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Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded domain and let \( p \) be a homogeneous prime ideal in \( R \). Let \( R_p \) be the localization of \( R \) at \( p \) and \( R_{(p)} = \{ r/s | r, s \in R_i \text{ and } s \notin p \} \). If \( R_i \cap (R - p) \neq \emptyset \), then \( R_p \) is the localization of a transcendental extension of \( R_{(p)} \). Thus \( R_p \) is normal (regular) if and only if \( R_{(p)} \) is normal (regular). Let \( \text{Proj}(R) = \{ p | p \text{ is a homogeneous prime ideal and } p \notin \bigoplus_{i \geq 0} R_i \} \). Under certain conditions a Noetherian graded domain \( R \) is normal if \( R_{(p)} \) is normal for each \( p \in \text{Proj}(R) \). If \( R = \bigoplus_{i \geq 0} R_i \) is reduced and \( F_0 = \{ r_i/u_i | r_i, u_i \in R_i \text{ and } u_i \in U \} \), where \( U \) is the set of all nonzero divisors, then the integral closure of \( R \) in the total quotient ring of \( R \) is also graded.

1. Introduction. Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded integral domain. Let \( \text{Spec}(R) \) be the set of all prime ideals in \( R \). Let \( R_+ = \bigoplus_{i > 0} R_i \); \( R_+ \) is an ideal in \( R \). An ideal \( \mathfrak{a} \) in \( R \) is said to be irrelevant if \( R_+ \subseteq \sqrt{\mathfrak{a}} \), the radical of \( \mathfrak{a} \). Let \( \text{Proj}(R) = \{ p \in \text{Spec}(R) | p \subseteq R_+ \text{ is homogeneous and nonirrelevant} \} \). For each \( p \in \text{Spec}(R) \), let \( R_p = \{ r/s | r, s \in R_i \text{ and } s \notin p \} \). (Note: \( R_{(p)} \) in [1] is defined for \( p \in \text{Proj}(R) \) only.) According to the terminology of Seidenberg [9], \( R_p \) is called the arithmetical local ring of \( R \) at \( p \) and \( R_{(p)} \), the geometrical local ring of \( R \) at \( p \). I prove that if \( R_i \cap (R - p) \neq \emptyset \) then \( R_p \) is the ring of quotients of a transcendental extension of \( R_{(p)} \) relative to a multiplicative set, \( R_p \) is normal (regular) if and only if \( R_{(p)} \) is normal (regular); see Theorem 2. In the case of an irreducible projective variety \( V \) over a field \( k \) in a projective \( n \)-space \( P^n_k \), \( V/k \) is normal if the geometrical local ring of \( V \) at each \( p \in V \), \( \mathcal{O}^*_V(p) \) is integrally closed. \( V \) is arithmetically normal if the ring of strictly homogeneous coordinates \( k[V] \) is integrally closed. The latter implies the former. For the converse, various cohomological criteria are developed; see [3], [8], [9]. I attempt to study the normality of a graded domain \( R \) if \( R_{(p)} \) is normal for every \( p \in \text{Proj}(R) \). In this paper, I also obtain the following theorem: Let \( R \) be a Noetherian graded domain, say \( R = R_0[x_0, \ldots, x_n] \) and \( x_0, \ldots, x_n \) are of homogeneous degree 1. Assume that \( R_0 \) contains a field \( k \) over which \( R_0 \) and \( k(x_0, \ldots, x_n) \) are linearly disjoint and separable. Let \( \mathfrak{B} \) be the kernel of the canonical map from the polynomial ring \( R_0[x_0, \ldots, x_n] \). Then \( R \) is normal if \( R_0 \) is normal, \( R_{(p)} \) is normal for every \( p \in \text{Proj}(R) \) and \( \text{coh.d.} \mathfrak{B} \cdot K[X_0, \ldots, X_n] < n - 1 \), where \( K \) is the quotient field of \( R_0 \).
In the §4, we prove that under certain conditions on a graded ring $R$ (not necessarily integral domain) the integral closure $\tilde{R}$ of $R$ in the total quotient ring of $R$ is also graded; see Theorem 6.

Our references on the elementary well known facts about graded rings can be found in [1] and [10].

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2. Normality and regularity of local domains. Let $R$ be a commutative ring with identity $1$. Let $p$ be a prime ideal in $R$. By height of $p$, we mean the supremum of the length of chains of prime ideals $p_0 \supsetneq p_1 \supsetneq p_2 \supsetneq \cdots \supsetneq p_n$ with $p_0 = p$ and denote it by $ht(p)$. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded integral domain. Let $K$ be the quotient field of $R$. We say that $R$ is integrally closed if $R$ is integrally closed in $K$.

The following theorem was originally proved in [9] for projective varieties. We observe that the same holds true for non-Noetherian graded domain also.

**Theorem 1.** Let $R = \bigoplus_{i \geq 0} R_i$ be a graded domain. Let $p \in \text{Spec}(R)$ be nonhomogeneous. If $ht(p) = 1$ then $R_p$ is integrally closed.

**Proof.** Let $p^*$ be the ideal generated by all the homogeneous elements of $p$. By [10, Lemma 3, p. 153] $p^*$ is a prime ideal and $p \not\supsetneq p^* \supsetneq 0$. Since $ht(p) = 1$, $p^* = 0$. Therefore $p$ contains no homogeneous element. Thus every nonzero homogeneous element $u$ is in $R - p$. It follows therefore $\bigoplus_{q \in \mathbb{Z}} K_q \subset R_p$. Let $f \in K$ be integral over $R_p$. Then there exists $h \in R - p$ such that $fh$ is integral over $R$. It follows from [10, Theorem 11, p. 157] that each of the homogeneous components is integral over $R$. By the preceding, each homogeneous component of $f \cdot h$ is in $R_p$. Therefore $f \cdot h \in R_p$ and $f \in R_p$. Thus $R_p$ is integrally closed.

Let $y \in K_i$ be any nonzero element. If $\xi \in K_q$, then $\xi/y^q \in K_0$. Moreover $R \subset K_0[y]$, $K = K_0(y)$, $y$ is transcendental over $K_0$, $K_q = K_0y^q$ and $\bigoplus_{q \in \mathbb{Z}} K_q = K_0[y, 1/y]$. We have the following theorem.

**Theorem 2.** Let $R = \bigoplus_{i \geq 0} R_i$ with that $R_1 \neq 0$. Let $p$ be a homogeneous prime ideal such that there exists an element $r_i \in R_i - p$. Then

Professor A. Seidenberg remarks that the present Theorem 2 strengthens Lemma 2 of [9; p. 618] and corrects its proof.
(a) $K_\theta$ is the quotient field of $R_{(p)}$ and $K_\theta \cap R_p = R_{(p)}$.
(b) $R_{(p)}$ is integrally closed in $K_\theta$ implies that $R_{(p)}$ is integrally closed in $K$.
(c) $R_p = (R_{(p)}[r_i])_S$, where $S = R - p$; $r_i$ is transcendental over $R_{(p)}$.
(d) $R_p$ is integrally closed in $K$ if and only if $R_{(p)}$ is integrally closed in $K_\theta$.
(e) $R_{(p)}$ is regular if and only if $R_p$ is regular.

Proof. By definition $R_{(p)} \subset K_\theta$. Let $x \in K_\theta$, $x = f_i/g_i$ for some $f_i, g_i \in R_i$ and $g_i \neq 0$. Then $x = f_i/g_i = (f_i/r_i)/(g_i/r_i)$, since $f_i/r_i$ and $f_i/r_i$ are both in $R_{(p)}$. Therefore $x$ is in the quotient field of $R_{(p)}$. Thus $K_\theta$ is the quotient field of $R_{(p)}$. For the second part of (a) we need only to prove that $K_\theta \cap R_p \subset R_{(p)}$. Let $x \in K_\theta \cap R_p$. Then $x = f_i/g_i$ for some $f_i, g_i \in R_i$ with $g_i \neq 0$. On the other hand $x = (s_i + s_{i+1} + \cdots + s_{i+m})/(s_i + s_{i+1} + \cdots + s_{i+m})$ with $s_i + s_{i+1} + \cdots + s_{i+m} \not\in p$. Then there exists an index $l + t$ such that $s_{l+t} \not\in p$. $f_i \cdot (s_i + s_{i+1} + \cdots + s_{i+m}) = g_i(r_i + r_{i+1} + \cdots + r_{i+l})$ implies that $l = j, m = k$ and $f_i \cdot s_{l+t} = g_i \cdot r_{i+t}$. Thus $x = f_i/g_i = r_{i+t}/s_{i+t}$ i.e. $x \in R_{(p)}$. Therefore $K_\theta \cap R_p = R_{(p)}$.

(b) If $R_{(p)}$ is integrally closed in $K_\theta$ then, since $K = K_\theta(r_i)$ and $r_i$ is transcendental over $K_\theta$ as noted in the preceding, $K_\theta$ is algebraically closed in $K$ and $R_{(p)}$ is thus integrally closed in $K$.

(c) As noted in (b), $r_i$ is transcendental over $R_{(p)}$. Let $f \in R$ be an element. Then $f = f_i + f_{i+1} + \cdots + f_m$ where $f_i \in R_i$ for some nonnegative integers $r$ and $n$. But $f = (f_i/r_i)r_i' + (f_{i+1}/r_{i+1}')r_{i+1}' + \cdots + (f_m/r_m')r_m'$ $\in (R_{(p)}[r_i])$. Therefore $R \subset R_{(p)}[r_i]$. Thus $S = R - p$ is a multiplicative set in $R_{(p)}[r_i]$. Now let $f/g \in R_p$, $g \in R - p$. Then for some nonnegative integer $i$ and $m$,

$$\frac{f}{g} = \frac{f_i}{g} + \cdots + \frac{f_m}{g} = \frac{1}{g} \left( f_i r_i' + \frac{f_{i+1}}{r_{i+1}'} r_{i+1}' + \cdots + \frac{f_m}{r_m'} r_m' \right).$$

Therefore $f/g \in (R_{(p)}[r_i])_S$ i.e. $R_p \subset (R_{(p)}[r_i])_S$. The other inclusion is obvious. Thus $R_p = (R_{(p)}[r_i])_S$.

(d) Now, if $R_{(p)}$ is integrally closed in $K$, then clearly $R_p = (R_{(p)}[r_i])_S$, being a localization of transcendental extension of an integrally closed domain, is integrally closed. Conversely if $R_p$ is integrally closed in $K$, let $f \in K_\theta$ be an integral element over $R_{(p)}$. Then $f \in R_p$. Thus $f \in R_p \cap K_\theta = R_{(p)}$, and $R_{(p)}$ is integrally closed.

(e) Recall that a ring $A$ is said to be regular if $A_\mathfrak{m}$ is a regular local ring for each maximal ideal $\mathfrak{m}$ in $A$. It follows from Serre's theorem [5; p. 139] that $A$ is regular if and only if $A_\mathfrak{p}$ is regular for every $\mathfrak{p} \in \text{Spec}(A)$.

If $R_{(p)}$ is a regular local ring, then by [5; Theorem 40, p. 126] the polynomial ring $R_{(p)}[r_i]$ is regular. Since localization of a regular ring is regular therefore $R_p = (R_{(p)}[r_i])_S$ is a regular local ring.
Conversely assume that $R_p = (R_{(p)}[r_i])_p$ is a regular local ring. Since $R_{(p)}[r_i]$ is a polynomial ring over $R_{(p)}$, therefore $R_{(p)}[r_i]$ is $R_{(p)}$-flat. $(R_{(p)}[r_i])_p$ is $R_{(p)}[r_i]$-flat therefore $R_p$ is $R_{(p)}$-flat. Thus $R_{(p)}$ is Noetherian. The inclusion map $R_{(p)} \to R_p$ is obviously a local homomorphism. Therefore it follows from [1; IV, 17.3.3 (i), p. 48] that $R_{(p)}$ is a regular local ring.

There are graded rings in which there are homogeneous prime ideals $p$ such that $p \cap R_i \neq R_i$. For example: (1) graded rings which are homogeneous coordinate rings of projective varieties. In this case $p \cap R_i \neq R_i$ for $p \in \text{Proj}(R)$. (2) $R = R_0[x]$, a graded ring generated over $R_0$; (3) Let $k[X, Y]$ be a polynomial ring in two indeterminates over a field $k$. Let $R = k[X] + (X Y) k[X, Y]$. $R$ has a graded structure $R = R_0 \oplus \cdots \oplus R_n$ with $R_0 = k$, $R_1 = kX$, $R_2 = kX^2 + kXY$, etc. It follows from the observation that $(X^i \cdot Y^j)^2 \in R_y$ if $j \geq 1$ that $p \cap R_i = 0$ for every $p \in \text{Proj}(R)$.

3. Normality of a graded domain. In this section, a graded domain $R$ is normal if it is integrally closed in its field of fractions.

Recall [6; Theorem 8, p. 400]: Let $\mathcal{O}$ and $\mathcal{O}'$ be two normal rings which contain a field $k$. If $\mathcal{O}$ and $\mathcal{O}'$ are separably generated over $k$ and if $\mathcal{O} \otimes_k \mathcal{O}'$ is an integral domain, then $\mathcal{O} \otimes_k \mathcal{O}'$ is a normal ring.

**Theorem 3.** Let $R_0$ be a normal integral domain containing a field $k$ such that $R_0$ is separable over $k$. Let $R = R_0[x] = R_0[x_1, \ldots, x_n]$ be an integral domain finitely generated over $R_0$ as an $R_0$-algebra such that the quotient field $K$ of $R_0$ and the quotient field $k(x)$ of $k[x_1, \ldots, x_n]$ are linearly disjoint over $k$, and $k(x)$ separable over $k$. Then $k[x]$ is normal if and only if $R$ is normal.

**Proof.** Let $X_1, \ldots, X_n$ be $n$ indeterminates over $R_0$. Let $\mathfrak{A}$ be the prime ideal in $k[X] = k[X_1, \ldots, X_n]$ such that $k[x_1, \ldots, x_n] = k[X_1, \ldots, X_n] / \mathfrak{A}$ and let $\mathfrak{B}$ be the prime ideal in $R_0[X] = R_0[x_1, \ldots, X_n]$ such that $R = R_0[X] / \mathfrak{B}$. Then $\mathfrak{B} \cdot K[X] \cap R_0[X] = \mathfrak{B}$ and $\mathfrak{A} = \mathfrak{B} \cap k[X]$. Since $K$ and $k(x)$ are linearly disjoint over $k$, it is well known that $\mathfrak{A} \cdot K[X] = \mathfrak{B} \cdot K[X]$ and $\mathfrak{A} \cap R_0[X] = \mathfrak{B}$, [4; Corollary 1, p. 67]. We shall use $\mathfrak{B}$ in both $R_0[X]$ and $K[X]$ as the prime ideal determined by $(x) = (x_1, \ldots, x_n)$. Since $R_0 \otimes_k k[x] = R_0[x]$, it follows that $R_0 \otimes_k k[x] = R_0[x]$, i.e. $R_0 \otimes_k k[x]$ is an integral domain. It follows from [6; Theorem 8, p. 400] that $R_0[x]$ is normal. Conversely if $R_0[x]$ is normal, then $R_0[x]_p$ is normal for each $p \in \text{Spec}(R_0[x])$. Let $p^c = p \cap k[x]$ for $p \in \text{Spec}(R_0[x])$ and $p \cap R_0 = \{0\}$. Then $k[x]_{p^c}$ is also normal. Indeed let $\xi \in k(x)$ be integral over $k[x]_{p^c}$. Since $k[x]_{p^c} \subset R_0[x]_{p^c}$, therefore $\xi \in R_0[x]_{p^c}$. Thus $\xi \in R_0[x]_{p^c} \cap k(x)$. It is sufficient to show that $R_0[x]_{p^c} \cap k(x) \subset k[x]_{p^c}$. Let $S = R_0 - \{0\}$. $K[x] = S^{-1}R_0[x]$ and
S^{-1}p is a prime ideal in K[x]. S^{-1}p \cap k[x] = p \cap k[x]. Since K and k(x) are linearly disjoint over k, it follows from [4; Proposition 6, p. 92] that K[x],_p \cap k(x) = k[x],. Thus k[x],_p \supset R_0[x],_p \cap k(x), and k[x],_p = R_0[x],_p \cap k(x). So \xi \in k[x],_p, and k[x],_p, is therefore normal.

We shall finish the proof by showing that Spec(k[x]) = \{p \cap k[x] | p \in Spec(R_0[x]) and p \cap R_0 = 0\}. Let q_x be a prime ideal. There exists a prime ideal Q_x in K[X] such that Q_x \cap k[X] = q_x. Indeed, using Zariski’s terminology [10; pp. 21–22 and pp. 161–176], we consider an algebraically closed field \Omega containing K and \Omega is of infinite transcendence degree over K. Let A^n be the n dimensional affine space, i.e. A^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in \Omega\}. Every prime ideal P in K[X] defines an irreducible algebraic variety V over K in A^n. Every irreducible algebraic variety V over K carries a generic point (\xi) = (\xi_1, \ldots, \xi_n) \in A^n over K, and P = \{g(X) \in K[X] | g(\xi) = 0\}. Let (\eta) = (\eta_1, \ldots, \eta_n) \in A^n be a generic point of q_x over k, i.e. q_x = \{f(X) \in k[X] | f(\eta) = 0\}. Let Q_x = \{F(X) \in K[X] | F(\eta) = 0\}. Then Q_x is a prime ideal and Q_x \cap k[X] = q_x. Let Q_x = Q_x \cap R_0[X], Q_x \cap R_0 = 0 and Q_x \cap k[X] = q_x. Since \mathfrak{A} \subset q_x \iff \mathfrak{B} \cdot K[X] \subset Q_x \iff \mathfrak{B} \subset Q_x. Let Q' = Q_x/\mathfrak{B} \subset R_0[x]. Then Q' \cap k[x] = q. Thus each prime ideal in k[x] is the contraction of a prime ideal in R_0[x] intersecting \mathfrak{B} at 0.

As the assertion in the last part of the proof of the above theorem will be referred later, we would like to state it as a corollary.

**Corollary.** Let R_0 be an integral domain containing a field k. Let R = R_0[x_1, \ldots, x_n] be an integral domain finitely generated over R_0 as an algebra such that the quotient field K of R_0 and the quotient field k(x) of k[x] = k[x_1, \ldots, x_n] are linearly disjoint over k. Then Spec(k[x]) = \{p \cap k[x] | p \in Spec(R_0[x]) and p \cap R_0 = 0\}. Moreover if R is graded with R_0 as the component of homogeneous degree 0, then Proj(k[x]) = \{p \cap k[x] | p \in Proj(R_0[x])\}.

**Proof** (of the last part). Let \mathfrak{A}, \mathfrak{B}, q, q_x, and Q_x be the same as those in the proof of Theorem 3. If R is a graded domain, then both \mathfrak{A} and \mathfrak{B} are homogeneous ideals. If q is a nonirrelevant and homogeneous prime ideal in k[x], then so is q_x. Let Q_x be the ideal in k[x] generated by the homogeneous elements belonging to Q_x. Then, by [10; Lemma 3, p. 153], Q_x is a prime ideal and clearly Q_x \cap k[x] = q_x. Since q_x is nonirrelevant, Q_x is also nonirrelevant, and Q_x \supset \mathfrak{B}. Let Q* = Q_x/\mathfrak{B}. We have Q* \cap k[x] = q. Therefore Proj(k[x]) = \{p \cap k[x] | p \in Proj(R) and p \cap R_0 = 0\}.

Let us recall some definitions and facts: Let R = \bigoplus_{i \geq 0} R_i be a graded integral domain. R is Noetherian if and only if R_0 is Noetherian and R is an R_0-algebra of finite type. Let \bar{R} be the integral closure of R in its field of quotients K. Let K be the homogeneous component of K of
degree $i$ as defined in §2. Then $\tilde{R}$ is graded with $\tilde{R}_i = \tilde{R} \cap K_i$. Thus if $R$ is normal then $R_\theta$ must be normal.

Corresponding to Krull's characterization of a Noetherian domain being normal [7; (12.9), p. 41], we have the following theorem for normality of a Noetherian graded domain.

**Theorem 4.** Let $R$ be a graded Noetherian domain such that $R_i - p \neq \emptyset$ for each homogeneous prime ideal $p$ of $ht$ 1 in $R$. If (1) $R_{(p)}$ is normal for every homogeneous prime ideal $p$ of height 1 and (2) the associated prime ideals of every nonzero homogeneous ideal are of height 1, then $R$ is normal.

**Proof.** We first note that it follows from condition (1), Theorem 1 and Theorem 2 that $R_p$ is normal for every $p \in \text{Spec}(R)$ and $ht(p) = 1$. Let $K, \tilde{R}$ and $R_i$ be the same as defined in the proceeding. Let $\alpha = \sum_{i=n}^{m} \alpha_i$ for some nonnegative integers $m$ and $n$ and $\alpha_i \in R_i$. Let $\alpha_i = b_{ij}/a_{ii}$ where $j - l = i$, $b_{ij} \in R_j$ and $a_{ii} \in R_i$. If $a_{ii}$ is a unit in $R$ then $\alpha_i \in R$. If $a_{ii}$ is a nonunit, then the nonzero homogeneous principal ideal $(a_{ii})R$ has a primary decomposition $\bigcap_{t=1}^{u} q_t$ with $p_1, \ldots, p_u$ as the associated prime ideals. In view of [10; Theorem 9 and Corollary; pp. 153–154] we may assume that $q_t$’s and $p_t$’s are homogeneous, (2) implies that $ht(p_t) = 1$ for $t = 1, 2, \ldots, u$. Thus $R_{p_t}$ is normal for $t = 1, 2, \ldots, u$. $\alpha_i$ is integral over $R$ implies that $\alpha_i$ is integral over $R_{p_t}$ for $t = 1, 2, \ldots, u$. Therefore $b_{ij} \in \bigcap_{t=1}^{u} ((a_{ii})R_{p_t} \cap R) = \bigcap_{t=1}^{u} q_t = (a_{ii})R$. Thus $\alpha_i = b_{ij}/a_{ii} \in R$ and $\alpha = \sum_{i=n}^{m} \alpha_i \in R$. $R$ is therefore normal.

Let $A = K[X_1, \ldots, X_n]$ be a polynomial ring over a field $K$. The smallest integer $d$ such that any chain of syzygies of the $A$-module $M$ terminates at $(d + 1)$th step is called the cohomological dimension of $M$ and is denoted by $\text{coh.d.}(M)$. Let $\mathfrak{A} \subset A$ be a homogeneous ideal such that $\mathfrak{A} \neq (0)$, $\neq (1)$. $\text{coh.d.}(\mathfrak{A}) \leq n$ and it is $n$ if and only if $(X_1, \ldots, X_n)A$ is an associated prime ideal of $\mathfrak{A}$. Let $l$ be a form in $A$, and $l \not\in K$. If $\mathfrak{A} : l = \mathfrak{A}$ then $\text{coh.d.}(\mathfrak{A}, l) = 1 + \text{coh.d.}(\mathfrak{A})$.

**Theorem 5.** Let $R = \bigoplus_{i \geq 0} R_i$ be a Noetherian graded integral domain generated over $R_\theta$ by nonzero homogeneous elements $x_1, \ldots, x_n$ of degree 1. Assume that $R_\theta$ contains a subfield $k$ over which $R_\theta$ and $k(x) = k(x_1, \ldots, x_n)$ are linearly disjoint and $R_\theta$ is normal. Assume $\text{tr.deg}_k k(x) > 0$. Let $R_\theta[X] = R_\theta[X_1, \ldots, X_n]$ be the polynomial ring over $R_\theta$ in indeterminates $X_1, \ldots, X_n$ and let $\mathfrak{B}$ be the ideal such that $R_{\theta}[x] = R_\theta[X]/\mathfrak{B}$. Let $\mathfrak{A} = \mathfrak{B} \cap k[X]$, and let $S = R_\theta - \{0\}$.

(1) If, for each $p \in \text{Proj}(R_\theta[x])$, $R_\theta[x]_{(p)}$ is normal and $\text{coh.d.}(S) < n - 1$, then $k[x]$ is normal.

(2) If $R_\theta$ and $k(x)$ are both separable over $k$, and if $R_\theta[x]_{(p)}$ is normal
for all \( p \in \text{Proj}(R_0[x]), \) and \( \text{coh.d.} S^{-1}\mathfrak{B} < n - 1 \) then \( R_0[x] \) is normal.

(3) If \( R_{(p)} \) is normal for each \( p \in \text{Proj}(R) \) and if \( \text{coh.d.} \mathfrak{B} \cdot S^{-1}R_0[X] = n - 1 \) then \( R_0[x] \) is not normal.

**Proof.** (1) Both \( \mathfrak{A} \) and \( \mathfrak{B} \) are homogeneous ideals, \( k[x] \) is graded.

As projective scheme \( \text{Proj}(R_0[x]) \equiv \text{Proj}((S^{-1}R_0)[x]) \) [1, Prop. (2.4.7), p. 30]. Therefore \( (S^{-1}R_0)[x] \) is locally normal, i.e. \( (S^{-1}R_0)[x]_{(p)} \) is normal for each \( p \in \text{Proj}(S^{-1}R_0[x]). \) Since \( \text{tr.deg.} S^{-1}R_0[x] > 0. \) If \( \text{coh.d.} S^{-1}\mathfrak{B} < n - 1, \) by [9, Theorem 3, p. 619], \( (S^{-1}R_0)[x] \) is normal. Therefore \( S^{-1}R_0[x]_p \) is normal for every \( p \in \text{Spec}(S^{-1}R_0[x]). \)

By the Corollary to Theorem 3, \( \text{Spec}(k[x]) = \{ p \in \text{Spec}(S^{-1}R_0[x]) \}, \) we have that \( k[x, q] \) is normal for every \( q \in \text{Spec}(k[x]). \) Therefore \( k[x] \) is normal.

(2) By (1), \( k[x] \) is normal. \( R_0 \) is normal. It follows from Theorem 3, \( R_0[x] \) is normal.

(3) If \( \text{coh.d.} \mathfrak{B} \cdot S^{-1}R_0[X] = n - 1, \) then it is well known that for a form \( l \) in \( R_0[X] \) prime to \( \mathfrak{B} \), i.e. \( \mathfrak{B}: l = \mathfrak{B} \), \( \text{coh.d.}(\mathfrak{B}, l) \cdot S^{-1}R_0[X] = n. \) Therefore \( (\mathfrak{B}, l) \cdot S^{-1}R_0[X] \) has \( (X) \cdot S^{-1}R_0[X] \) as an associated prime ideal. Since \( \dim \mathfrak{B} \cdot S^{-1}R_0[X] > 0, \) \( (\mathfrak{B}, l)S^{-1}R_0[X] \) has an embedded associated prime. On the other hand, it is easy to see that \( (X)S^{-1}R_0[X] \cap R_0[X] = (X)R_0[X]. \) Therefore it follows from [5, Lemma 7c, p. 50] that \( (\mathfrak{B}, l)R_0[X] \) has \( (X)R_0[X] \) as an embedded associated prime ideal. Let \( (l)R_0[X] = (\mathfrak{B}, l)R_0[X]/\mathfrak{B}. \) Therefore \( (l)R_0[X] \) is a principal homogeneous ideal having \( (x) \cdot R_0[x] \) as an embedded associated prime ideal. It follows from Theorem 4 that \( R \) is not normal.

4. Integral closure of a graded ring. In this section, we study a general graded ring, \( R = \bigoplus_{i \geq 0} R_i. \) Let \( F \) be the total quotient ring of \( R, \) and let \( \widetilde{R} \) be the integral closure of \( R \) in \( F. \) In case of a graded domain, the integral closure \( \widetilde{R} \) of \( R \) in its quotient field \( K \) is again graded and \( \widetilde{R}_i = \widetilde{R}_i \cap K_i \) for \( i \geq 0. \) We investigate \( \widetilde{R} \) when \( R \) is not an integral domain. A ring \( R \) is normal if \( R_p \) is an integral domain and integrally closed in its quotient field for each \( p \in \text{Spec}(R). \)

Let \( R = \bigoplus_{i \geq 0} R_i. \) Let \( U \) be the set of all nonzero divisors of \( R. \) Let \( F \) be the total quotient ring and let \( F_i = \{ r_i/u_i | r_i \in R_i, u_i \in R_j \cap U, l - j = i \}. \) These are the notations going to be used in the sequel.

**Theorem 6.** Assume \( U \cap R_i \neq \emptyset \) and let \( u_i \in U \cap R_i. \) Then (1) the ring \( \bigoplus_{i \in \mathbb{Z}} F_i \) is a direct sum, and \( \bigoplus_{i \in \mathbb{Z}} F_i = F_0[u_1, 1/u_1], \) \( F = F_0[u_1]_{u_1}, \) \( u_1 \) is algebraically independent over \( F_0, \) and \( F_i = F_0 \cdot u_i, \) for all \( i \in \mathbb{Z}. \) If \( F_0 \) is Noetherian then so is \( F. \) (2) \( F_0 \) is reduced, i.e. \( F_0 \) has no nonzero nilpotent element, if and only if \( R \) is reduced. (3) If \( R \) is reduced and \( F_0 \) is
Noetherian, then $F_0[\{u_i\}]$ is integrally closed in $F$. (4) If $R$ is reduced and $F_0$ is Noetherian, then $R$ is a graded subring of $\bigoplus_{i \in \mathbb{Z}} F_i$.

Proof. (1) It follows from the definition of $F_i$’s that each $F_i$ is an additive group and $F_i \cdot F_j \subset F_{i+j}$, $\Sigma_{i \in \mathbb{Z}} F_i$ is a ring. Let $f_1 + \cdots + f_s \in \Sigma_{i \in \mathbb{Z}} F_i$. Suppose $f_1 + \cdots + f_s = 0$. Let $f_m = r_m/u_{j_m}$ where $l_m - j_m = m$ and $m = k, \cdots, s$. Let $u = \prod_{m-k}^{s} u_{j_m}$. Then $u f_k + \cdots + u f_s = 0$ in $R$, and $u f_k, \cdots, u f_s$ are homogeneous elements of distinct degrees. Therefore $u f_k = \cdots = u f_s = 0$. Thus $f_k = \cdots = f_s = 0$, and the sum $\Sigma F_i$ is therefore a direct sum.

Let $f_k \in F_k$. Then $f_k/u_i \in F_0$. Therefore $f_k \in F_0 \cdot u_i$ and $f_k = F_0 \cdot u_i$. Hence $\bigoplus_{i \in \mathbb{Z}} F_i = F_0[\{u_i\}, 1/u_i]$. For any $f \in F$,

$$f = (f_k + \cdots + f_s)/u = \frac{1}{u} \left( \frac{f_k}{u_i} u_i + \cdots + \frac{f_s}{u_i} u_i \right).$$

Therefore $F = F_0[\{u_i\}, 1/u_i] = F_0[\{u_i\}]$. $u_i$ is algebraically independent over $F_0$. Indeed, let $a_0 u_i + a_1 u_{i-1} + \cdots + a_n = 0$, where $a_i \in F_0$ and $a_0 \neq 0$. Writing $a_i = r_i/u_{j_i}$ with $l_i - j_i = i$, we have $a_i u_{i-1} \in F_{i-1}$. Therefore $a_i u_{i-1} = 0$, and $a_i = 0$ for $i = 0, 1, \cdots, n$. Therefore $u_i$ is algebraically independent over $F_0$.

If $F_0$ is Noetherian, then so is $F_0[\{u_i\}]$. Now $F = F_0[\{u_i\}]$. Therefore $F$ is also Noetherian.

(2) It is obvious that $R$ is reduced implies that $F_0$ is reduced. Conversely, we note if $(x_m/u_i)$" = 0, then $x_m = 0$. Also if $y_m \in R_m$ such that $y_m^* = 0$ then $(y_m/u_i)^* = 0$. Thus $y_m = 0$. Now let $y$ be a nilpotent element in $R$. Write $y = y_k + \cdots + y_s$. For some positive integer $b$, $y^b = (y_k + \cdots + y_s)^b = 0$. Thus $y^b = 0$ and then $(y_{k+1} + \cdots + y_s)^b = 0$ and so on we get $y_{k+1} = \cdots = y_s = 0$, so $y_m = \cdots = y_s = 0$. Therefore $y = 0$ and $R$ is reduced.

(3) $F_0$ is reduced. It follows from that $F = F_0[\{u_i\}]$ and that $u_i$ is transcendental over $F_0$, the nonzero divisors of $F_0$ are the same as the nonzero divisors of $R$ in $F_0$. Let $U_0$ be the set of all nonzero divisors of $F_0$. Let $u_0 \in U_0$, then $u_0 = r_m/u_m$ where $u_m \in U$ and $r_m \in R_m$. Moreover $r_m \in U$ also. Thus $u_0$ is a unit i.e. $U_0$ is a multiplicative group in $F_0$. Hence the total quotient ring $(F_0)_{U_0} = F_0$. Since $F_0$ is Noetherian and reduced, therefore, $F_0 = \bigoplus_{i = 1}^{s} G_i$ where $G_i$’s are fields. It follows from [2; Proposition (6.5.2), p. 146] that $F_0$ is normal.

It follows from [5; Proposition (1.7.8), p. 116] that $F_0[\{u_i\}]$ is normal. Since $F_0[\{u_i\}]$ is a polynomial ring in $u_i$, and $F_0$ is reduced, therefore $F_0[\{u_i\}]$ is also reduced. $F_0$ is Noetherian implies that $F$ is Noetherian. Then $F = \bigoplus_{i = 1}^{s} H_i$ where $H_i$’s are fields. Thus it follows from [2; Proposition (6.5.2), p. 146] that $F_0[\{u_i\}]$ is integrally closed.

Note: Let $A = Z/(4)[X]$, the polynomial ring in $X$ over $Z/(4)$. $Z/(4)$ is integrally closed, while $A$ is not. Indeed, let $y = (x + 1)/(x - 1)$, $y^2 - 1 = 0$, $y \not\in A$. 

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(4) Let \( x \in \widetilde{R} \). Since \( R \subset R_0[u_i] \), \( x \) is integral over \( F_0[u_i] \). By (3), \( R \subset F_0[u_i] \). The rest of the proof is practically the same argument used in the proof of [10; Theorem 11, p. 157]. We summarize the proof: Let \( x \in \widetilde{R}, x = x_k + \cdots + x_s, k \leq s, x_k \neq 0 \) is called the initial homogeneous term. We want to show that each \( x_i, i = k, \ldots, s \), is integral over \( R \) also. Since \( x \in \widetilde{R} \subset \sum F_i \) there exists \( u_m \in R_m \cap U \) for some positive integer \( m \), such that \( u_m x \in R \). Case (a), if \( R \) is Noetherian, then \( R[x] \) is a finite \( R \)-module. There exists an integer \( \lambda > 0 \) such that \( u_m^i x^i \in R \) for all integer \( i \geq 0 \). Let \( d = u_m^\lambda \). Then \( dR[x] \subset R \). The initial homogeneous term \( dx^i \) is \( dx^i \). \( dx^i \in R \) implies \( dx^i \in R \). Therefore \( x^i \in (1/d)R \), a Noetherian \( R \)-module. Therefore \( R[x_i] \subset R \cdot 1/d \) is a Noetherian \( R \)-submodule. Therefore \( x_k \) is integral over \( R \). Repeating that argument to \( x - x_k = x_{k+1} + \cdots + x_s \), we conclude that \( x_i \in \widetilde{R} \) for \( i = k, \ldots, s \). Therefore \( \widetilde{R} \) is graded in this case. Next we look at case (b): \( R \) is not Noetherian. Let \( x \in \widetilde{R}, \) and \( x^n + a_1x^{n-1} + \cdots + a_n = 0 \) where \( a_1, \ldots, a_n \in R \). As in case (a), there is a homogeneous nonzero divisor \( d \in R \) such that \( dx^i \in R \). Let \( \{y_1, \ldots, y_s\} = \{d, dx_k, \) and homogeneous components of \( a_i \}'s \}. Let \( A = k[y_1, \ldots, y_s], \) where \( k = \mathbb{Z} \) or \( \mathbb{Z}/(n) \) according to whether \( R \) is of characteristic 0 or \( n > 0 \). \( \mathbb{A} \subset R \). Let \( A_q = A \cap R_q \). Then \( A = \Sigma A_q \) is a graded subring of \( R \). \( U \cap A \) contains \( d \). Therefore \( A_{U \cap A} \), the total quotient ring of \( A \), contains \( x_k \), and hence contains \( x \) also. Thus the above integral relation takes place in \( A_{U \cap A} \). Since \( A \) is Noetherian, therefore case (a) is applicable. Therefore \( x_k \) is integral over \( A \) hence \( x_k \) is integral over \( R \).

References


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