QUASITRIANGULAR OPERATOR ALGEBRAS

JOAN KATHRYN PLASTIRAS
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JOAN K. PLASTIRAS

Fix a sequence $\mathcal{P} = \{P_n\}_{n=1}^\infty$ of finite dimensional projections increasing to the identity on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$. The quasitriangular algebra associated with $\mathcal{P}$ and denoted as $\mathcal{QT}(\mathcal{P})$ is defined to be the set of those operators $T$ in $\mathcal{L}(\mathcal{H})$ for which $\|P_n TP_n\| \to 0$.

In this paper we will examine the structure of the $\mathcal{QT}(\mathcal{P})$ algebras. Specifically, if $\mathcal{R} = \{R_n\}_{n=1}^\infty$ is another sequence of finite dimensional projections increasing to the identity on the same Hilbert space, when is $\mathcal{QT}(\mathcal{R})$ equal to $\mathcal{QT}(\mathcal{P})$? By an algebraic isomorphism between two algebras we shall mean a bijection which preserves algebraic structure: that is to say — addition, scalar multiplication, multiplication, but we do not impose any topological condition. When are two quasitriangular algebras isomorphic?

In [5] we asked the same questions of $\mathcal{D}(\mathcal{E}) + \mathcal{C}(\mathcal{H}) = \{T + K : T$ belongs to the commutant of $E$ and $K$ is compact} and answered them completely by arguments very different from those presented here; the conclusions were different too. The concept of quasitriangularity for operators was first isolated for systematic study in [3]. The quasitriangular algebra was introduced later in [1] and a formula expressing the distance from such an algebra to an arbitrary operator was obtained. We begin our discussion with an algebraic property:

**Definition 1.** A subset $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ is said to be inverse-closed if whenever $T$ in $\mathcal{I}$ is invertible in $\mathcal{L}(\mathcal{H})$ then $T^{-1}$ belongs to $\mathcal{I}$.

**Lemma 2.** $\mathcal{QT}(\mathcal{P})$ is inverse-closed for every sequence $\mathcal{P} = \{P_n\}_{n=1}^\infty$ of finite dimensional projections increasing to the identity on a Hilbert space.

Before verifying Lemma 2 we remark that the assumption that the $P_k$ be finite dimensional is essential.

**Proof.** From [1, Corollary following 2.2] we know that $\mathcal{QT}(\mathcal{P}) = \mathcal{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$, where $\mathcal{T}(\mathcal{P})$ is the set of operators $T$ such that $P_n TP_n = 0$ for all $n$. Hence, it suffices to assume that $S$ belongs to $\mathcal{T}(\mathcal{P}) + \mathcal{C}(\mathcal{H})$ and is invertible in $\mathcal{L}(\mathcal{H})$ and show that $S^{-1}$ belongs to $\mathcal{QT}(\mathcal{P})$. So, $S = T + C$, where $T \in \mathcal{T}(\mathcal{P})$ and $C \in \mathcal{C}(\mathcal{H})$. Since $S_m = T + P_mC_P$ tends in norm to $S$, $S_m$ is invertible for all $m$ greater than a positive
integer $l$. Fix $m > l$ and note that $S_m P_n = P_n S_m P_n$ for all $n \geq m$, and since $\dim P_n < \infty$, $S_m$ maps $P_n \mathcal{H}$ onto itself, so that $P_n S_m^{-1} P_n = S_m^{-1} P_n$ (or equivalently, $P_n S_m^{-1} P_n = 0$). Hence, $S_m^{-1}$ belongs to $\mathcal{F}(\mathcal{P})$ by definition. As $S_m^{-1}$ tends in norm to $(T + C)^{-1}$ and $\mathcal{F}(\mathcal{P})$ is norm-closed [1, Proposition 2.1], we conclude that $(T + C)^{-1}$ belongs to $\mathcal{F}(\mathcal{P})$.

**Theorem 3.** Suppose that $T$ is an invertible operator in $L(\mathcal{H})$. Then $T$ implements an automorphism of $\mathcal{F}(\mathcal{P})$ (i.e. $T \mathcal{F}(\mathcal{P}) T^{-1} = \mathcal{F}(\mathcal{P})$) if and only if $T$ belongs to $\mathcal{F}(\mathcal{P})$.

**Proof.** $\Leftarrow$: Assume that $T$ belongs to $\mathcal{F}(\mathcal{P})$. To show that $T$ implements an inner automorphism of $\mathcal{F}(\mathcal{P})$ it will suffice to show that $T^{-1}$ also belongs to $\mathcal{F}(\mathcal{P})$. But that is immediate from Lemma 2.

$\Rightarrow$: Assume that $T$ implements an automorphism of $\mathcal{F}(\mathcal{P})$. First we conclude from [1, Theorem 3.3] that $T$ admits a factorization $T = UA$, where $A$ belongs to $\mathcal{F}(\mathcal{P})$ and $U$ is a partial isometry. Note that $A = U^* T$ has closed range; since $\ker A = \{0\}$, $A$ is semi-Fredholm by definition. Since $A$ belongs to $\mathcal{F}(\mathcal{P})$ the index of $A$ is nonnegative [2] so that $\ker A^* = \{0\}$ and $A$ is consequently invertible. This forces $U$ to be unitary. Since $A \in \mathcal{F}(\mathcal{P})$ is invertible, then by the previous argument, $A$ implements an automorphism of $\mathcal{F}(\mathcal{P})$ so that we are reduced to showing that if $U$ is a unitary operator which implements an automorphism of $\mathcal{F}(\mathcal{P})$, then $U$ belongs to $\mathcal{F}(\mathcal{P})$.

So, we assume that $U$ does not belong to $\mathcal{F}(\mathcal{P})$ and arrive at a contradiction. Since $U$ does not belong to $\mathcal{F}(\mathcal{P})$ then by the definition of $\mathcal{F}(\mathcal{P})$ there is an $\alpha > 0$ and a subsequence $\{P_{n(k)}\}_{k=1}^\infty$ of $\mathcal{P}$ for which $\lim_k \|P_{n(k)} U P_{n(k)}\| \geq \alpha$. From Lemma 2 we know that $U^*$ does not belong to $\mathcal{F}(\{P_{n(k)}\}_{k=1}^\infty)$, so that by definition, there is $\beta > 0$ and a subsequence $\{m(k)\}_{k=1}^{\infty}$ of $\{n(k)\}_{k=1}^{\infty}$ for which $\lim_k \|P_{m(k)} U^* P_{m(k)}\| \geq \beta$. If we let $\epsilon = \min(\alpha, \beta)/2$, then we can conclude that $\|P_{n} U P_{n}\|$ and $\|P_{n} U P_{n}\|$ are both greater than $\epsilon$ for all $n$ in an infinite subset $M$ of $\mathbb{N}$.

We will obtain a sequence $\{m_n, n_n\}_{n=1}^{\infty}$ of positive integers such that $0 < m_1 < n_1 < m_2 < n_2 < \cdots$ and projections $\{F_k, E_k\}_{k=1}^{\infty}$ such that $F_k = P_{m_k} P_{n_k}$ and $E_k = P_{n_k} P_{m_k}$ for which $\|F_k U E_k\|$ and $\|E_k U F_k\|$ are both greater than $\epsilon/2$. We do so inductively.

For $k = 1$, define $F_1 = P_{m_1}$, where $m_1$ is the first integer in $M$. Let $n_1$ be the first integer such that $\|P_{m_1} P_{m_1} U P_{n_1}\|$ and $\|P_{m_1} U P_{m_1} P_{n_1}\|$ are both greater than $\epsilon/2$ (such an $n_1$ exists because $\|P_{m_1} U P_{m_1}\|$ and $\|P_{m_1} U P_{m_1}\|$ are greater than $\epsilon$ and the $P_n$ tend strongly to the identity).

Assume that we have obtained $\{E_k, F_k\}_{k=1}^{\infty}$. To obtain $m_{i+1}$ and $n_{i+1}$, note that $U P_{n_i}$ and $P_{n_i} U$ are compact; hence, there is a positive integer $j$ such that $\|P_{n_i}^j U P_{n_i}\|$ and $\|P_{n_i} U P_{n_i}^j\|$ are both less than $\epsilon/4$ for all $n \geq j$. Let $m_{i+1}$ be the first integer in $M$ greater than $j$. 


Then

\[ \| P_{m_{i+1}} \|_{UP_{m_{i+1}}} \geq \| P_{m_{i+1}} \|_{UP_{m_{i+1}}} - \| P_{m_{i+1}} \|_{UP_{m_{i+1}} P_{m_{i+1}} P_n} \]
\[ \geq \epsilon - \| P_{m_{i+1}} \|_{UP_{n}} \]
\[ \geq \epsilon - \epsilon / 4 = \frac{3}{4} \epsilon. \]

Similarly, \( \| P_{m_{i+1}} \|_{P_{n} UP_{m_{i+1}}} \geq 3 \epsilon / 4 \) by the same argument. Let \( n_{i+1} \) be the first positive integer greater than \( m_{i+1} \) for which \( \| P_{n_{i+1}} \|_{P_{n_{i+1}} UP_{m_{i+1}}} \) and \( \| P_{m_{i+1}} \|_{P_{n} UP_{m_{i+1}} P_{n_{i+1}}} \) are both greater than \( \epsilon / 2 \). Let \( F_{i+1} = P_{m_{i+1}} \) and let \( E_{i+1} = P_{n_{i+1}} P_{m_{i+1}} \). Continue inductively.

We select a subsequence \( \{ E_{i}, F_{i} \}_{i=1}^{\infty} \) of \( \{ E_{i}, F_{i} \}_{i=1}^{\infty} \) as follows: first, we let \( \{ \alpha_{i} \}_{i=1}^{\infty} \) be any sequence of positive real numbers such that \( \sum \alpha_{i} \leq \epsilon^2 / 16 \). Let \( \alpha_{i} = 1 \). Assuming that we have obtained \( \alpha_{k} \), let \( \alpha_{k+1} \) be the next positive integer such that for all \( l \leq k + 1 \), \( \| E_{k+1} \|_{U F_{l}} \) and \( \| F_{k+1} \|_{U E_{l}} \) are less than \( \alpha_{k+1} \) while \( \| E_{l} \|_{U F_{k+1}} \) and \( \| F_{l} \|_{U E_{k+1}} \) are less than \( \alpha_{l} \). This is possible because \( U F_{l}, F_{l} U \) (respectively \( U E_{l}, E_{l} U \)) are compact and the \( E_{l} \) (respectively \( F_{l} \)) tend weakly to zero. Continue inductively. Now for each \( \alpha_{k} \) there is a rank one partial isometry \( T_{k} \in \mathcal{L}(E_{k} \mathcal{H}, F_{k} \mathcal{H}) \) such that \( \| E_{k} \|_{U T_{k} U} \| U F_{k+1} \| \geq \epsilon^2 / 4 \). Clearly, \( T = \sum_{k=1}^{\infty} T_{k} \) is a partial isometry in \( \mathcal{F}(\mathcal{P}) \). So, for arbitrary \( l \) in \( \mathbb{N} \),

\[ E_{i} (U T_{i} U^*) F_{i} = \sum_{k=1}^{\infty} E_{i} U T_{k} U^* F_{i} = E_{i} U T_{i} U^* F_{i} + \sum_{k=1}^{\infty} E_{i} U T_{k} U^* F_{i}. \]

Hence,

\[ \| E_{i} (U T_{i} U^*) F_{i} \| + \left\| \sum_{k=1}^{\infty} E_{i} U T_{k} U^* F_{i} \right\| \geq \| E_{i} U T_{i} U^* F_{i} \|. \]

\[ \| E_{i} (U T_{i} U^*) F_{i} \| + \sum_{k=1}^{\infty} \| E_{i} U T_{k} U^* F_{i} \| \geq (\epsilon / 2)^2 = \epsilon^2 / 4. \]

Therefore,

\[ \| E_{i} (U T_{i} U^*) F_{i} \| \geq \frac{\epsilon^2}{4} - \sum_{k \neq i} \| E_{i} U F_{k} \| \cdot \| E_{i} U^* F_{i} \| \]
\[ \geq \frac{3 \epsilon^2}{16}. \]

Since \( \alpha_{i} \) was arbitrary, it follows from the construction that

\[ \frac{3 \epsilon^2}{16} \leq \| E_{i} (U T_{i} U^*) F_{i} \| \leq \| P_{m_{i}} (U T_{i} U^*) P_{m_{i}} \|. \]
Hence,

$$\lim_{k} \left\| P_k(UTU^*)P_k \right\| > 0$$

and it follows by definition of $\mathcal{H}(\mathcal{P})$ that $UTU^*$ does not belong to $\mathcal{H}(\mathcal{P})$. This contradicts our assumption that $U$ implements an automorphism of $\mathcal{H}(\mathcal{P})$ and thus concludes the argument of the proof of Theorem 3.

**Definition 4.** Let $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ be a sequence of finite dimensional projections increasing to the identity on a Hilbert space $\mathcal{H}$. An operator $T$ is said to be strictly upper triangular for $\mathcal{P}$ if $P_n TP_{n+1} = 0$ for all $n$ in $\mathbb{N}$.

**Remark 5.** Note that in the proof of Theorem 3 we showed that if $U$ does not belong to $\mathcal{H}(\mathcal{P})$ then there is an operator $T$, which is strictly upper triangular for $\mathcal{P}$, and such that $UTU^*$ does not belong to $\mathcal{H}(\mathcal{P})$.

**Remark 6.** Let $\mathcal{I} = \{S_n\}_{n=1}^{\infty}$ be any sequence of finite dimensional projections increasing to the identity on $\mathcal{H}$. Let $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ be a subsequence of $\mathcal{I}$. Then $\mathcal{H}(\mathcal{I}) \subseteq \mathcal{H}(\mathcal{P})$. Equality may fail; however, if $T$ is strictly upper triangular for $\mathcal{P}$ then $T$ belongs to $\mathcal{H}(\mathcal{I})$.

**Definition 7.** A sequence of projections $\mathcal{I} = \{S_n\}_{n=1}^{\infty}$ increasing to the identity on a Hilbert space $\mathcal{H}$ is said to be a defining sequence for a quasitriangular algebra $\mathcal{A}$ if and only if $\mathcal{A} = \{T \in \mathcal{L}(\mathcal{H}) : \left\| S_nTS_n \right\| \to 0\}$.

**Remark 8.** Suppose that $U$ is a unitary operator which implements an isomorphism $T \rightarrow UTU^*$ from $\mathcal{H}(\mathcal{P})$ onto $\mathcal{H}(\mathcal{I})$. Then $U$ maps defining sequences of $\mathcal{H}(\mathcal{P})$ to defining sequences of $\mathcal{H}(\mathcal{I})$.

**Lemma 9.** Suppose that $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ and $\mathcal{I} = \{S_n\}_{n=1}^{\infty}$ are sequences of finite dimensional projections increasing to the identity such that $\mathcal{P} \cup \mathcal{I}$ is totally ordered by inclusion. Then $\mathcal{H}(\mathcal{P}) = \mathcal{H}(\mathcal{I})$ if and only if there exist positive integers $m_0$ and $n_0$ such that $P_{m_0+k} = S_{n_0+k}$ for all $k$ in $\mathbb{N}$.

**Proof.** $\Leftarrow$ : This conclusion is clear.

$\Rightarrow$ : Assume that $\mathcal{H}(\mathcal{P}) = \mathcal{H}(\mathcal{I})$. Then $\mathcal{H}(\mathcal{P}) = \mathcal{H}(\mathcal{P} \cup \mathcal{I})$. We assert that $\mathcal{P}$ contains all but perhaps finitely many of the projections in $\mathcal{P} \cup \mathcal{I}$. Contrapositively, assume not. Let $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ be a total ordering of $\mathcal{P} \cup \mathcal{I}$ and choose an infinite subsequence $\{n_k\}_{k=1}^{\infty}$ for which $R_{n_k} \not\in \mathcal{P}$ but $R_{n_k+1} \in \mathcal{P}$. Let $T_k$ be any rank one partial isometry with initial space $(R_{n_k} \ominus R_{n_k-1})\mathcal{H}$ and final space $(R_{n_k+1} \ominus R_{n_k})\mathcal{H}$. Then $T = \Sigma_{k=1}^{\infty} T_k$ is a partial isometry which belongs to $\mathcal{H}(\mathcal{P})$ but not to $\mathcal{H}(\mathcal{P} \cup \mathcal{I})$. 

Hence, $\mathcal{D}(\mathcal{P} \cup \mathcal{I}) \subseteq \mathcal{D}(\mathcal{P})$. We conclude that $\mathcal{P}$ contains all but perhaps finitely many of the projections in $\mathcal{P} \cup \mathcal{I}$.

By symmetry, $\mathcal{I}$ contains all but perhaps finitely many of the projections in $\mathcal{P} \cup \mathcal{I}$. So there exists a positive integer $k$ such that $\{P_n: \dim P_n \geq k\} \subseteq \mathcal{I}$ and $\{S_n: \dim S_n \geq k\} \subseteq \mathcal{P}$. Let $m_0$ be the first positive integer such that $\dim(P_{m_0}) \geq k$ and let $n_0$ be the first integer such that $\dim(S_{n_0}) \geq k$. Then $P_{m_0+k} = S_{n_0+k}$ for all $k \in \mathbb{N}$.

**Theorem 10.** $\mathcal{I} = \{S_n\}_{n=1}^\infty$ is a defining sequence for $\mathcal{D}(\mathcal{P})$ if and only if there exist positive integers $m_0$ and $n_0$ such that $\lim_k \|P_{m_0+k} - S_{n_0+k}\| = 0$.

**Proof.** $\Leftarrow$: We note that $\mathcal{D}(\mathcal{I}) \subseteq \mathcal{D}(\mathcal{P})$ since for $T$ in $\mathcal{D}(\mathcal{I})$, $P_{m_0+k} TP_{m_0+k} \leq \|S_{n_0+k} TS_{n_0+k}\| + \|P_{m_0+k} - S_{n_0+k}\| TS_{n_0+k}$ and the other inclusion follows by symmetry.

$\Rightarrow$: We assume that $\mathcal{I} = \{S_n\}_{n=1}^\infty$ is a defining sequence for $\mathcal{D}(\mathcal{P})$. Let $V$ be any unitary operator such that $\{VS_n V^*\}_{n=1}^\infty \cup \{P_n\}_{n=1}^\infty$ is a sequence of projections totally ordered by set inclusion.

Let $\mathcal{W} = \{W_n\}_{n=1}^\infty$ with $W_n = VS_n V^*$ for each $n$. We assert that $V$ belongs to $\mathcal{D}(\mathcal{W})$. So assume that $T$ is strictly upper triangular for $\mathcal{W}$; it suffices to show that $VT V^*$ belongs to $\mathcal{D}(\mathcal{W})$ by Remark 5. By Remark 6, $T$ belongs to $\mathcal{D}(\mathcal{P} \cup \mathcal{I}) \subseteq \mathcal{D}(\mathcal{P})$ so that it remains to observe that $\mathcal{D}(\mathcal{P}) V^* \subseteq \mathcal{D}(\mathcal{W})$: $W_n(VT V^*) W_n = (VS_n V^*)(VT V^*)(VS_n V^*) = VS_n TS_n V^*$, so that $\|W_n(VT V^*) W_n\| = \|VS_n TS_n V^*\| = \|S_n TS_n\| \rightarrow 0$.

Hence, we conclude that $V$ belongs to $\mathcal{D}(\mathcal{W})$. Since $\mathcal{D}(\mathcal{W})$ is inverse-closed by Lemma 2, it follows that $\|W_n V W_n\| \rightarrow 0$ and $\|W_n V W_n\| = \|V_n V^* W_n\| \rightarrow 0$.

1. Since $W_n V = VS_n$, we have that $W_n V W_n = VS_n W_n$ so that $\|W_n V W_n\| = \|VS_n W_n\| = \|S_n W_n\| \rightarrow 0$ and

2. Since $W_n V = VS_n$, we have that $W_n V W_n = VS_n W_n$ so that $\|W_n V W_n\| = \|VS_n W_n\| = \|S_n W_n\| \rightarrow 0$.

Since $\|S_n - W_n\| = \max\{|S_n W_n|, \|S_n W_n\|\}$ [5, Lemma 6] it follows that $\lim_n \|S_n - W_n\| = 0$ and by a previous argument that $\mathcal{W}$ is a defining sequence for $\mathcal{D}(\mathcal{P})$. It follows from Lemma 9 that there are integers $m_0$ and $n_0$ such that $W_{m_0+k} = P_{m_0+k}$ for all $k \in \mathbb{N}$. Hence
\[ \lim_k \| S_{m+k} - P_{m+k} \| = 0, \]
which concludes the proof.

**Example 11.** As an easy consequence of Theorem 10, it follows that there exist defining sequences \( \mathcal{P} = \{P_n\}_{n=1}^\infty \) and \( \mathcal{R} = \{R_n\}_{n=1}^\infty \) for a quasitriangular algebra \( \mathcal{A} \) such that \( \{P_n \vee R_n\}_{n=1}^\infty \) is not a defining sequence for \( \mathcal{A} \) ("\( \vee \)" denotes the supremum of two projections). This phenomenon is suggested by an example in [3, p. 285].

We shall say that two subsets of \( \mathcal{L}(\mathcal{H}) \), \( \mathcal{I} \) and \( \mathcal{J} \), are **locally isomorphic** if each operator in \( \mathcal{I} \) is unitarily equivalent to an operator in \( \mathcal{J} \) and conversely. Because every quasitriangular operator is a compact perturbation of a triangular operator, it follows that any two quasitriangular algebras are locally isomorphic; from Theorem 12 we conclude that they are not necessarily isomorphic.

**Theorem 12.** Let \( \mathcal{H}(\mathcal{P}) \) and \( \mathcal{H}(\mathcal{I}) \) be quasitriangular algebras. Then \( \mathcal{H}(\mathcal{P}) \) and \( \mathcal{H}(\mathcal{I}) \) are algebraically isomorphic if and only if there exist positive integers \( j_0 \) and \( l_0 \) such that \( \dim(P_{j+k}) = \dim(S_{l+k}) \) for all \( k \) in \( \mathbb{N} \).

**Proof.** \( \Leftarrow \): If we assume that there exist positive integers \( j_0 \) and \( l_0 \) such that \( \dim(P_{j+k}) = \dim(S_{l+k}) \) for all \( k \) in \( \mathbb{N} \), then we can define a unitary operator \( U \) such that \( U P_{j+k} U^* = S_{l+k} \) for all \( k \) in \( \mathbb{N} \). We assert that \( U \) implements an isomorphism from \( \mathcal{H}(\mathcal{P}) \) to \( \mathcal{H}(\mathcal{I}) \).

\( \Rightarrow \): Assume that there is a map \( \alpha \) from \( \mathcal{H}(\mathcal{P}) \) to \( \mathcal{H}(\mathcal{I}) \) which preserves algebraic structure. Since \( \mathcal{H}(\mathcal{P}) \) and \( \mathcal{H}(\mathcal{I}) \) are Banach algebras, each of which contains the set of finite rank operators, it follows from [6, Theorem 2.5.19] that there exists an invertible operator \( S \) such that \( \alpha(T) = STS^{-1} \) for all \( T \) in \( \mathcal{H}(\mathcal{P}) \).

We conclude from [1, Theorem 3.3] that \( S \) has a factorization \( S = UA \) where \( A \) belongs to \( \mathcal{F}(\mathcal{P}) \) and \( U \) is unitary. Then we note that \( R_n = UP_n U^* \) is a defining sequence for \( \mathcal{H}(\mathcal{I}) \); by Theorem 10, we note that there exist positive integers \( m_0 \) and \( n_0 \) such that \( \| R_{m+k} - S_{m+k} \| \to 0 \). So, there exists a positive integer \( d \) such that \( \| R_{m+d+k} - S_{m+d+k} \| < 1 \) for all \( k \) in \( \mathbb{N} \). Hence, \( \dim(R_{m+d+k}) = \dim(S_{m+d+k}) \) for all \( k \) in \( \mathbb{N} \). Since \( \dim(P_n) = \dim(R_n) \) for all \( n \) in \( \mathbb{N} \), let \( j_0 = m_0 + d \) and let \( l_0 = n_0 + d \) to obtain the theorem.

**References**


Received February 23, 1976.

UNIVERSITY OF PENNSYLVANIA
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The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

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<td>559</td>
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