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THE GEOMETRY OF $p(S^1)$

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J. R. QUINE

Let p be a polynomial of degree n . The image of the unit circle, $p(S^1)$, can be thought of as a subset of the real part of an algebraic curve W of degree $2n$. This paper outlines some facts about $p(S^1)$ which can be obtained using classical algebraic geometry, for example Bézout's theorem.

Introduction. We wish to study the image of the unit circle S^1 in the complex plane under mapping by a polynomial of degree n . If we let $x^2 + y^2 = 1$ be the equation of the unit circle in R^2 , then if x and y vary over the complex numbers C , we can think of the unit circle as the real part of an algebraic variety V in C^2 . We show that similarly $p(S^1)$ can be thought of as a subset of the real part of an algebraic variety W in C^2 . We use the method of absolute coordinates as outlined in Winger [12] and Morley [7], and we discuss W in terms of the Schwarz function as used by Davis [2].

We obtain the equation for the real part of W in the form $h(\xi, \bar{\xi}) = 0$, where h is a polynomial of degree $2n$. We show that if all the zeros of p' are in $|z| < 1$, then $p(S^1)$ is actually all of the real part of W . We show that the circular points are of multiplicity n on W and that W has at most $(n - 1)^2$ simple nodes. If no singular point of W is on $p(S^1)$ then p is univalent, i.e., one-to-one in $|z| < 1$. We give this condition in terms of a Hermitian form.

1. Definitions. Let C denote the complex numbers. In the following, we consider C as a subset of C^2 , identifying the complex number z with the point $(z, \bar{z}) \in C^2$. We say (z, \bar{z}) are absolute coordinates of z (Winger [12] p. 324). If V is a set in C^2 we will call $C \cap V = \{(z, \zeta) \in V \mid \zeta = \bar{z}\}$ the real part of V .

Let $S^1 = \{z \mid |z| = 1\}$ be the unit circle in C . The equation of S^1 in absolute coordinates is $z\bar{z} = 1$, so we may consider S^1 as the real part of the variety $V \subseteq C^2$ given by the equation $z\zeta = 1$.

Let $p(z) = a_0 + a_1z + \cdots + a_nz^n$ be a polynomial of degree n . Let $\bar{p}(z) = \bar{a}_0 + \bar{a}_1z + \cdots + \bar{a}_nz^n$. We consider p as a map from C to C . Since $(z, \bar{z}) \rightarrow (p(z), \bar{p}(z))$ gives the mapping in absolute coordinates, we may look at p as the restriction to C of the mapping $\bar{p}: C^2 \rightarrow C^2$ defined by $(z, \zeta) \rightarrow (p(z), \bar{p}(\zeta))$.

2. $\bar{p}(V)$. We now look at $W = \bar{p}(V)$, which is the rational curve in C^2 given by parametric equations $\xi = p(z)$, $\eta = \bar{p}(1/z)$. We find

S^1 . We see that $V = \{(z, \overline{z^*}) \mid z \in \mathbb{C}\}$. The function $z \rightarrow \overline{z^*} = 1/z$ is called the Schwarz function for S^1 (Davis [2]), and V may be considered as the graph in \mathbb{C}^2 of the Schwarz function.

Likewise near a nonsingular point of $p(S^1) \subseteq W$, the function $p(z) \rightarrow p(1/\bar{z})$ is reflection in the analytic arc $p(S^1)$, and locally this function followed by conjugation is called the Schwarz function for $p(S^1)$. Writing $\eta = S(\xi)$ for the Schwarz function, we see that the complete analytic function that it determines is algebraic satisfying $h(\xi, \eta) = 0$, where h is as in the previous section. Thus W may be considered as the graph of the Schwarz function for $p(S^1)$.

4. $W \cap \mathbb{C} - p(S^1)$. We have seen that $p(S^1) \subseteq W \cap \mathbb{C}$. If $\xi = p(z) = p(1/\bar{z})$ for $|z| \neq 1$, then $\xi \in W \cap \mathbb{C}$, but ξ is not on $p(S^1)$. We may say $\xi \in W \cap \mathbb{C} - p(S^1)$ if ξ is not on $p(S^1)$ but is its own reflection in $p(S^1)$, i.e., $S(\xi) = \bar{\xi}$ and $\xi \notin p(S^1)$. It would be interesting to know more about $W \cap \mathbb{C} - p(S^1)$, and in particular the relationship to the zeros of the derivative of p . We prove the following

THEOREM 1. *If all the zeros of $p'(z)$ are in $|z| < 1$, then $W \cap \mathbb{C} = p(S^1)$.*

Proof. Suppose to the contrary that there is a complex number a such that $|a| \neq 1$ and $p(a) = p(1/\bar{a})$. Then

$$\int_{1/\bar{a}}^a p'(t) dt = 0$$

where the integral is over the line segment from $1/\bar{a}$ to a . Therefore $p'(z)$ is apolar to

$$q(z) = \int_{1/\bar{a}}^a (t - z)^{n-1} dt = \frac{(z - a)^n}{n} - \frac{(z - 1/\bar{a})^n}{n}$$

(see Marden [6] p. 61). The zeros of q are on the perpendicular bisector L of the line segment joining a and $1/\bar{a}$. The distance of L from 0 is $\frac{1}{2}(1/r + r) > 1$, where $r = |a|$. Let A be the closed half-plane determined by L , and not containing the disc $|z| < 1$. By Grace's theorem (Marden [6] p. 61), A contains at least one zero of p' . But this contradicts the hypothesis of the theorem, and we have proof by contradiction.

As a consequence of the theorem, for example, the image of the unit circle under $p(z) = z^2 + z$ is

$$\begin{aligned}
 h(\xi, \bar{\xi}) &= \begin{vmatrix} -\bar{\xi} & 1 & 1 & 0 \\ 0 & -\bar{\xi} & 1 & 1 \\ 1 & 1 & -\xi & 0 \\ 0 & 1 & 1 & -\xi \end{vmatrix} \\
 &= |1 - \xi|^2 - (1 - |\xi|^2)^2 \\
 &= 0
 \end{aligned}$$

since the only zero of the derivative is at $-1/2$.

5. Points of W on the line at ∞ . We consider \mathbf{C}^2 as a subspace of the projective space $P_2(\mathbf{C})$ in the usual way by identifying the point (z, ζ) with the point in $P_2(\mathbf{C})$ with homogeneous coordinates $(z, \zeta, 1)$. Let $\tilde{h}(\xi, \eta, \chi)$ be the ternary form defined by $\tilde{h}(\xi, \eta, \chi) = \chi^{2n} h(\xi/\chi, \eta/\chi)$. Let W^* be the projective closure of W in $P_2(\mathbf{C})$, i.e., let W^* be the projective variety given by $\tilde{h}(\xi, \eta, \chi) = 0$. From the determinant expression for h in §1, we see that $\tilde{h}(\xi, \eta, 0) = (\xi\eta)^n$. Therefore the points with homogeneous coordinates $(0, 1, 0)$ and $(1, 0, 0)$ are on W^* . These are just the circular points given in absolute coordinates (Winger [12] p. 52). We also see that $\tilde{h}(\xi, 1, \chi) = (-1)^n (a_0\chi - \xi)^n + (\text{forms in } (\chi, \xi) \text{ of degree } > n)$. Thus $(0, 1, 0)$ is on W^* of multiplicity n . Likewise $(1, 0, 0)$ is on W^* of multiplicity n . The effect of this is to reduce the number of real intersections of $p(S^1)$ with curves through the circular points. For example, by Bézout's theorem (Walker [11] p. 111; Fulton [3] p. 112) W^* intersects a circle exactly $2(2n)$ times. Now $2n$ of these intersections are at circular points, therefore the number of real intersections is at most $2n$. Since $p(S^1) \subseteq W \cap \mathbf{C}$, the number of intersections of a circle with $p(S^1)$ is at most $2n$. For more on this see Quine [10].

6. Multiple points of W . We investigate points of W with more than one preimage under \bar{p} . Suppose that $p(\alpha) = p(\beta)$ and $\bar{p}(1/\alpha) = \bar{p}(1/\beta)$. Write

$$G(z, \zeta) = \frac{p(z) - p(\zeta)}{z - \zeta} = \sum_{k=1}^n a_k \phi_k(z, \zeta)$$

where ϕ_k is the form of degree $k - 1$ defined by

$$\phi_k(z, \zeta) = (z^k - \zeta^k)/(z - \zeta).$$

We note that G is of degree $n - 1$ and $G(z, z) = p'(z)$. Now writing

$$G^*(z, \zeta) = z^{n-1} \zeta^{n-1} \bar{G}(1/z, 1/\zeta) \\ = \sum_{k=1}^n \bar{a}_k \phi_k(z, \zeta) (z\zeta)^{n-k}$$

we note that G^* is of degree $2(n - 1)$. We see that (α, β) is on the intersection of the curves given by $G(z, \zeta) = 0$ and $G^*(z, \zeta) = 0$. By Bézout's theorem, if G and G^* have no common component, then they have at most $2(n - 1)^2$ intersections. We have the following theorem

THEOREM 2. *If G and G^* have a common component, then $p(z) = q(z^k)$ where k is an integer greater than 1 and q is a polynomial.*

Proof. Make the change of variables $z = uv, \zeta = u$. We have $G(z, \zeta) = g(u, v)$ where

$$g(u, v) = \sum_{k=1}^n a_k \frac{v^k - 1}{v - 1} u^{k-1}$$

and $G^*(z, \zeta) = u^{n-1} g^*(u, v)$ where

$$g^*(u, v) = \sum_{k=1}^n \bar{a}_k v^{n-k} \frac{v^k - 1}{v - 1} u^{n-k}.$$

Now G and G^* have a common component iff g and g^* have a common component. Let $R(v)$ be the resultant of g and g^* as polynomials in u . From the determinant expression for the resultant we have

$$R(v) = |a_n|^{2(n-1)} \left(\frac{v^n - 1}{v - 1} \right)^{2(n-1)} + \dots$$

so that R is of degree $2(n - 1)^2$. Thus any common factor of g and g^* is a polynomial in v alone. Therefore let $f = f(v)$ and suppose f divides g . Then f divides $(v^n - 1)/(v - 1)$ and so f has as a zero some primitive k th root of unity, where k divides n . Denote this root by ω , then

$$g(u, \omega) = \frac{p(u) - p(u\omega)}{u(1 - \omega)}$$

is identically 0 in u . Therefore $p(u) \equiv p(u\omega)$ hence $p(z) = q(z^k)$ for some polynomial q and the proof follows by contradiction.

If $p(z) = q(z^k)$ then $p(S^1) = q(S^1)$. Therefore without loss of generality in studying $p(S^1)$, we may assume that p is reduced so that $p(z)$ is not of the form $q(z^k)$, and we will henceforth make this assumption. We note that if $a_1 = 1$ the assumption holds automatically.

COROLLARY 1. *The equation $\tilde{p}(v_1) = \tilde{p}(v_2) = w$ for $v_1, v_2 \in V$ and $v_1 \neq v_2$ holds for at most $(n - 1)^2$ points in W .*

COROLLARY 2. *$p(S^1)$ has at most $(n - 1)^2$ self-intersections.*

The last corollary is sharp as we showed in Quine [8]. We note that self-intersections of $p(S^1)$ correspond to real singularities of the algebraic curve W .

7. Univalent polynomials. Let $p(z) = z + a_2z^2 + \dots + a_nz^n$. Let $V_n = \{(a_2, \dots, a_n) \mid p \text{ is 1-1 in } |z| < 1\}$ be the domain of variability for polynomials of degree n . Now (a_2, \dots, a_n) is in the interior of V_n iff W has no singular points on $p(S^1)$ (see Quine [8]). We determine the condition algebraically as follows: Let $R(z)$ be the resultant of $G(z, \zeta)$ and $G^*(z, \zeta)$ as polynomials in ζ . R is of degree $2(n - 1)^2$, and the condition that $(a_2, \dots, a_n) \in \text{Int } V_n$ becomes $R(z) \neq 0$ for $|z| = 1$. By the symmetry of G and G^* we see that $R(z) = 0$ iff $R(1/\bar{z}) = 0$, therefore without loss of generality, we may assume that R is self-inversive, i.e., $z^{2(n-1)^2} \bar{R}(1/z) = R(z)$. The condition that a self-inversive polynomial have no zeros on $|z| = 1$ can be expressed in terms of a Hermitian form following Krein [5]. Let $R_1(z) = (n - 1)^2 R(z) - zR'(z)$. Let

$$B(x, y) = \frac{R(x)\bar{R}_1(y) + R_1(x)\bar{R}(y)}{1 - xy}$$

$$= \sum_{j, k=0}^{2(n-1)^2-1} b_{jk}x^jy^k.$$

The matrix (b_{jk}) determines a Hermitian form B on $\mathbb{C}^{2(n-1)^2}$ in the usual way. Let π be the number of positive squares and ν the number of negative squares of B reduced to canonical form. Krein showed that $R(z)$ has no zeros on $|z| = 1$ iff $\pi = \nu$. Therefore we have

THEOREM 3. *$(a_2, \dots, a_n) \in \text{Int } V_n$ iff $\pi = \nu$ for the Hermitian form B .*

For more information on V_n , see Koessler [4], Quine [9], Brannan [1].

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