SOME RESULTS ON NORMALITY OF A GRADED RING

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Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded domain and let \( p \) be a homogeneous prime ideal in \( R \). Let \( R_p \) be the localization of \( R \) at \( p \) and \( R_p = \{ r/s \mid r, s \in R_i \text{ and } s \notin p \} \). If \( R_i \cap (R - p) \neq \emptyset \), then \( R_p \) is a localization of a transcendental extension of \( R_{(p)} \). Thus \( R_p \) is normal (regular) if and only if \( R_{(p)} \) is normal (regular). Let \( \text{Proj}(R) = \{ p \mid p \text{ is a homogeneous prime ideal and } p \not\subset \bigoplus_{i=0} \cdots \} \). Under certain conditions a Noetherian graded domain \( R \) is normal if \( R_p \) is normal for each \( p \in \text{Proj}(R) \). If \( R = \bigoplus_{i \geq 0} R_i \) is reduced and \( F_0 = \{ r/s \mid r, s \in R_i \text{ and } u_i \in U \} \) where \( U \) is the set of all nonzero divisors is Noetherian, then the integral closure of \( R \) in the total quotient ring of \( R \) is also graded.

1. Introduction. Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded integral domain. Let \( \text{Spec}(R) \) be the set of all prime ideals in \( R \). Let \( R_+ = \bigoplus_{i=0} R_i \). \( R_+ \) is an ideal in \( R \). An ideal \( \mathfrak{A} \) in \( R \) is said to be irrelevant if \( R_+ \subset \mathfrak{A} \), the radical of \( \mathfrak{A} \). Let \( \text{Proj}(R) = \{ p \in \text{Spec}(R) \mid p \subset R_+ \text{ is homogeneous and nonirrelevant} \} \). For each \( p \in \text{Spec}(R) \), let \( R_p = \{ r/s \mid r, s \in R \text{ and } s \notin p \} \), and for each homogeneous prime ideal \( p \), let \( R_{(p)} = \{ r/s \mid r, s \in R_i \text{ and } s \notin p \} \). (Note: \( R_{(p)} \) in [1] is defined for \( p \in \text{Proj}(R) \) only.) According to the terminology of Seidenberg [9], \( R_p \) is called the arithmetical local ring of \( R \) at \( p \) and \( R_{(p)} \) the geometrical local ring of \( R \) at \( p \). I prove that if \( R_i \cap (R - p) \neq \emptyset \) then \( R_p \) is the ring of quotients of a transcendental extension of \( R_{(p)} \) relative to a multiplicative set, \( R_p \) is normal (regular) if and only if \( R_{(p)} \) is normal (regular); see Theorem 2. In the case of an irreducible projective variety \( V \) over a field \( k \) in a projective \( n \)-space \( P^n_k \), \( V/k \) is normal if the geometrical local ring of \( V \) at each \( p \in V \), \( \mathcal{O}(p) \) is integrally closed. \( V \) is arithmetically normal if the ring of strictly homogeneous coordinates \( k[V] \) is integrally closed. The latter implies the former. For the converse, various cohomological criteria are developed; see [3], [8], [9]. I attempt to study the normality of a graded domain \( R \) if \( R_{(p)} \) is normal for every \( p \in \text{Proj}(R) \). In this paper, I also obtain the following theorem: Let \( R \) be a Noetherian graded domain, say \( R = R_0[x_1, \ldots, x_n] \) and \( x_1, \ldots, x_n \) are of homogeneous degree 1. Assume that \( R_0 \) contains a field \( k \) over which \( R_0 \) and \( k(x_1, \ldots, x_n) \) are linearly disjoint and separable. Let \( \mathfrak{B} \) be the kernel of the canonical map from the polynomial ring \( R_0[x_1, \ldots, x_n] \). Then \( R \) is normal if \( R_0 \) is normal, \( R_{(p)} \) is normal for every \( p \in \text{Proj}(R) \) and \( \text{coh.d.} \mathfrak{B} \cdot K[X_1, \ldots, X_n] < n - 1 \), where \( K \) is the quotient field of \( R_0 \).
In the §4, we prove that under certain conditions on a graded ring \( R \) (not necessarily integral domain) the integral closure \( \hat{R} \) of \( R \) in the total quotient ring of \( R \) is also graded; see Theorem 6.

Our references on the elementary well known facts about graded rings can be found in [1] and [10].

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2. Normality and regularity of local domains. Let \( R \) be a commutative ring with identity \( 1 \). Let \( p \) be a prime ideal in \( R \). By height of \( p \), we mean the supremum of the length of chains of prime ideals \( p_0 \supsetneq p_1 \supsetneq \cdots \supsetneq p_n \) with \( p_0 = p \) and denote it by \( h(p) \). Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded integral domain. Let \( K \) be the quotient field of \( R \). We say that \( R \) is integrally closed if \( R \) is integrally closed in \( K \).

The following theorem was originally proved in [9] for projective varieties. We observe that the same holds true for non-Noetherian graded domain also.

**Theorem 1.** Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded domain. Let \( p \in \text{Spec}(R) \) be nonhomogeneous. If \( h(p) = 1 \) then \( R_p \) is integrally closed.

**Proof.** Let \( p^* \) be the ideal generated by all the homogeneous elements of \( p \). By [10, Lemma 3, p. 153] \( p^* \) is a prime ideal and \( p \not\subseteq p^* \subseteq p \). Since \( h(p) = 1 \), \( p^* = 0 \). Therefore \( p \) contains no homogeneous element. Thus every nonzero homogeneous element \( u \) is in \( R - p \). It follows therefore \( \bigoplus_{q \in \mathbb{Z}} K_q \subseteq R_p \). Let \( f \in K \) be integral over \( R_p \). Then there exists \( h \in R - p \) such that \( fh \) is integral over \( R \). It follows from [10, Theorem 11, p. 157] that each of the homogeneous components is integral over \( R \). By the preceding, each homogeneous component of \( f \cdot h \) is in \( R_p \). Therefore \( f \cdot h \in R_p \) and \( f \in R_p \). Thus \( R_p \) is integrally closed.

Let \( y \in K \) be any nonzero element. If \( \xi \in K_p \), then \( \xi/y^q \in K_p \). Moreover \( R \subseteq K_p[y] \), \( K = K_p(y) \), \( y \) is transcendental over \( K_p \), \( K_q = K_q[y] \) and \( \bigoplus_{q \in \mathbb{Z}} K_q = K_p[y, 1/y] \). We have the following theorem.

**Theorem 2.** Let \( R = \bigoplus_{i \geq 0} R_i \) with that \( R_1 \neq 0 \). Let \( p \) be a homogeneous prime ideal such that there exists an element \( r_i \in R_i - p \). Then

Professor A. Seidenberg remarks that the present Theorem 2 strengthens Lemma 2 of [9; p. 618] and corrects its proof.
(a) \( K_\theta \) is the quotient field of \( R(p) \) and \( K_\theta \cap R_p = R(p) \).
(b) \( R(p) \) is integrally closed in \( K_\theta \) implies that \( R(p) \) is integrally closed in \( K \).
(c) \( R_p = (R(p)[r_t])_S \), where \( S = R - p \); \( r_t \) is transcendental over \( R(p) \).
(d) \( R_p \) is integrally closed in \( K \) if and only if \( R(p) \) is integrally closed in \( K_\theta \).
(e) \( R(p) \) is regular if and only if \( R_p \) is regular.

**Proof.** By definition \( R(p) \subset K_\theta \). Let \( x \in K_\theta \), \( x = f_i/g_i \) for some \( f_i, g_i \in R_i \) and \( g_i \neq 0 \). Then \( x = f_i/g_i = (f_i/r_i')(g_i/r_i') \), since \( f_i/r_i' \) and \( f_i/r_i' \) are both in \( R(p) \).

Therefore \( x \) is in the quotient field of \( R(p) \). For the second part of (a) we need only to prove that \( K_\theta \cap R_p \subset R(p) \). Let \( x \in K_\theta \cap R_p \). Then \( x = f_i/g_i \), for some \( f_i, g_i \in R_i \) with \( g_i \neq 0 \). On the other hand \( x = (f_i/r_i') \), \( (f_i/r_i') \), \( s_i + s_{r+t} + \cdots + s_{r+t+m} \) with \( s_i + s_{r+t} + \cdots + s_{r+t+m} \) \( \subset p \). Then there exists an index \( l + t \) such that \( s_i + s_{r+t} + \cdots + s_{r+t+m} \) \( g_i(r_i + r_{i+1} + \cdots + r_{i+m}) \) implies that \( l = j, m = k \) and \( f_i \cdot s_{r+t} = g_i \cdot r_{i+t} \). Thus \( x = f_i/g_i = r_{i+t}/s_{r+t} \) i.e. \( x \in R(p) \). Therefore \( K_\theta \cap R_p = R(p) \).

(b) If \( R(p) \) is integrally closed in \( K_\theta \), then, since \( K = K_\theta (r_t) \) and \( r_t \) is transcendental over \( K_\theta \) as noted in the preceding, \( K_\theta \) is algebraically closed in \( K \) and \( R(p) \) is thus integrally closed in \( K \).

(c) As noted in (b), \( r_t \) is transcendental over \( R(p) \). Let \( f \in R \) be an element. Then \( f = f_r + f_{r+1} + \cdots + f_n \) where \( f_i \in R_i \) for some nonnegative integers \( r \) and \( n \). But \( f = (f_i/r_i')r_i' + (f_{r+t}/r_{i+t}')r_{i+t}' + \cdots + (f_n/r_n')r_n' \). Therefore \( R \subset (R(p)[r_t])_S \). Thus \( S = R - p \) is a multiplicative set in \( R(p)[r_t] \). Now let \( f/g \in R_p, g \in R - p \). Then for some nonnegative integer \( t \) and \( m \),

\[
\frac{f}{g} = \frac{f_t}{g} + \cdots + \frac{f_m}{g} = \frac{1}{g} \left( \left( \frac{f_t}{r_t} \right) r_t' + \left( \frac{f_{r+t}}{r_{i+t}'} \right) r_{i+t}' + \cdots + \left( \frac{f_m}{r_{r+t}'} \right) r_{r+t}' \right).
\]

Therefore \( f/g \in (R(p)[r_t])_S \) i.e. \( R_p \subset (R(p)[r_t])_S \). The other inclusion is obvious. Thus \( R_p = (R(p)[r_t])_S \).

(d) Now, if \( R(p) \) is integrally closed in \( K \), then clearly \( R_p = (R(p)[r_t])_S \), being a localization of transcendental extension of an integrally closed domain, is integrally closed. Conversely if \( R_p \) is integrally closed in \( K \), let \( f \in K_\theta \) be an integral element over \( R(p) \). Then \( f \in R_p \). Thus \( f \in R_p \cap K_\theta = R(p) \), and \( R(p) \) is integrally closed.

(e) Recall that a ring \( A \) is said to be regular if \( A_\mathfrak{m} \) is a regular local ring for each maximal ideal \( \mathfrak{m} \) in \( A \). It follows from Serre's theorem [5; p. 139] that \( A \) is regular if and only if \( A_\mathfrak{p} \) is regular for every \( \mathfrak{p} \in \text{Spec}(A) \).

If \( R(p) \) is a regular local ring, then by [5; Theorem 40, p. 126] the polynomial ring \( R(p)[r_t] \) is regular. Since localization of a regular ring is regular therefore \( R_p = (R(p)[r_t])_S \) is a regular local ring.
Conversely assume that \( R_p = (R_p[r_1])_s \) is a regular local ring. Since \( R_p[r_1] \) is a polynomial ring over \( R_p \) therefore \( R_p[r_1] = R_p \)-flat. \((R_p[r_1])_s \) is \( R_p \)-flat therefore \( R_p \) is \( R_p \)-flat. Thus \( R_p \) is Noetherian. The inclusion map \( R_p \rightarrow R_p \) is obviously a local homomorphism. Therefore it follows from [1; IV, 17.3.3 (i), p. 48] that \( R_p \) is a regular local ring.

There are graded rings in which there are homogeneous prime ideals \( p \) such that \( p \cap R \neq R \). For example: (1) graded rings which are homogeneous coordinate rings of projective varieties. In this case \( p \cap R \neq R \) for \( p \in \text{Proj}(R) \). (2) \( R = R_0[x] \), a graded ring generated over \( R_0 \), (3) Let \( k[x, y] \) be a polynomial ring in two indeterminates over a field \( k \). Let \( R = k[x, y] \) be a polynomial ring in two indeterminates over a field \( k \). Let \( R = k[Y] + (X \cdot Y) \cdot k[X, Y] \). \( R \) has a graded structure \( R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots \) with \( R_0 = k, R_1 = k \cdot Y; R_2 = kY^2 + k(X \cdot Y), \ R_3 = kY^3 + kX^2Y + kXY^2, \) etc. It follows from the observation that \((X' \cdot Y')^2 \in R_y \) if \( j \geq 1 \) that \( p \cap R_i = 0 \) for every \( p \in \text{Proj}(R) \).

3. Normality of a graded domain. In this section, a graded domain \( R \) is normal if it is integrally closed in its field of fractions.

Recall [6; Theorem 8, p. 400]: Let \( \mathcal{O} \) and \( \mathcal{O}' \) be two normal rings which contain a field \( k \). If \( \mathcal{O} \) and \( \mathcal{O}' \) are separably generated over \( k \) and if \( \mathcal{O} \otimes_k \mathcal{O}' \) is an integral domain, then \( \mathcal{O} \otimes_k \mathcal{O}' \) is a normal ring.

Theorem 3. Let \( R_0 \) be a normal integral domain containing a field \( k \) such that \( R_0 \) is separable over \( k \). Let \( R = R_0[x] = R_0[x_1, \ldots, x_n] \) be an integral domain finitely generated over \( R_0 \) as an \( R_0 \)-algebra such that the quotient field \( K \) of \( R_0 \) and the quotient field \( k(x) \) of \( k[x_1, \ldots, x_n] \) are linearly disjoint over \( k \), and \( k(x) \) separable over \( k \). Then \( k[x] \) is normal if and only if \( R \) is normal.

Proof. Let \( X_1, \ldots, X_n \) be \( n \) indeterminates over \( R_0 \). Let \( \mathcal{A} \) be the prime ideal in \( k[X] = k[X_1, \ldots, X_n] \) such that \( k[x_1, \ldots, x_n] \subseteq k[X_1, \ldots, X_n] / \mathcal{A} \) and let \( \mathcal{B} \) be the prime ideal in \( R_0[X] = R_0[X_1, \ldots, X_n] \) such that \( R = R_0[X] / \mathcal{B} \). Then \( \mathcal{B} \cdot K[X] \cap R_0[X] = \mathcal{B} \) and \( \mathcal{A} = \mathcal{B} \cap k[X] \). Since \( K \) and \( k(x) \) are linearly disjoint over \( k \), it is well known that \( \mathcal{A} \cdot K[X] = \mathcal{B} \cdot K[X] \) and \( \mathcal{A} \cdot R_0[X] = \mathcal{B} \), [4; Corollary 1, p. 67]. We shall use \( \mathcal{B} \) in both \( R_0[X] \) and \( K[X] \) as the prime ideal determined by \( (x) = (x_1, \ldots, x_n) \). Since \( R_0 \otimes_k k[X] = R_0[X] \), it follows that \( R_0 \otimes_k k[x] = R_0[x] \), i.e. \( R_0 \otimes_k k[x] \) is an integral domain. It follows from [6; Theorem 8, p. 400] that \( R_0[x] \) is normal. Conversely if \( R_0[x] \) is normal, then \( R_0[x] \) is normal for each \( p \in \text{Spec}(R_0[x]) \). Let \( p' = p \cap k[x] \) for \( p \in \text{Spec}(R_0[x]) \) and \( p \cap R_0 = \{0\} \). Then \( k[x]_{p'} \) is also normal. Indeed let \( \xi \in k(x) \) be integral over \( k[x]_{p'} \). Since \( k[x]_{p'} \subseteq R_0[x]_{p'} \), therefore \( \xi \in R_0[x]_{p'} \). Thus \( \xi \in R_0[x] \cap k(x) \). It is sufficient to show that \( R_0[x]_{p} \cap k(x) \subseteq k[x]_{p'} \). Let \( S = R_0 - \{0\} \). \( K[x] = S^{-1}R_0[x] \) and
$S^{-1}p$ is a prime ideal in $K[x]$. $S^{-1}p \cap k[x] = p \cap k[x]$. Since $K$ and $k(x)$ are linearly disjoint over $k$, it follows from [4; Proposition 6, p. 92] that $k[x]_{S^{-1}p} \cap k(x) = k[x]_p$. Thus $k[x]_p \supset R_0[x]_p \cap k(x)$, and $k[x]_p = R_0[x]_p \cap k(x)$. So $\xi \in k[x]_p$ and $k[x]_p$ is therefore normal.

We shall finish the proof by showing that $\text{Spec}(k[x]) = \{p \cap k[x] | p \in \text{Spec}(R_0[x]) \text{ and } p \cap R_0 = 0\}$. Let $q_x$ be a prime ideal. There exists a prime ideal $Q_x$ in $K[X]$ such that $Q_x \cap k[X] = q_x$. Indeed, using Zariski's terminology [10; pp. 21-22 and pp. 161-176], we consider an algebraically closed field $\Omega$ containing $K$ and $\Omega$ is of infinite transcendence degree over $K$.

Let $\mathbb{A}^n$ be the $n$ dimensional affine space, i.e. $\mathbb{A}^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in \Omega\}$. Every prime ideal $P$ in $K[X]$ defines an irreducible algebraic variety $V$ over $K$ in $\mathbb{A}^n$. Every irreducible algebraic variety $V$ over $K$ carries a generic point $(\xi) = (\xi_1, \ldots, \xi_n) \in \mathbb{A}^n$ over $K$, and $P = \{g(X) \in K[X] | g(\xi) = 0\}$. Let $(\eta) = (\eta_1, \ldots, \eta_n) \in \mathbb{A}^n$ be a generic point of $q_x$ over $k$, i.e. $q_x = \{f(X) \in k[X] | f(\eta) = 0\}$. Let $Q_\eta = \{F(X) \in K[X] | F(\eta) = 0\}$. Then $Q_\eta$ is a prime ideal and $Q_\eta \cap k[X] = q_x$. Let $Q_x = Q_\eta \cap R_0[x]$, $Q_x \cap R_0 = 0$ and $Q_x \cap k[x] = q_x$. Since $\mathbb{A} \subset q_x \iff \mathbb{B} \cdot K[X] \subset Q_x \iff \mathbb{B} \subset Q_x$. Let $Q' = Q_x/\mathbb{B} \subset R_0[x]$. Then $Q' \cap k[x] = q$. Thus each prime ideal in $k[x]$ is the contraction of a prime ideal in $R_0[x]$ intersecting $R_0$ at 0.

As the assertion in the last part of the proof of the above theorem will be referred later, we would like to state it as a corollary.

**Corollary.** Let $R_\circ$ be an integral domain containing a field $k$. Let $R = R_\circ[x_1, \ldots, x_n]$ be an integral domain finitely generated over $R_\circ$ as an algebra such that the quotient field $K$ of $R_\circ$ and the quotient field $k(x)$ of $k[x] = k[x_1, \ldots, x_n]$ are linearly disjoint over $k$. Then $\text{Spec}(k[x]) = \{p \cap k[x] | p \in \text{Spec}(R_\circ[x]) \text{ and } p \cap R_\circ = 0\}$. Moreover if $R$ is graded with $R_\circ$ as the component of homogeneous degree 0, then $\text{Proj}(k[x]) = \{p \cap k[x] | p \in \text{Proj}(R_\circ[x])\} = \{p \cap k[x] | p \in \text{Proj} K[x]\}$.

**Proof** (of the last part). Let $\mathbb{A}, \mathbb{B}, q, q_x$, and $Q_\eta$ be the same as those in the proof of Theorem 3. If $R$ is a graded domain, then both $\mathbb{A}$ and $\mathbb{B}$ are homogeneous ideals. If $q$ is a nonirrelevant and homogeneous prime ideal in $k[x]$, then so is $q_x$. Let $Q_\eta^x$ be the ideal in $K[x]$ generated by the homogeneous elements belonging to $Q_\eta$. Then, by [10; Lemma 3, p. 153], $Q_\eta^x$ is a prime ideal and clearly $Q_\eta^x \cap k[X] = q_x$. Since $q_x$ is nonirrelevant, $Q_\eta^x$ is also nonirrelevant, and $Q_\eta^x \supset \mathbb{B}$. Let $Q^x = Q_\eta^x/\mathbb{B}$. We have $Q^x \cap k[x] = q$. Therefore $\text{Proj}(k[x]) = \{p \cap k[x] | p \in \text{Proj}(R) \text{ and } p \cap R_\circ = 0\}$.

Let us recall some definitions and facts: Let $R = \bigoplus_{i=0} R_i$ be a graded integral domain. $R$ is Noetherian if and only if $R_\circ$ is Noetherian and $R$ is an $R_\circ$-algebra of finite type. Let $\bar{R}$ be the integral closure of $R$ in its field of quotients $K$. Let $K_\circ$ be the homogeneous component of $K$ of...
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for all \( p \in \text{Proj}(R_0[x]) \), and coh.d.\( S^{-1}\mathfrak{B} < n - 1 \) then \( R_0[x] \) is normal.

(3) If \( R_{(p)} \) is normal for each \( p \in \text{Proj}(\mathfrak{B}) \) and if coh.d.\( \mathfrak{B} \cdot S^{-1}R_0[X] = n - 1 \) then \( R_0[x] \) is not normal.

**Proof.** (1) Both \( \mathfrak{A} \) and \( \mathfrak{B} \) are homogeneous ideals, \( k[x] \) is graded. As projective scheme \( \text{Proj}(R_0[x]) \equiv \text{Proj}((S^{-1}R_0)[x]) \) [1, Prop. (2.4.7), p. 30]. Therefore \( (S^{-1}R_0)[x] \) is locally normal, i.e. \( (S^{-1}R_0)[x]_{(p)} \) is normal for each \( p \in \text{Proj}(S^{-1}R_0[x]) \). Since tr.deg.\( S^{-1}R_0[x] > 0 \), if coh.d.\( S^{-1}\mathfrak{B} < n - 1 \), by [9, Theorem 3, p. 619], \( S^{-1}R_0[x] \) is normal. Therefore \( S^{-1}R_0[x]_{(p)} \) is normal for every \( p \in \text{Spec}(S^{-1}R_0[x]) \). Since \( (S^{-1}R_0)[x]_{(p)} \cap k(x) = k[x]_{(p)} \) as shown in the preceding, where \( p^c = p \cap k[x] \), \( k[x]_{(p)} \) is normal. By the Corollary to Theorem 3, \( \text{Spec}(k[x]) = \{ p^c \mid p \in \text{Spec}(S^{-1}R_0[x]) \} \), we have that \( k[x] \) is normal for every \( q \in \text{Spec}(k[x]) \). Therefore \( k[x] \) is normal.

(2) By (1), \( k[x] \) is normal. \( R_0 \) is normal. It follows from Theorem 3, \( R_0[x] \) is normal.

(3) If coh.d.\( \mathfrak{B} \cdot S^{-1}R_0[X] = n - 1 \), then it is well known that for a form \( l \) in \( R_0[X] \) prime to \( \mathfrak{B} \) i.e. \( \mathfrak{B} : l = \mathfrak{B}, \) coh.d.\( (\mathfrak{B}, l) \cdot S^{-1}R_0[X] = n \). Therefore \( (\mathfrak{B}, l) \cdot S^{-1}R_0[X] \) has \( (X) \cdot S^{-1}R_0[X] \) as an associated prime ideal. Since \( \dim \mathfrak{B} \cdot S^{-1}R_0[X] > 0 \), \( (\mathfrak{B}, l)S^{-1}R_0[X] \) has an embedded associated prime. On the other hand, it is easy to see that \( (X)S^{-1}R_0[X] \cap R_0[X] = (X)R_0[X] \). Therefore it follows from [5, Lemma 7c, p. 50] that \( (\mathfrak{B}, l)R_0[X] \) has \( (X)R_0[X] \) as an embedded associated prime ideal. Let \( (\bar{I})R_0[X] = (\mathfrak{B}, l)R_0[X]/\mathfrak{B} \). Therefore \( (\bar{I})R_0[x] \) is a principal homogeneous ideal having \( (X) \cdot R_0[x] \) as an embedded associated prime ideal. It follows from Theorem 4 that \( R \) is not normal.

### 4. Integral closure of a graded ring

In this section, we study a general graded ring, \( R = \bigoplus_{i \geq 0} R_i \). Let \( F \) be the total quotient ring of \( R \), and let \( \bar{R} \) be the integral closure of \( R \) in \( F \). In case of a graded domain, the integral closure \( \bar{R} \) of \( R \) in its quotient field \( K \) is again graded and \( \bar{R}_i = \bar{R} \cap K_i \) for \( i \geq 0 \). We investigate \( \bar{R} \) when \( R \) is not an integral domain. A ring \( R \) is normal if \( R_\wp \) is an integral domain and integrally closed in its quotient field for each \( \wp \in \text{Spec}(R) \).

Let \( R = \bigoplus_{i \geq 0} R_i \). Let \( U \) be the set of all nonzero divisors of \( R \). Let \( F \) be the total quotient ring and let \( F_i = \{ r_i/u_i | r_i \in R_i, u_i \in R_i \cup U, i - j = i \} \). These are the notations going to be used in the sequel.

**Theorem 6.** Assume \( U \cap R_i \neq \emptyset \) and let \( u_i \in U \cap R_i \). Then (1) the ring \( \Sigma_{i \in \mathbb{Z}} F_i \) is a direct sum, and \( \bigoplus_{i \in \mathbb{Z}} F_i = F_0[u_i, 1/u_i], F = F_0[u_i], u_i \) is algebraically independent over \( F_0 \), and \( F_i = F_0 \cdot u_i \) for all \( i \in \mathbb{Z} \). If \( F_0 \) is Noetherian then so is \( F \). (2) \( F_0 \) is reduced, i.e. \( F_0 \) has no nonzero nilpotent element, if and only if \( R \) is reduced. (3) If \( R \) is reduced and \( F_0 \) is
Noetherian, then $F_0[u_t]$ is integrally closed in $F$. (4) If $R$ is reduced and $F_0$ is Noetherian, then $R$ is a graded subring of $\bigoplus_{i \in \mathbb{Z}} F_i$.

Proof. (1) It follows from the definition of $F_i$'s that each $F_i$ is an additive group and $F_i : F_j \subset F_{i+j}, \Sigma_{i \in \mathbb{Z}} F_i$ is a ring. Let $f_k + \cdots + f_i \in \Sigma_{i \in \mathbb{Z}} F_i$. Suppose $f_k + \cdots + f_i = 0$. Let $f_m = r_m/u_m$ where $l_m - j_m = m$ and $m = k, \cdots, s$. Let $u = \Pi_{m=k}^s u_{m}$. Then $uf_k + \cdots + uf_i = 0$ in $R$, and $uf_k, \cdots, uf_i$ are homogeneous elements of distinct degrees. Therefore $uf_k = \cdots = uf_i = 0$. Thus $f_k = \cdots = f_i = 0$, and the sum $\Sigma F_i$ is therefore a direct sum. Let $f_k \in F_k$. Then $f_k/u_t \in F_0$. Therefore $f_k \in F_0 \cdot u_t^i$ and $F_k = F_0 \cdot u_t^i$. Hence $\bigoplus_{i \in \mathbb{Z}} F_i = F_0[u_t, 1/u_t]$. For any $f \in F$, 

$$f = (f_k + \cdots + f_i)/u = \frac{1}{u} \left( \frac{f_k}{u_t^i} + \cdots + \frac{f_i}{u_t^i} u_t^i \right).$$

Therefore $F = F_0[u_t, 1/u_t] \subset F_0[u_t]$. $u_t$ is algebraically independent over $F_0$. Indeed, let $a_o u_t^o + a_i u_t^{i-1} + \cdots + a_s = 0$, where $a_i \in F_o$ and $a_o \neq 0$. Writing $a_i = r_i/u_i$ with $l_i - j_i = i$, we have $a_i u_t^{i-1} \in F_{n-i}$. Therefore $a_i u_t^{i-1} = 0$, and $a_i = 0$ for $i = 0, 1, \cdots, n$. Therefore $u_t$ is algebraically independent over $F_0$.

If $F_0$ is Noetherian, then so is $F_0[u_t]$. Now $F = F_0[u_t]$ Therefore $F$ is also Noetherian.

(2) It is obvious that $R$ is reduced implies that $F_0$ is reduced. Conversely, we note if $(x_m/u_t^m)^* = 0$, then $x_m = 0$. Also if $y_m \in R_m$ such that $y_m^m = 0$ then $(y_m/u_t^m)^* = 0$. Thus $y_m = 0$. Now let $y$ be a nilpotent element in $R$. Write $y = y_k + \cdots + y_s$. For some positive integer $b$, $y^b = (y_k + \cdots + y_s)^b = 0$. Thus $y_k^b = 0$ and then $(y_{k+1} + \cdots + y_s)^b = 0$ and so on we get $y_{k+1}^b = \cdots = y_s^b = 0$, so $y_m = \cdots = y_s = 0$. Therefore $y = 0$ and $R$ is reduced.

(3) $F_0$ is reduced. It follows from that $F = F_0[u_t]$ and that $u_t$ is transcendental over $F_0$, the nonzero divisors of $F_0$ are the same as the nonzero divisors of $R$ in $F_0$. Let $U_0$ be the set of all nonzero divisors of $F_0$. Let $u_o \in U_0$, then $u_o = r_m/u_m$ where $u_m \in U$ and $r_m \in R_m$. Moreover $r_m \in U$ also. Thus $u_o$ is a unit i.e. $U_0$ is a multiplicative group in $F_0$. Hence the total quotient ring $(F_0)_{U_0} = F_0$. Since $F_0$ is Noetherian and reduced, therefore, $F_o = \bigoplus_{i \in \mathbb{Z}} G_i$, where $G_i$'s are fields. It follows from [2; Proposition (6.5.2), p. 146] that $F_0$ is normal.

It follows from [5; Proposition (1.7.8), p. 116] that $F_0[u_t]$ is normal. Since $F_o[u_t]$ is a polynomial ring in $u_t$, and $F_0$ is reduced, therefore $F_o[u_t]$ is also reduced. $F_0$ is Noetherian implies that $F$ is Noetherian. Then $F = \bigoplus_{i \in \mathbb{Z}} H_i$ where $H_i$'s are fields. Thus it follows from [2; Proposition (6.5.2), p. 146] that $F_0[u_t]$ is integrally closed.

Note: Let $A = Z/(4)[X]$, the polynomial ring in $X$ over $Z/(4)$. $Z/(4)$ is integrally closed, while $A$ is not. Indeed, let $y = (x + 1)/(x - 1), y^2 - 1 = 0, y \notin A$. 


(4) Let \( x \in \bar{R} \). Since \( R \subset R_0[u_t] \), \( x \) is integral over \( F_0[w/u] \). By (3), \( \bar{R} \subset F_0[u_t] \). The rest of the proof is practically the same argument used in the proof of [10; Theorem 11, p. 157]. We summarize the proof: Let \( x \in \bar{R} \), \( x = x_k + \cdots + x_s, k \leq s \), \( x_k \neq 0 \) is called the initial homogeneous term. We want to show that each \( x_i, i = k, \ldots, s \), is integral over \( R \) also. Since \( x \in \bar{R} \subset \Sigma F_0 \) there exists \( u_m \in R_m \cap U \) for some positive integer \( m \), such that \( u_m x \in R \). Case (a), if \( R \) is Noetherian, then \( R[x] \) is a finite \( R \)-module. There exists an integer \( \lambda > 0 \) such that \( u_{m+i}^{d+i} \in R \) for all integer \( i \geq 0 \). Let \( d = u_m^k \). Then \( dR[x] \subset R \). The initial homogeneous term \( dx^k \) is \( dx^i \subset R \) implies \( dx^i \in R \). Therefore \( x_k \in (1/d)R \), a Noetherian \( R \)-module. Therefore \( R[x_k] \subset R \cdot 1/d \) is a Noetherian \( R \)-submodule. Therefore \( x_k \) is integral over \( R \). Repeating that argument to \( x - x_k = x_{k+1} + \cdots + x_s \), we conclude that \( x_i \in \bar{R} \) for \( i = k, \ldots, s \). Therefore \( \bar{R} \) is graded in this case. Next we look at case (b): \( R \) is not Noetherian. Let \( x \in \bar{R} \), and \( x^n + a_1 x^{n-1} + \cdots + a_n = 0 \) where \( a_1, \ldots, a_n \in R \). As in case (a), there is a homogeneous nonzero divisor \( d \in R \) such that \( dx_k^i \in R \). Let \( \{y_1, \ldots, y_s\} = \{d, dx_k, \ldots, dx_s \} \), homogeneous components of \( a_i \)'s. Let \( A = k[y_1, \ldots, y_s] \), where \( k = \mathbb{Z} \) or \( \mathbb{Z}/(n) \) according to whether \( R \) is of characteristic \( 0 \) or \( n > 0 \). \( A \subset R \). Let \( A_k = A \cap R_k \). Then \( A = \Sigma A_k \) is a graded subring of \( R \). \( U \cap A \) contains \( d \). Therefore \( A_{U \cap A} \), the total quotient ring of \( A \), contains \( x_k \) and hence contains \( x \) also. Thus the above integral relation takes place in \( A_{U \cap A} \). Since \( A \) is Noetherian, therefore case (a) is applicable. Therefore \( x_k \) is integral over \( A \). hence \( x_k \) is integral over \( R \).

References


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