ANOTHER MARTINGALE CONVERGENCE THEOREM

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A classical martingale theorem is generalized to “martingale like” sequences. The method of proof is a generalization of Doob’s proof by “downcrossings”.

**Introduction.** Let \((\Omega, B, P)\) be a probability space, \(\{B_n\}\) an increasing sequence of sub sigma fields of \(B\). Let \(\{f_n, B_n, n \geq 1\}\) be an adapted sequence of \(P\)-integrable random variables.

The sequence is said to be a martingale in the limit if

\[
\limsup_{n \to \infty} \left| f_n - E(f_n \mid B_n) \right| = 0 \quad P \text{ a.e.}
\]

If was proven in an earlier paper, Mucci [3] that every uniformly integrable martingale in the limit converges both \(L_1\) and \(P\) a.e., generalizing the corresponding martingale theorem. The purpose of the present note is to prove that every \(L_1\)-bounded martingale in the limit converges pointwise to an integrable random variable, thereby generalizing another classical martingale theorem. We recall that a sequence \(\{f_n\}\) is said to be \(L_1\)-bounded if \(\sup_n \int |f_n| < \infty\).

**Theorem.** Let \(\{f_n, B_n, n \geq 1\}\) be an \(L_1\)-bounded martingale in the limit. Then there exists \(f \in L_1\) with \(f_n \to f\) \(P\) a.e.

**Proof.** Fix \(a < b\), two arbitrary real numbers. We define, following the classical proof:

\(\varphi(a, b)\) is the number of “downcrossings” of \(\{f_n\}\) from above \(b\) to below \(a\). Our objective will be to show that \(P(\varphi(a, b) = \infty) = 0\) so that \(P(\lim f_n \leq a < b \leq \lim f_n) = 0\), thereby determining that \(f = \lim_n f_n\) exists almost everywhere, and since

\[
\int |f| < \lim \int |f_n| < \infty; \quad \text{that } f \in L_1.
\]

Our procedure consists in defining a “modified” number of downcrossings \(\varphi(a, b)\) and showing that \(P(\varphi(a, b) = \infty) = 0\) and further that, almost everywhere,

\[
\varphi(a, b) < \infty \quad \text{implies } \varphi(a, b) < \infty.
\]
We begin by defining a sequence of stopping times:

\[ \tau_0 = 0. \]

Now let \( \{\alpha_n\} \) be a decreasing sequence of positive numbers with \( \sum \alpha_n < \infty \), and let \( N \) be a fixed positive integer.

Define \( \tau_{2n-1} \) as the first \( m \leq N \) such that:

1. \( m > \tau_{2n-2} \)
2. \( f_m > b \)
3. \( \sup_{m > \tau_{2n-1}} |f_m - E(f_m | B_m)| < \alpha_n \).

If no such \( m \) exists, set \( \tau_{2n-1} = N \).

Likewise, define \( \tau_{2n} \) as the first \( m \leq N \) such that:

1. \( m > \tau_{2n-1} \)
2. \( f_m < a \)
3. \( \sup_{m > \tau_{2n}} |f_m - E(f_m | B_m)| < \alpha_n \).

If no such \( m \) exists, set \( \tau_{2n} = N \). We have

\[
\int f_{\tau_{2n-1}} - \int f_{\tau_{2n}} = \sum_{k=1}^{N} \int_{(\tau_{2n-1} = k)} (f_k - E(f_N | B_k)) + \sum_{k=1}^{N} \int_{(\tau_{2n} = k)} (E(f_N | B_k) - f_k) < 2\alpha_n.
\]

Thus

\[
(*) \sum_1^\infty (f_{\tau_{2n-1}} - f_{\tau_{2n}}) < 2 \sum_1^\infty \alpha_n = 2\alpha.
\]

We want an inequality in the other direction.

Define

\[
\phi(N, a, b) = \sum_1^\infty (I_{(f_{\tau_{2n-1}} > b)} \cdot I_{(f_{\tau_{2n}} < a)} \cdot I_{(\sup_{m > \tau_{2n}} |E(f_{\tau_{2n}}|B_{\tau_{2n}}) - f_{\tau_{2n}}| < \alpha_n)}
\]

the number of times we make a "downcrossing" subject to conditions (3), (3) on our stopping times.

We have

\[
\sum_1^\infty (f_{\tau_{2n-1}} - f_{\tau_{2n}}) \geq (b - a) \phi(N, a, b) - |b| - |f_N|.
\]

Taking integrals, defining

\[
\phi(a, b) = \lim_{N \to \infty} \phi(N, a, b),
\]
and using Fatou’s lemma and (*), we have

\[ \int \tilde{\varphi}(a, b) < \frac{1}{b-a} \left[ |b| + 2a + \sup_n \int |f_n| \right] < \infty. \]  

Therefore \( P(\varphi(a, b) < \infty) = 1. \)

Let us now define

\[ \Omega_0 = (\varphi(a, b) < \infty) \cap \left( \limsup_{n \to \infty} \sup_{n > m} |f_n - E(f_n | B_n)| = 0 \right). \]

Clearly \( P(\Omega_0) = 1 \) and we will be finished if we can show that \( \varphi(a, b) < \infty \)
on \( \Omega_0. \) Now, for a particular \( \omega \in \Omega_0, \) let \( \varphi(a, b) = M. \) Suppose \( \varphi(a, b) = \infty. \)

Then we can find a sequence \( \{n_k\} \) where \( f_{n_{2k-1}} \geq b, f_{n_{2k}} \leq a \) and where \( (3), (3) \) hold. This contradicts \( \varphi(a, b) = M. \)

**Corollary 1.** Let \( \{f_n, B_n, n \geq 1\} \) be a martingale in the limit, and let \( r \geq 1. \) Then there exists \( f \in L_r \), such that \( f_n \to f \) both \( P \) a.e. and in the \( L_r \)-norm \( \iff \{|f_n|^r\} \) is uniformly integrable.

**Proof.** If \( \{|f_n|^r\} \) is uniformly integrable, then \( \{f_n, B_n\} \) is \( L_1 \)-bounded, hence \( f_n \to fP \) a.e. The rest follows by the usual classical arguments. (See Neveu, p. 57.)

**Corollary 2.** Let \( s_n = \sum_1^n \xi_k \) where \( \{\xi_k\} \) is an independent sequence. Then \( s_n \to s \in L_1 \) both \( P \) a.e. and \( L_1 \) provided \( \{s_n\} \) is Cauchy in the \( L_1 \)-norm.

**Proof.** The Cauchy condition is equivalent to \( \{s_n, B_n, n \geq 1\} \) being a martingale in the limit (here \( B_n = \sigma(\xi_1 \cdots \xi_n) \)). Further,

\[ \sup_n \int |s_n| \leq \int |s_M| + \sup_{n \geq M} \int |s_n - s_M| < \infty. \]

**References**


Received July 28, 1975 and in revised form January 13, 1976.

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