

# Pacific Journal of Mathematics

**RELATIONS BETWEEN CONVERGENCE OF SERIES AND  
CONVERGENCE OF SEQUENCES**

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## RELATIONS BETWEEN CONVERGENCE OF SERIES AND CONVERGENCE OF SEQUENCES

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Let  $A = (a_n)_{n \in \mathbf{N}}$  be a sequence of real numbers. For  $\xi \in (0, 1)$  define

$$S_n(\xi, A) := \sum_{k=[n\xi]+1}^n a_k, \quad n \in \mathbf{N}$$

where  $[x]$  is the greatest integer less than or equal to  $x$ . If no ambiguity can arise we write  $S_n(\xi)$  instead of  $S_n(\xi, A)$ . In the theory of regularly varying sequences the problem arose of concluding from the convergence of the sequence  $S_n(\xi)$ ,  $n \in \mathbf{N}$ , for all  $\xi$  in an appropriate set  $K \subset (0, 1)$  of real numbers, that the sequence  $a_n$ ,  $n \in \mathbf{N}$ , converges to zero. In this paper we give some positive results for the case that  $K$  consists of two elements.

In [3] it was shown that such a conclusion is not possible if  $K$  consists only of a single rational number and that the conclusion is possible if  $K = \{\xi, 1 - \xi\}$  with  $\xi \in (0, 1)$  irrational. The question whether such a conclusion is possible if  $K$  consists of one irrational or all rational numbers was answered negatively in [4].

DEFINITION 1. If  $a_n \in \{0, 1\}$ ,  $n \in \mathbf{N}$ , and  $a_n = 1$  for infinitely many  $n \in \mathbf{N}$ , then we call  $A := (a_n)_{n \in \mathbf{N}}$  a 0-1 sequence.

Let  $A = (a_n)_{n \in \mathbf{N}}$  be a sequence of real numbers such that  $S_n(\xi_1, A)$ ,  $n \in \mathbf{N}$ , and  $S_n(\xi_2, A)$ ,  $n \in \mathbf{N}$ , are convergent for different  $\xi_1, \xi_2 \in (0, 1)$ . Let  $\alpha = \liminf_{n \in \mathbf{N}} a_n$  and  $\beta = \limsup_{n \in \mathbf{N}} a_n$ . Since  $\alpha = \beta$  implies  $\lim_{n \in \mathbf{N}} a_n = 0$  — as otherwise  $\lim_{n \in \mathbf{N}} |S_n(\xi_1, A)| = \infty$  — the Lemma below shows that only the following three cases are possible:

- (I)  $\lim_{n \in \mathbf{N}} a_n = 0$
- (II)  $\alpha < \beta$  and each  $\gamma \in (\alpha, \beta)$  is an accumulation point of  $a_n$ ,  $n \in \mathbf{N}$
- (III)  $\alpha < \beta$  and there exists a 0-1 sequence  $B$  such that  $S_n(\xi_i, B)$ ,  $n \in \mathbf{N}$  converges for  $i = 1, 2$ .

LEMMA 2. Let  $A = (a_n)_{n \in \mathbf{N}}$  be a sequence of real numbers such that not every point between  $\alpha := \liminf_{n \in \mathbf{N}} a_n$  and  $\beta := \limsup_{n \in \mathbf{N}} a_n$  is an accumulation point of the sequence  $a_n$ ,  $n \in \mathbf{N}$ . If  $\xi_i \in (0, 1)$ ,  $i = 1, \dots, k$ , and  $S_n(\xi_i, A)$ ,  $n \in \mathbf{N}$ , is convergent for  $i = 1, \dots, k$ , then there exists a 0-1 sequence  $B = (b_n)_{n \in \mathbf{N}}$ , such that  $S_n(\xi_i, B)$ ,  $n \in \mathbf{N}$ , is convergent for  $i = 1, \dots, k$ .

*Proof.* Since not every point between  $\alpha$  and  $\beta$  is an accumulation point of the sequence  $a_n, n \in \mathbb{N}$ , there exist  $\gamma, \delta$  with  $\alpha < \gamma < \delta < \beta$  such that  $a_n \notin (\gamma, \delta)$  for all  $n \in \mathbb{N}$ . Since we can consider the sequence  $(a_n)_{n \in \mathbb{N}}$  or the sequence  $(-a_n)_{n \in \mathbb{N}}$ , we may w.l.g. assume that  $\gamma \geq 0$ .

Let  $b_n = 0$  if  $a_n \leq \gamma$  and  $b_n = 1$  if  $a_n \geq \delta$ . Then  $B = (b_n)_{n \in \mathbb{N}}$  is a 0-1 sequence. According to our assumption there exists  $n_0 \in \mathbb{N}$  such that

$$(+) \quad |S_{n+1}(\xi_i, A) - S_n(\xi_i, A)| < \delta - \gamma \quad \text{if } n \geq n_0, \quad i = 1, \dots, k.$$

Since

$$S_{n+1}(\xi_i, A) - S_n(\xi_i, A) = \begin{cases} a_{n+1}, & \text{if } [n\xi_i] = [(n+1)\xi_i] \\ a_{n+1} - a_{[n\xi_i]+1}, & \text{otherwise} \end{cases}$$

we obtain from (+) that  $S_{n+1}(\xi_i, B) = S_n(\xi_i, B)$  for all  $n \geq n_0, i = 1, \dots, k$ , whence  $S_n(\xi_i, B), n \in \mathbb{N}$ , converges for  $i = 1, \dots, k$ .

Now we shall prove that for most pairs of real numbers — more exactly for all  $\xi_1, \xi_2 \in (0, 1)$  with  $\xi_1^r \neq \xi_2^s$  for all  $r, s \in \mathbb{N}$  — case (III) cannot occur.

**THEOREM 3.** *Let  $a_n, n \in \mathbb{N}$ , be a sequence of real numbers and  $\xi_1, \xi_2 \in (0, 1)$  such that  $S_n(\xi_i), n \in \mathbb{N}$ , is convergent for  $i = 1, 2$ .*

*Assume that:*

$$(*) \quad \xi_1^r \neq \xi_2^s \quad \text{for all } r, s \in \mathbb{N}.$$

*Then  $a_n, n \in \mathbb{N}$ , converges to zero or every real number between  $\liminf_{n \in \mathbb{N}} a_n$  and  $\limsup_{n \in \mathbb{N}} a_n$  is an accumulation point of  $a_n, n \in \mathbb{N}$ .*

*Proof.* Assume that the assertion is false.

Hence according to Lemma 2 we may assume that  $a_n, n \in \mathbb{N}$ , is a 0-1 sequence. Let  $\xi_1 < \xi_2$  and put  $\eta_i := 1/\xi_i$ .

Then there exists  $z > 1$  with  $\eta_1 = \eta_2^z$ . We have to prove that  $z$  is rational. Since  $S_n(\xi_i), n \in \mathbb{N}$ , converges for  $i = 1, 2$  and  $a_n, n \in \mathbb{N}$ , is a 0-1 sequence, there exists  $n_0 \in \mathbb{N}$  with

$$(1) \quad S_n(\xi_i) = S_{n_0}(\xi_i) \quad \text{for } n \geq n_0 \quad (i = 1, 2)$$

$$(2) \quad n_0(1 - \eta_2^{-1/\beta_j}) > 2/(\eta_2 - 1)$$

where  $j := \lim_{n \in \mathbb{N}} S_n(\xi_2) \in \mathbb{N}$ .

Let  $N_1 := \{n \in \mathbb{N} : n > n_0 \text{ and } a_n = 1\}$  and let  $\langle a \rangle := \min\{n \in \mathbb{N} : a \leq n\}$  for  $n \geq 1$ .

Since  $\langle t \cdot \eta \rangle = \inf\{n \in \mathbf{N}: [n \cdot 1/\eta] = t\}$ ,  $t \in \mathbf{N}$ ,  $\eta > 1$ , we have

$$S_{\langle t \cdot \eta \rangle} \left( \frac{1}{\eta} \right) - S_{\langle t \cdot \eta \rangle - 1} \left( \frac{1}{\eta} \right) = a_{\langle t \cdot \eta \rangle} - a_t \quad (t \in \mathbf{N}, \eta > 1)$$

and hence we obtain from (1) that

$$(3) \quad t \in \mathbf{N}_1 \text{ implies } \langle t \cdot \eta_i \rangle \in \mathbf{N}_1 \text{ for } i = 1, 2.$$

Define inductively for  $t \in \mathbf{N}_1$ ,  $\eta > 1$

$$\tau^0(t, \eta) := t$$

and

$$\tau^n(t, \eta) := \langle \tau^{n-1}(t, \eta) \cdot \eta \rangle.$$

According to (3) we directly obtain that

$$(4) \quad t \in \mathbf{N}_1 \text{ implies } \tau^n(t, \eta_i) \in \mathbf{N}_1 \text{ for } n \in \mathbf{N} \text{ and } i = 1, 2.$$

Since  $j = S_n(\xi_2) \in \mathbf{N}$  for all  $n \geq n_0$  according to (1), there exist exactly  $j$  elements  $t_i \in \mathbf{N}_1$ ,  $i = 1, \dots, j$  with

$$(5) \quad n_0 < t_1 < t_2 < \dots < t_j \leq \langle n_0 \cdot \eta_2 \rangle.$$

Since  $\eta_2 > 1$ , (5) implies

$$(6) \quad \tau^n(n_0, \eta_2) < \tau^n(t_1, \eta_2) < \dots < \tau^n(t_j, \eta_2) \leq \tau^{n+1}(n_0, \eta_2)$$

for all  $n \in \mathbf{N}$ . Now we obtain from relations (1), (4), (5) and (6) that

$$(7) \quad \mathbf{N}_1 = \{\tau^n(t_i, \eta_2): i = 1, \dots, j, n \in \mathbf{N} \cup \{0\}\}.$$

As by (4)  $\tau^n(t_1, \eta_1) \in \mathbf{N}_1$ , according to (7) for each  $n \in \mathbf{N}$  there exist  $k(n) \in \mathbf{N}$ ,  $i(n) \in \{1, \dots, j\}$  with

$$(8) \quad \tau^n(t_1, \eta_1) = \tau^{k(n)}(t_{i(n)}, \eta_2).$$

By induction it is easily proved that

$$(9) \quad |\tau^n(t, \eta) - t\eta^n| \leq 1 + \eta + \dots + \eta^{n-1} = (\eta^n - 1)/(\eta - 1)$$

for  $t \in \mathbf{N}$  and  $n > 1$ .

Since  $t_i < \eta_2 t_1$  for  $i = 1, \dots, j$  (see (5)) there exist  $x_i \in [0, 1]$  with

$t_i = t_1 \eta_2^{x_i}$ . Then  $x_1 = 0 < x_2 < \dots < x_j < 1 =: x_{j+1}$ . Hence there exists  $l \in \{1, \dots, j\}$  with

$$(10) \quad x_{l+1} - x_l \geq \frac{1}{j}.$$

Let us now assume that  $z$  is irrational. According to ([2], p. 69) there exists an element  $m \in \mathbb{N}$  with

$$(11) \quad x_l + \frac{1}{3j} < mz - [mz] < x_l + \frac{2}{3j}.$$

Since  $\eta_1 = \eta_2^z$  we obtain from (8) and (9) that

$$|t_1 \eta_2^{mz} - t_{i(m)} \eta_2^{k(m)}| \leq \frac{1}{\eta_1 - 1} \eta_2^{mz} + \frac{1}{\eta_2 - 1} \eta_2^{k(m)}$$

and hence

$$(12) \quad |t_1 \eta_2^{mz} - t_{i(m)} \eta_2^{k(m)}| \leq \frac{2}{\eta_2 - 1} \eta_2^{\max(mz, k(m))}.$$

Now we distinguish four cases

(i) If  $mz < k(m)$  then  $mz - k(m) \leq -1/3j$  according to (10) and (11). Hence we obtain from (5) and (2) that

$$|t_1 \eta_2^{mz - k(m)} - t_{i(m)}| \geq t_1 - t_1 \eta_2^{mz - k(m)} \geq t_1 (1 - \eta_2^{-1/3j}) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

In the following three cases we assume that  $mz > k(m)$ .

(ii) Let  $i(m) \leq l$ : As  $mz - k(m) \geq x_l + 1/3j$  by (11) we obtain from (5) and (2) that

$$|t_1 - t_{i(m)} \eta_2^{k(m) - mz}| \geq t_1 - t_1 \eta_2^{x_{i(m)} + k(m) - mz} \geq t_1 (1 - \eta_2^{-1/3j}) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

(iii) Let  $i(m) > l$  and  $[mz] = k(m)$ : Then  $mz - k(m) \leq x_{l+1} - 1/3j$  by (10) and (11), and we obtain from (5) and (2) that

$$|t_1 - t_{i(m)} \eta_2^{k(m) - mz}| \geq t_{l+1} \eta_2^{-(x_{l+1} - 1/3j)} - t_1 = t_1 (\eta_2^{1/3j} - 1) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

(iv) If  $i(m) > l$  and  $[mz] > k(m)$ , then  $mz - k(m) \geq 1 + 1/3j$  by (11), and we obtain from (5) and (2)

$$|t_1 - t_{i(m)}\eta_2^{k(m)-mz}| \geq t_1 - t_i\eta_2^{-(1+1/3i)} \geq t_1 - t_1\eta_2^{-1/3i} = t_1(1 - \eta_2^{-1/3i}) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

Thus we have shown that the assumption of  $z$  being irrational leads to a contradiction.

If  $r, s \in \mathbf{N}$  denote by  $(r, s)$  the greatest common divisor of  $r$  and  $s$ .

The following remark shows that for two rational numbers condition (\*) of Theorem 3 is nearly always fulfilled.

REMARK 4. If  $\xi_1, \xi_2 \in (0, 1)$  are rational numbers and  $\xi_1^r = \xi_2^s$  for  $r, s \in \mathbf{N}$  with  $(r, s) = 1$  then there exist  $t, u \in \mathbf{N}$  such that  $\xi_1 = (t/u)^s$  and  $\xi_2 = (t/u)^r$ .

*Proof.* Let w.l.g.  $\xi_i = l_i/m_i$  where  $l_i, m_i \in \mathbf{N}$  and  $(l_i, m_i) = 1$  for  $i = 1, 2$ . If  $\xi_1^r = \xi_2^s$  i.e.  $l_1^r m_2^s = l_2^s m_1^r$ , then  $l_1^r = l_2^s$  and  $m_1^r = m_2^s$ .

We may choose  $r$  and  $s$  such that  $(r, s) = 1$ . Then by representation of  $l_i, m_i$  as a product of prime numbers we obtain  $t, u \in \mathbf{N}$  with

$$t^s = l_1, \quad t^r = l_2 \quad \text{and} \quad u^s = m_1, \quad u^r = m_2.$$

According to Theorem 3 Cases I and II can occur. According to Example 2 of [4] it is not possible to exclude Case II. Even if  $S_n(\xi, A)$ ,  $n \in \mathbf{N}$ , converges for each rational number  $\xi \in (0, 1)$  the sequence  $a_n, n \in \mathbf{N}$ , need not converge to zero.

We remark that the following questions remain unsolved:

(1) If  $\xi_1$  and  $\xi_2$  are two different irrational numbers, does the convergence of  $S_n(\xi_i, A)$ ,  $n \in \mathbf{N}$ , (for  $i = 1, 2$ ) imply that  $a_n, n \in \mathbf{N}$ , converges to zero?

(2) Give an exact characterization of those pairs of rational numbers  $\xi_1, \xi_2$  for which only Case I or II is possible; is for instance the condition (\*) of Theorem 3 such an exact characterization?

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