{j-1}||B_{pj} - a_{pj}| \leq M\sum_{i=1}^{\rho} |A_{j-1}a_{pj}|
\]

\[ \leq M( |h_q - h_p| + |h_p - K_{q0} - K_{p0}| ||\Delta g_i|| + \sum_{i=2}^{p} |K_{q,j-1} - K_{p,j-1}||\Delta g_i||)
\]

\[ \leq M \left[ \epsilon/8M + \sum_{i=1}^{\rho} \left( \epsilon/8mM \left( \int_{\alpha}^{\beta} |dg| + 1 \right) \right) |\Delta g_i|| \right] < \epsilon/4. \]

Hence,

\[ |F(y) - F(x)| \leq |F(y) - A| + |A - B| + |B - A_p| + |A_p - F(x)| \leq \epsilon/8 + \epsilon/4 + \epsilon/4 + \epsilon/8 < \epsilon. \]

The proof of Conclusion 2 is similar to the proof of Conclusion 2 of Theorem 7.

**Theorem 9.** Given. The function \( K \) is bounded on \([a, b] \times [a, b]\) and on \([a, b]\) \( g \) has bounded variation and \( F(x) = V(a, b; 1, K, dg) \) exists, where \( 1 \) denotes the identity function. Conclusion. (1) \( F \) is quasicontinuous on \([a, b]\); (2) \( (1, K, g) \in OM\) on \([a, b]\); and (3) if \( dh \in O B^0 \) on \([a, b]\), then \( V(a, b; h, K, dg) \) exists and is \( h(b) - (L) \int_{a}^{b} h(t)dF(t) \).

**Proof of Conclusion 1.** Let \( \epsilon > 0 \), let \( M \) be a bound for \( |K| \) and let \( Q \) be a number having the properties defined in Theorem 5. Since \( dg \in O B^0 \), there is a subdivision \( D = \{x_i\} \) of \([a, b]\) such that, if \( 0 < i \leq m \) and \( x_{i-1} < x < y < x_n \), then \( \int_{x}^{y} |dg| < \epsilon/6QM \left( 1 + \int_{a}^{b} |dg| \right) \). Suppose that \( 0 < r \leq m \) and \( x_{r-1} < x < y < x_r \). Since \( F(x) = V(x, b; 1, K, dg) \) and \( F(y) = V(y, b; 1, K, dg) \) exist, then there exists a subdivision \( R = \{x_i\} \) of \([x, b]\) and an integer \( p \) such that \( 0 < p < n \) and a subdivision \( P = \{y_i\} \) of \([y, b]\) such that \( x = z_0, y = z_{p-1}, |F(x) - V(R, 1, K, \Delta g)| < \epsilon/6, \) and \( |F(y) - V(P, 1, K, \Delta g)| < \epsilon/6. \)
Let \( A = V(R, 1, K, \Delta g) = |a_{ij}| \) and \( C = V(P, 1, K, \Delta g) = |a_{ij}|^p = *A_p \); let \( B = |b_{ij}| \) be the \( n \times n \) determinant such that \( b_{ii} = a_{pp} \) for \( p \leq i \leq n \), \( b_{ij} = 0 \) for \( j < i \) \( \leq p \), \( b_{ij} = b_{ij} \) for \( p < i \leq n \) and \( 2 \leq j \leq p \), and \( b_{ij} = a_{ij} \) otherwise. In the following manipulations, \( |\Delta g_{i+1}| = 1 \) and \( |a_{ij}| \) denotes the norm of the element \( a_{ij} \); hence,

\[
|A - B| \leq Q \sum_{i=1}^n \sum_{j=1}^n |a_{ij} - b_{ij}| |\Delta g_{i+1}|
\]  
(Th. 5)

\[
= Q\left[|a_{n1} - b_{n1}| + \sum_{j=2}^n |a_{nj}|\right] + Q \sum_{i=p+1}^{n-1} |a_{i1} - b_{i1}| |\Delta g_{i+1}|
+ Q\left[|a_{i1} - b_{i1}| + \sum_{j=p+1}^{n-1} |a_{ij} - b_{ij}| |\Delta g_{i+1}|\right]
+ Q\left[|a_{i1} - b_{i1}| + \sum_{j=p+1}^{n-1} |a_{ij} - b_{ij}| |\Delta g_{i+1}|\right]
\]

\[
\leq 2QM \int_a^b |dg| + QM(|\Delta g_1| + |\Delta g_p|) \int_a^b |dg|
+ QM \int_a^b |dg| \int_a^b |dg|
\]

\[
< \epsilon/3 + \epsilon/6 + \epsilon/6 = 2\epsilon/3.
\]

It follows from Theorem 1f that \( B = *A_p = C \); hence,

\[
|F(x) - F(y)| \leq |F(x) - A| + |A - B| + |B - C| + |C - F(y)|
\]

\[
< \epsilon/6 + 2\epsilon/3 + 0 + \epsilon/6 = \epsilon.
\]

Therefore, \( F \) is quasicontinuous on \([a, b]\).

The proof of Conclusion 2 is similar to the proof of Conclusion 2 of Theorem 7.

**Proof of Conclusion 3.** Suppose that \( \epsilon > 0 \). Since \( dh \in OB^0 \) and \( F \) is quasicontinuous, then \((R) \int_a^b dhF\) exists. Since \((R) \int_a^b dhF\) and \( V(a, b; 1, K, dg) \) exist and \((h, K, g) \in OM^{**}\), there exists a subdivision \( D \) of \([a, b]\) such that if \( D' = \{x_i\}^n_i \) is a refinement of \( D \) and \( 0 < i \leq n \), then

\[
|\left(\begin{array}{c} (R) \int_a^b dhF - \sum_i \Delta h_i F_i \end{array}\right)| < \epsilon/3
\]

and

\[
|F(x_i) - V(D, 1, K, \Delta g)| < \epsilon/3 \left(\int_a^b |dh| + 1\right)
\]
for \( i = 0, 1, 2, \ldots, n \), where \( D_t = \{ x_p \}_{p=0}^n \) be a refinement of \( D \) and let \( D_r = \{ x^r_p \}_{p=0}^n \) for \( i = 1, 2, \ldots, n \). Also, let \( V(D, h, K, \Delta g) = A = \left| a_y \right|_1 \) and let \( V^*(D, h, K, \Delta g) = B = \left| b_y \right|_1 \) be the \( n \times n \) Volterra determinant such that (1) \( b_{11} = a_{11} \), and (2) if \( 1 < i \leq n \), then \( b_{ij} = a_{ij} - a_{i-1,j} \) for \( j = 1, 2, \ldots, n \). Note that \( A = \left| a_y \right|_1 \) can be transformed into \( B = \left| b_y \right|_1 \) by adding the negative of the elements of the \( n - 1 \)st row of \( A \) to the \( n \)th row of \( A \), the negative of the elements of the \( n - 2 \)nd row to the \( n - 1 \)st row, etc. Hence, \( A = B \), and for \( i = 1, 2, 3, \ldots, n \), the determinant \( *B_{i+1} \) can be transformed into \( V(D, 1, K, \Delta g) \) by adding the elements of the first row of \( *B_{i+1} \) to the 2nd row, the elements of the new 2nd row to the 3rd row, etc. Hence, there exists an element \( a \) of \( N \) such that \( |a| < \epsilon \) and

\[
V(D, h, K, \Delta g) = \left| b_y \right|_1 = \sum_{i=1}^{n} b_{yi} * B_{i+1}^{*} \quad \text{(Th. 1b)}
\]

\[
= (h_0 + h_0 K_{10} \Delta g_1) * B_2 + \sum_{i=2}^{n} \Delta h_i + (h_0 K_{10} - K_{11,0}) \Delta g_1 * B_{i+1}
\]

\[
= h_0 (1 + K_{10} \Delta g_1) * B_2 + \sum_{i=2}^{n} (K_{10} - K_{11,0}) \Delta g_1 * B_{i+1} + \sum_{i=1}^{n} \Delta h_i * B_{i+1}
\]

\[
= h_0 V(D, 1, K, \Delta g) + \sum_{i=1}^{n} \Delta h_i F(x_i) + \sum_{i=1}^{n-1} \Delta h_i [ *B_{i+1} - F(x_i) ]
\]

\[
= h_0 V(a, b; 1, K, \Delta g) + (R) \int_a^b dhF + \alpha
\]

\[
= h(a) F(a) + h(x) F(x) |b_a^b - (L) \int_a^b hdF + \alpha
\]

\[
= h(b) - (L) \int_a^b hdF + \alpha.
\]

Therefore, \( V(a, b; h, K, \Delta g) \) exists and

\[
V(a, b; h, K, \Delta g) = h(a) V(a, b; 1, K, \Delta g) + (R) \int_a^b dhF
\]

\[
= h(b) - (L) \int_a^b hV(t, b; 1, K, \Delta g).
\]

In Theorem 11 we prove a set of sufficient conditions for the existence of the limit \( V(a, b; h, K, \Delta g) \). Theorem 10 is a lemma which is used in the proof of Theorem 11.

**Theorem 10.** Given. The symbols \( n, r \) and \( p \) represent positive integers, \( p < n \), and \( A = \left| a_y \right|_1 \) is an \( n \times n \) Volterra determinant and \( B = \left| b_y \right|_1 \) is an \( (n + r) \times (n + r) \) Volterra determinant such that

1. if \( 0 < j \leq i \leq p \), then \( b_{yi} = a_{yi} \);
2. if \( p < i \leq p + r \) and \( 0 < j \leq p \), then \( b_{yi} = a_{yi} \);
3. if \( p + r < i \leq n + r \) and \( 0 < j \leq p \), then \( b_{yi} = a_{i-r,j} \);
4. if \( p + r < i \leq n + r \), then \( \sum_{j=p+r}^{p+r+1} b_{yi} = a_{i-r,p+1} \).
(5) if \( p + r + 1 < j \leq i \leq n + r \), then \( b_{ij} = a_{i-r,j-r} \); and

(6) if \( p < j \leq i \leq p + r \), then \( b_{ij} = 0 \).

**Conclusion.** \( A = B \), where \( A \) and \( B \) represent elements of \( N \).

**Proof.** Note that \( A_i = B_i \) for \( i = 1, 2, \ldots, p \) and \( *A_i = *B_i \) for \( i > p + 1 \). It follows from (6) and (4) above that \( *B_{i+1} = *B_{p+r+1,i+1} \) for \( i = p, p+1, \cdots, p+r \) and \( \sum_{i=p}^{p+r} *B_{i+1} = \sum_{i=p}^{p+r} *B_{p+r+1,i+1} = *A_{p+1} \). Hence,

\[
B = \sum_{j=1}^{p} B_{j-1}(\sum_{i=p}^{p+r} *b_{i,j} *B_{i+1}) \tag{Th. 1d}
\]

\[
= \sum_{j=1}^{p} B_{j-1}(\sum_{i=p}^{p+r} *a_{i,j} *B_{i+1} + \sum_{i=p+r+1}^{p+r} *b_{i,j} *B_{i+1})
\]

\[
= \sum_{j=1}^{p} A_{j-1}(\sum_{i=p}^{p+r} a_{i,j} *B_{i+1} + \sum_{i=p+1}^{p+r} a_{i,j} *A_{i+1}) \tag{2,3,5}
\]

\[
= \sum_{j=1}^{p} A_{j-1}(\sum_{i=p}^{p+r} a_{i,j} *A_{i+1}) = A, \quad \text{(Th. 1d)}
\]

**Theorem 11.** Given \([a, b]\) is a number interval, \( K \) is a bounded function from \( R \times R \) to \( N \) and \( h \) and \( g \) are functions from \( R \) to \( N \) such that \( \frac{dh}{dt} \) and \( \frac{dg}{dt} \in OB_0 \) on \([a, b]\),

\[
\int_{a}^{b} \int_{a}^{b} |dK| \cdot |dg| \cdot |dg| = 0 \quad \text{and} \quad \int_{a}^{b} |dK(b,t)| \cdot |dg(t)| = 0. \]

**Conclusion.** (1) \( V(a, b; h, K, dg) \) exists, and (2) if \( \int_{a}^{b} |dK(x,t)| \cdot |dg(t)| = 0 \) uniformly on \([a, b]\), then on \([a, b]\) the function \( f(x) = V(a, x; h, K, dg) \) exists, \( (h, K, g) \in OM^* \) and \( f \) is the solution of the equation

\[
f(x) = h(x) + (L) \int_{a}^{x} f(t)K(x, t)dg(t).\]

**Proof.** We will show that the limit \( V(a, b; h, K, dg) \) exists by showing that the following Cauchy criterion condition is satisfied: if \( \varepsilon > 0 \) then there is a subdivision \( D \) of \([a, b]\) such that, if \( D' \) is a refinement of \( D \), then \( |V(D, h, K, \Delta g) - V(D', h, K, \Delta g)| < \varepsilon \). Let \( \varepsilon > 0 \) and let \( M \) be a bound for \((1 + |h|)(1 + |K|)(1 + \int_{a}^{b} |dg|)\) on \([a, b]\). It follows from Theorem 5 that there is a number \( Q \) such that, if \( U, W, u \) and \( w \) are functions bounded by \( M \) on \([a, b]\) and \( D = \{x_i\}_0^n \) is a subdivision of \([a, b]\), then

\[
|A - B| \leq Q \sum_{p=1}^{n} \sum_{i=1}^{p} |a_{ip} - b_{ip}| |g(x_{p+1}) - g(x_p)|,
\]

where \( A = |a_{ij}| = V(D, u, U, \Delta g), \ B = |b_{ij}| = V(D, w, W, \Delta g) \) and \( |\Delta g_{n+1}| = 1 \).
Since \( dg \) and \( dh \) lie in \( \mathcal{OB}^\circ \) and \( \int_a^b \int_a^b |dK| \, |dg| \, |dg| = 0 \) and \( \int_a^b |dK(b, t)| \, |dg(t)| = 0 \), there is a subdivision \( D = \{ x_i \}_{i=0}^n \) of \([a, b]\) such that

\[
(1) \quad \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |dg| \int_{x_{i-1}}^{x_i} |dg| < \varepsilon/9MQ,
\]

\[
\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |dh| \int_{x_{i-1}}^{x_i} |dg| < \varepsilon/9Q, \quad \text{and} \quad \int_a^b |dg| < \varepsilon/18MQ;
\]

\[
(2) \quad \sum_{i=1}^n \sum_{j=1}^n M_{ij} \int_{x_{i-1}}^{x_i} |dg| \int_{x_{j-1}}^{x_j} |dg| < \varepsilon/9MQ,
\]

where for each \( i \) and \( j \), \( M_{ij} \) is the lub of \( |K(x_{i-1}, x_{j-1}) - K(x, y)| \) for \( x_{i-1} \leq x < x_i \) and \( x_{j-1} \leq y < x_j \); and

\[
(3) \quad \sum_{i=1}^n M_i \int_{x_{i-1}}^{x_i} |dg| < \varepsilon/9Q,
\]

where for each \( i, M_i \) is the lub of \( |K(b, x_{i-1}) - K(b, x)| \) for \( x_{i-1} \leq x < x_i \).

Let \( D' = \{ z_i \}_{i=0}^m \) be a refinement of \( D \), \( A = |a_y|_n = V(D, h, K, \Delta g) \) and \( B = |b_y|_n = V(D', h, K, \Delta g) \). Let \( \{ n_i \}_{i=0}^n \) be the sequence of integers such that \( x_i = z_{n_i} \) for \( i = 1, 2, \cdots, n \). We now define an \( m \times m \) determinant \( C = |c_{ij}|_n \) such that \( C = A \) and \( |B - C| < \varepsilon \) and, hence, \( |A - C| + |C - B| < \varepsilon \). In the following paragraphs, the symbols \( h_n, K_n \) and \( \Delta g \) represent \( h(z_i), K(z_i, z_j) \) and \( g(z_i) - g(z_{j-1}) \), respectively.

Let \( P_1 \) be the set of integer pairs such that \( i, j \in P_1 \) iff \( i = 1 \) and \( 1 \leq i < n_2 \). Let \( c_y = a_{11} \) for \( i, j \in P_1 \); then

\[
\sum_{i, j \in P_1} |b_{ij} - c_{ij}| \, |\Delta g_{i+1}| \leq \sum_{i=1}^{n_2-1} 2M \, |\Delta g_{i+1}| \leq 2M \int_a^b |dg| < 2M(\varepsilon/18MQ) = \varepsilon/9Q.
\]

Let \( P_2 \) be the set of integer pairs such that \( i, j \in P_2 \) iff \( j = 1 \) and \( n_2 \leq i < m \). If \( i, j \in P_2 \) and \( 2 < p \leq n \) and \( n_{p-1} \leq i < n_p \), then \( c_{ij} = a_{p-1,1} \). Let \( N_p = \{ n_{p-1}, n_p \} \). Since \( x_0 = z_0 = a \), then

\[
\sum_{i, j \in P_2} |b_{ij} - c_{ij}| \, |\Delta g_{i+1}| = \sum_{i=1}^{n_p-3} \sum_{i \in N_p} |h(z_i) + h(z_0)K(z_0, z_0)(\Delta g_1) - h(x_{p-1}) + h(x_0)K(x_0, x_0)(\Delta g_1)|
\]

\[
- K(x_{p-1}, x_0)[g(x_i) - g(x_0)]|\Delta g_{i+1}|
\]

\[
= \sum_{i=1}^{n_p-3} \sum_{i \in N_p} |[h(z_i) - h(x_{p-1})] + h(x_0)[K(z_0, z_0)(g(z_i) - g(z_0)]
\]

\[
- K(x_{p-1}, x_0)[g(x_i) - g(x_0)]|\Delta g_{i+1}|
\]
\[ \leq \sum_{p=3}^{n} \sum_{i \in \mathbb{N}_p} |h(z_i) - h(x_{p-1})| |\Delta g_{i+1}| + \sum_{p=3}^{n} \sum_{i \in \mathbb{N}_p} |h(x_0)| |K(z_n, x_0) - K(x_{p-1}, x_0)| |g(x_i) - g(x_0)| |\Delta g_{i+1}| + |g(z_i) - g(x_i)| |\sum_{p=3}^{n} \sum_{i \in \mathbb{N}_p} h(x_0)| |K(z_n, z_0)| |\Delta g_{i+1}| + (\varepsilon/18MQ) |h(x_0)| |K(z_n, z_0)| \left( \int_a^b |dg| + 1 \right) \leq \varepsilon/9Q + M(\varepsilon/9MQ) + \varepsilon/9Q = \varepsilon/3Q, \]

where \( M_{p0} \) is the lub of \( |K(x_{p-1}, x_0) - K(x, y)| \) for \( x_{i-1} \leq x < x_i \) and \( x_{j-1} \leq y < x_j \).

Let \( P_3 \) be the set of integer pairs such that \( i, j \in P_3 \) iff \( 1 < j \leq n_1 \) and \( j \leq i \leq m \). Let \( c_{ij} = 0 \) for \( i, j \in P_3 \); then

\[ \sum_{i,j \in P_3} |b_{ij} - c_{ij}| |\Delta g_{i+1}| = \sum_{i,j \in P_3} |K(z_n, z_{j-1})| \Delta g_j |\Delta g_{i+1}| \]
\[ \leq \sum_{i,j \in P_3} M |g(z_i) - g(z_{j-1})| \leq M \int_a^b |dg| < M(\varepsilon/9MQ) = \varepsilon/9Q. \]

Let \( P_4 \) be the set of integer pairs such that the pair \( i, j \in P_4 \) iff \( 1 < j \leq n_1 \) and \( n_1 < j \leq m \). Let \( c_{ij} = 0 \) for \( i, j \in P_4 \); then

\[ \sum_{i,j \in P_4} |b_{ij} - c_{ij}| |\Delta g_{i+1}| = \sum_{i,j \in P_4} |K(z_n, z_{j-1})| \Delta g_j |\Delta g_{i+1}| \]
\[ \leq M \sum_{i,j \in P_4} \sum_{p=1}^{n-1} \sum_{i \neq j} |g(z_i) - g(z_{j-1})||g(z_{j-1}) - g(z_j)| \]
\[ \leq M \sum_{i,j \in P_4} \int_{x_{p-1}}^{x_p} |dg| \int_{x_{p-1}}^{x_p} |dg| < M(\varepsilon/9MQ). \]

Let \( P_5 \) be the set of integer pairs such that \( i, j \in P_5 \) iff \( i = m \) and also \( j = 1 \) or \( n_1 < j \leq m \). Let \( c_{mj} = a_{n_1} = h(x_n) + h(x_0)K(x_n, x_0)[g(x_i) - g(x_0)] \) and, if \( 1 < p \leq n \) and \( n_{p-1} < j \leq n_p \) let \( c_{mj} = K(x_n, x_{p-1})[g(z_i) - g(z_{j-1})] \).

Since \( z_m = x_n \) and \( z_0 = x_0 \), then

\[ \sum_{i,j \in P_5} |b_{ij} - c_{ij}| = |b_{m1} - c_{m1}| + \sum_{p=2}^{n} \sum_{i \neq j} |b_{mj} - c_{mj}| \]
\[ = |h(z_m) + h(z_0)K(z_m, z_0)[g(z_i) - g(z_0)] - h(x_n) - h(x_0)K(x_n, x_0)[g(x_i) - g(x_0)]| \]
\[ + \sum_{p=2}^{n} \sum_{i \neq j} |K(z_m, z_{j-1})[g(z_i) - g(z_{j-1})] - K(x_n, x_{p-1})[g(z_i) - g(z_{j-1})]| \]
\[
\leq |h(z_0)| |K(z_m, z_0)||g(x_i) - g(z_i)| \\
+ \sum_{p=2}^{n} \sum_{p=n_{p-1} + 1}^{n} |K(x_m, z_{i-1}) - K(x_m, x_{p-1})||g(z_i) - g(z_{j-1})| \\
\leq M |g(x_i) - g(z_i)| + \sum_{p=2}^{n} \sum_{p=n_{p-1} + 1}^{n} M_p |g(z_i) - g(z_{j-1})| \\
\leq M(\epsilon/18MQ) + \sum_{p=2}^{n} M_p \int_{x_{p-1}}^{x_p} |dg| < \epsilon/3Q,
\]

where \(M_p\) is the lub of \(|K(b, x_{p-1}) - K(b, z)|\) for \(x_{p-1} \leq z < x_p\).

Let \(P_6\) be the set of integer pairs such that \(i, j \in P_6\) iff there are integers \(p\) and \(q\) such that \(2 \leq q < p \leq n\) and such that \(n_{p-1} \leq i < n_p\) and \(n_{q-1} < j \leq n_q\). If \(i, j \in P_6\) and \(n_{p-1} \leq i < n_{p+1}\) and \(n_{q-1} < j \leq n_q\), let 
\[c_i = K(x_{p-1}, x_{q-1})[g(z_i) - g(z_{j-1})];\]
then
\[
\sum_{i,j \in P_6} |b_{ij} - c_{ij}| \Delta g_{i+1} = \sum_{p=2}^{n} \sum_{q=2}^{n} \sum_{p=n_{p-1} + 1}^{n} \sum_{q=n_{q-1} + 1}^{n} |K_{i,j} - K(x_{p-1}, x_{q-1})||\Delta g_{i}||\Delta g_{i+1}| \\
\leq \sum_{p=2}^{n} \sum_{q=2}^{n} \sum_{p=n_{p-1} + 1}^{n} \sum_{q=n_{q-1} + 1}^{n} M_{pq} |\Delta g_{i}||\Delta g_{i+1}| \\
\leq \sum_{p=2}^{n} \sum_{q=2}^{n} \sum_{p=n_{p-1} + 1}^{n} \sum_{q=n_{q-1} + 1}^{n} M_{pq} \int_{x_{p-1}}^{x_p} |dg| \int_{x_{q-1}}^{x_q} |dg| < \epsilon/9Q,
\]

where \(M_{pq}\) is the lub of \(|K(x_{p-1}, x_{q-1}) - K(x, y)|\) for \(x_{p-1} \leq x < x_p\) and \(x_{q-1} \leq y < x_q\).

The determinant \(|c_{ij}|^n\) can be reduced to the determinant \(|a_{ij}|^n\) by the following steps.

(1) If \(n_i > 1\), use Theorem 1f and obtain a determinant of lower order.

(2) For each integer \(p\) such that \(2 < p \leq n\) and \(n_p > n_{p-1} + 1\), use Theorem 10 and the definition of the determinant \(|c_{ij}|^n\) to obtain a determinant of lower order. Note that, if \(1 < n_p \leq i < n_{p+1}\), then
\[
\sum_{i,n_{q-1} + 1}^{n} c_i = K(x_{p-1}, x_{q-1})[g(x_i) - g(x_{q-1})] = a_{pq},
\]

Hence,
\[
|B - A| \leq |B - C| + |C - A| = |B - C| \\
\leq Q \sum_{i=1}^{m} \sum_{j=1}^{n} |b_{ij} - c_{ij}| |\Delta g_{i+1}| \quad \text{(Th. 5)} \\
= Q(\sum_{i=1}^{n} \sum_{j \in P} |b_{ij} - c_{ij}| |\Delta g_{i+1}|) < Q(\epsilon/Q) = \epsilon.
\]

Therefore, if \(\epsilon > 0\) then there is a subdivision \(D\) of \([a, b]\) such that if \(D'\) is a refinement of \(D\), then \(|V(D, h, K, \Delta g) - V(D', h, K, \Delta g)| < \epsilon\); hence, the limit \(V(a, b; h, K, dg)\) exists.
Suppose that \( \int_a^b |dK(, t)||dg(t)| = 0 \) uniformly on \([a, b]\). If \( a < x \leq b \), it follows from Conclusion 1 that \( V(a, x; h, K, dg) \) exists. We now prove that \( (h, K, g) \in OM^* \) on \([a, b]\). Let \( \epsilon > 0 \) and define a subdivision \( D \) of \([a, b]\) in the same manner as in Conclusion 1 except that \( \int_a^b |dK(b, t)||dg(t)| = 0 \) uniformly is used in defining \( D \) in place of \( \int_a^b |dK(b, t)||dg(t)| = 0 \).

If \( \{x_i\}_{i=0}^n \) is a refinement of \( D \) and \( 0 < p \leq n \), then a repetition of the steps in the proof of Conclusion 1 shows that, if \( Q' \) is a refinement of \( Q = \{x_i\}_{i=0}^n \), then

\[
|V(Q', h, K, \Delta g) - V(Q, h, K, \Delta g)| < \epsilon.
\]

Since \( V(a, x_p; h, K, dg) \) exists, there is a refinement \( Q' \) of \( Q \) such that \( |V(Q', h, K, \Delta g) - V(a, x_p; h, K, dg)| < \epsilon \); hence,

\[
|V(a, x_p; h, K, dg) - V(Q, h, K, \Delta g)| \leq |V(a, x_p; h, K, dg) - V(Q')| + |V(Q') - V(Q)| < 2\epsilon.
\]

Therefore, \( (h, K, g) \in OM^* \) on \([a, b]\). It follows from Theorem 3 that \( f \) is bounded on \([a, b]\) and from Theorem 6 that \( f \) is the solution on \([a, b]\) of the equation \( f(x) = h(x) + (L) \int_a^x f(t)K(x, t)dg(t) \).

REFERENCES


Received October 15, 1974 and in revised form February 11, 1976.

Southwest Texas State University
Richard Fairbanks Arnold and A. P. Morse, *Plus and times* ................. 297
Edwin Ogilvie Buchman and F. A. Valentine, *External visibility* ........ 333
R. A. Czerwinski, *Bonded quadratic division algebras* .................... 341
William Richard Emerson, *Averaging strongly subadditive set functions in unimodular amenable groups. II* ........................................... 353
Lynn Harry Erbe, *Existence of oscillatory solutions and asymptotic behavior for a class of third order linear differential equations* .............. 369
Kenneth R. Goodearl, *Power-cancellation of groups and modules* ........ 387
J. C. Hankins and Roy Martin Rakestraw, *The extremal structure of locally compact convex sets* ......................................................... 413
Burrell Washington Helton, *The solution of a Stieltjes-Volterra integral equation for rings* ............................................................ 419
Frank Kwang-Ming Hwang and Shen Lin, *Construction of 2-balanced (n, k, \( \lambda \)) arrays* ................................................................. 437
Wei-Eihn Kuan, *Some results on normality of a graded ring* .......... 455
Dieter Landers and Lothar Rogge, *Relations between convergence of series and convergence of sequences* ........................................... 465
Lawrence Louis Larmore and Robert David Rigdon, *Enumerating immersions and embeddings of projective spaces* .......................... 471
Douglas C. McMahon, *On the role of an abelian phase group in relativized problems in topological dynamics* .................................... 493
Robert Wilmer Miller, *Finitely generated projective modules and TTF classes* ................................................................. 505
Yashaswini Deval Mittal, *A class of isotropic covariance functions* ........ 517
Anthony G. Mucci, *Another martingale convergence theorem* ........... 539
Joan Kathryn Plastiras, *Quasitriangular operator algebras* ............... 543
John Robert Quine, Jr., *The geometry of \( p(S^1) \)* ......................... 551
Tsuang Wu Ting, *The unloading problem for severely twisted bars* .... 559