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**ON CONTINUOUS IMAGE AVERAGING OF PROBABILITY  
MEASURES**

SUN MAN CHANG

## ON CONTINUOUS IMAGE AVERAGING OF PROBABILITY MEASURES

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Let  $M$  be a compact space, and  $X$  a complete separable metric space. Let  $P(X)$  denote the probability measures on  $X$ . Let  $\lambda$  be a probability measure on  $M$ . Define a function  $\varphi_\lambda$  from  $C(M, P(X))$  to  $P(X)$  by  $\varphi_\lambda(T)(f) = \int T(t)(f)d\lambda(t)$  for every  $T \in C(M, P(X))$ ,  $f \in C(X)$ . We show that  $\varphi_\lambda$  is an open mapping.

1. Introduction. By a measure on a space  $X$ , we mean a regular Borel measure on  $X$ . A nonnegative measure is called a probability measure if its total mass is 1.

Let  $M$  be a compact space, and let  $X$  be a complete separable metric space. Let  $P(X)$  denote the collection of all probability measures on  $X$ . Let  $C(X)$  denote the set of all bounded continuous real-valued functions on  $X$ . Give  $P(X)$  the weak topology as functionals on  $C(X)$ . Let  $C(M, P(X))$  denote the set of all continuous functions from  $M$  into  $P(X)$ . Give  $C(M, P(X))$  the topology of uniform convergence. Let  $\lambda$  be a fixed probability measure on  $M$ . For each  $T \in C(M, P(X))$ , define a functional  $\varphi_\lambda(T)$  on  $C(X)$  by

$$\varphi_\lambda(T)(f) = \int T(t)(f)d\lambda(t).$$

By [3, p. 35 and p. 47],  $\varphi_\lambda(T)$  may be considered as a measure in  $P(X)$ . Write  $\varphi_\lambda(T) = \int T(t)d\lambda(t)$ . Denote the mapping  $T \rightarrow \varphi_\lambda(T)$  by  $\varphi_\lambda$ . Then  $\varphi_\lambda$  is a continuous function from  $C(M, P(X))$  into  $P(X)$ . This paper is to show that  $\varphi_\lambda$  is an open mapping. This result contains a result due to Eifler [2, Theorem 2.4] as a special case when  $M$  consists of two points.

For a metric space  $X$ , we write  $x_n \rightarrow x$  if  $(x_n)_{n=1}^\infty$  converges to  $x$  in  $X$ .

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2. Basic lemmas. We will use the following notation in Lemma 2.1: Let  $X$  and  $Y$  be complete separable metric spaces, and  $\pi: Y \rightarrow X$  a continuous function. Then  $\pi$  induces a mapping also denoted by  $\pi$ , from  $P(Y)$  to  $P(X)$  and defined by  $\pi\mu(E) = \mu(\pi^{-1}(E))$ .

LEMMA 2.1. *Let  $X$  be a complete separable metric space. Then there exist a totally disconnected complete separable metric space  $G$ , a continuous function  $\varphi: G \rightarrow X$ , and a continuous function  $\tilde{\varphi}: P(X) \rightarrow P(G)$  such that  $\varphi\tilde{\varphi}(\mu) = \mu$  for all  $\mu \in P(X)$ . Moreover,  $\tilde{\varphi}$  is affine:*

$$\tilde{\varphi}(a\mu + (1-a)\nu) = a\tilde{\varphi}(\mu) + (1-a)\tilde{\varphi}(\nu)$$

for every  $0 < a < 1$ , and measures  $\mu, \nu \in P(X)$ .

*Proof.* Such a space  $G$  is constructed by using a sequence  $(F_n)_{n=1}^{\infty}$  of partitions of unity on  $X$  having the property that each  $F_n$  is subordinate to a cover of diameter less than  $1/n$ . The details of its construction can be found in [1].

Let  $X$  be a totally disconnected complete separable metric space. Consider sets of the form

$$(*) \quad M_{\mu, \varepsilon}(G_1, \dots, G_n) = \{\nu \in P(X): |\nu(G_i) - \mu(G_i)| < \varepsilon \\ \text{for } i = 1, \dots, n\}$$

where  $\varepsilon > 0$ ,  $\mu \in P(X)$ , and  $G_1, G_2, \dots, G_n$  are mutually disjoint, both open and closed subsets of  $X$  such that  $\bigcup_{i=1}^n G_i = X$ .

LEMMA 2.2. *The collection of sets of the form (\*) is a base for the topology on  $P(X)$ .*

*Proof.* For any open subset  $U$  of  $X$ , let

$$N_{\mu, \varepsilon}(U) = \{\nu \in P(X): \nu(U) + \varepsilon > \mu(U)\}.$$

Since sets of the form  $N_{\mu, \varepsilon}(U)$  is a sub-base for the topology on  $P(X)$ , it suffices to show that

$$N_{\mu, \varepsilon}(U) \cap M_{\mu, \varepsilon}(G_1, \dots, G_n)$$

contains a set of the form (\*). Let  $V \subseteq U$  be a both open and closed subset of  $X$  such that  $\mu(V) + \varepsilon/2 > \mu(U)$ . Then  $N_{\mu, \varepsilon/2}(V) \subseteq N_{\mu, \varepsilon}(U)$ , and it is easy to check that

$$M_{\mu, \varepsilon/2n}(G_1 \cap V, \dots, G_n \cap V, G_1 \setminus V, \dots, G_n \setminus V) \\ \subseteq N_{\mu, \varepsilon/2}(V) \cap M_{\mu, \varepsilon}(G_1, \dots, G_n).$$

This completes the proof.

### 3. Main result.

THEOREM 3.1. *Let  $M$  be a compact space, and let  $X$  be a complete separable metric space. Let  $\lambda$  be a probability measure on  $M$ . Then the function  $\varphi_\lambda: C(M, P(X)) \rightarrow P(X)$  defined by*

$$\varphi_\lambda(T) = \int T(t)d\lambda(t)$$

is open.

*Proof.* The proof will be accomplished in two steps: (A) We establish the result when  $X$  is totally disconnected. (B) We use (A) to complete the proof.

(A) Let  $X$  be a totally disconnected complete separable metric space. Let  $T \in C(M, P(X))$ , and let  $\mathcal{U}_T$  be a neighborhood of  $T$ . It suffices to show that  $\varphi_\lambda(\mathcal{U}_T)$  is a neighborhood of  $\varphi_\lambda(T)$ . By Lemma 2.2, we may take  $\mathcal{U}_T$  to be a set of the form:

$$\mathcal{U}_T = \{S \in C(M, P(X)): S(M_i) \subseteq \mathcal{V}_i, \text{ for } i = 1, \dots, m\}$$

where for each  $i$ ,  $M_i$  is a compact subset of  $M$ , and  $\mathcal{V}_i$  is a basic open subset of  $P(X)$  of the form:

$$\mathcal{V}_i = \{\theta \in P(X): |\theta(G_{ij}) - \theta_i(G_{ij})| < \varepsilon, \text{ for } j = 1, \dots, n_i\}$$

where  $\theta_i \in P(X)$  and  $\{G_{ij}: j = 1, \dots, n_i\}$  is an open cover for  $X$  consisting of mutually disjoint open subsets of  $X$ .

Let  $\mathcal{E}$  be the collection of all nonempty subsets  $U$  of  $X$  such that  $U = G_{1j_1} \cap G_{2j_2} \cap \dots \cap G_{mj_m}$ . Write  $\mathcal{E} = \{U_1, \dots, U_n\}$ . Then  $\mathcal{E}$  is an open cover for  $X$  and  $U_i \cap U_j = \emptyset$  if  $i \neq j$ .

Since each  $G_{ij}$  is both open and closed, we have

$$\delta = \text{Max}_{ij} \text{Max}_{t \in M_i} |T(t)(G_{ij}) - \theta_i(G_{ij})| < \varepsilon.$$

Let  $\varepsilon_0 = \varepsilon - \delta > 0$ . One sees immediately that if  $S \in C(M, P(X))$  is such that  $\text{Max}_{i \in M} |S(t)(G_{ij}) - T(t)(G_{ij})| < \varepsilon_0$  for all  $i, j$ , then  $S \in \mathcal{U}_T$ .

Let  $\mu = \int T(t)d\lambda(t)$ , and  $a_i = \mu(U_i)$ ,  $1 \leq i \leq n$ . Then  $\sum a_i = 1$  and we may assume that  $a_n > 0$ . Let  $N$  be an integer such that  $N \cdot a_i > n^2$  whenever  $a_i > 0$ ,  $1 \leq i \leq n$ . Define

$$\mathcal{V} = \{\nu \in P(X): |\nu(U_i) - a_i| < \varepsilon_0/2N \text{ for } i = 1, \dots, n\}.$$

It suffices to show that  $\varphi_\lambda(\mathcal{U}_T) \supseteq \mathcal{V}$ .

Let  $\nu \in \mathcal{V}$ . Then  $\nu = \nu_1 + \dots + \nu_n$ , where  $\nu_i$  is a measure on  $X$  defined as  $\nu_i(A) = \nu(A \cap U_i)$ . Let  $b_i = \nu(U_i)$ . Then  $|a_i - b_i| < \varepsilon_0/2N$ , and  $b_i > 0$  whenever  $a_i > 0$ .

Now, go back to the function  $T$ . Let  $f_i(t) = T(t)(U_i)$ . Then all  $f_i$ ,  $i = 1, \dots, n$ , are continuous functions on  $M$ , and  $\int f_i(t)d\lambda(t) = a_i$ . We will construct continuous functions  $g_1, \dots, g_n$  on  $M$  such that

$$(1) \int g_i(t)d\lambda(t) = b_i,$$

(2)  $\text{Max}_{t \in M} |g_i(t) - f_i(t)| < \varepsilon_0/n$ , and

(3)  $0 \leq g_i(t) \leq 1$  and  $\sum_{i=1}^n g_i(t) = 1$  for all  $t$ .

Given  $i = 1, \dots, n-1$ , define  $g_i$  as follows:

(a) If  $b_i = a_i$ , let  $g_i(t) = f_i(t)$  for all  $t$ .

(b) If  $b_i > a_i$ , set  $\delta_i = b_i - a_i < \varepsilon_0/2N$ . Let  $g_i(t) = f_i(t) + (\delta_i/a_n)f_n(t)$ .

Then,

$$\begin{aligned} f_i(t) &\leq g_i(t) \leq f_i(t) + (\varepsilon_0/2N \cdot a_n)f_n(t) \\ &\leq f_i(t) + (\varepsilon_0/2n^2)f_n(t). \end{aligned}$$

(c) If  $b_i < a_i$ , set  $\delta_i = a_i - b_i < \varepsilon_0/2N$ . Since  $a_i > 0$ , so that  $b_i > 0$ . Define  $h_i(t) = 0$ , if  $f_i(t) \leq \delta_i$ ;  $h_i(t) = f_i(t) - \delta_i$ , otherwise. Then  $b_i \leq \int h_i(t)d\lambda(t) \leq a_i$ . Let  $b'_i = \int h_i(t)d\lambda(t)$  and  $g_i(t) = (b_i/b'_i)h_i(t)$ . Then  $g_i(t) \leq f_i(t)$  and

$$\begin{aligned} f_i(t) - g_i(t) &\leq \delta_i + h_i(t)(1 - b_i/b'_i) \\ &\leq \delta_i + \varepsilon_0/2N \cdot a_i < \varepsilon_0/n^2. \end{aligned}$$

Thus for  $i = 1, \dots, n-1$ ,  $0 \leq g_i \leq 1$ ,  $\int g_i(t)d\lambda(t) = b_i$ , and

$$\text{Max}_{t \in M} |g_i(t) - f_i(t)| < \varepsilon_0/n^2.$$

Moreover,  $g_i(t) \leq f_i(t) + (\varepsilon_0/2n^2)f_n(t)$ . Hence,  $g_1(t) + \dots + g_{n-1}(t) \leq 1$  for all  $t$ . Let  $g_n(t) = 1 - g_1(t) - \dots - g_{n-1}(t)$ . Then the functions  $g_1, \dots, g_n$  are as required. This completes the construction.

Now let  $I, J$  be subsets of  $\{1, 2, \dots, n\}$  such that  $I = \{i: b_i > 0\}$ ,  $J = \{j: b_j = 0\}$ . For each  $j \in J$ , pick a measure  $\alpha_j \in P(U_j)$ . Define a continuous function  $S: M \rightarrow P(X)$  by  $S(t) = \sum_{i \in I} (g_i(t)/b_i)\nu_i + \sum_{j \in J} g_j(t)\alpha_j$ . Clearly,

$$\begin{aligned} \varphi_\lambda(S) &= \sum_{i \in I} \left( \int \frac{g_i(t)}{b_i} d\lambda(t) \right) \nu_i + \sum_{j \in J} \left( \int g_j(t) d\lambda(t) \right) \alpha_j \\ &= \sum_{i \in I} \nu_i = \nu, \quad \text{and} \quad \text{Max}_{t \in M} |S(t)(U_i) - T(t)(U_i)| < \varepsilon_0/n \end{aligned}$$

for all  $i$ . Since each  $G_{ij}$  is a disjoint union of  $U_k$ , it follows that  $\text{Max}_{t \in M} |S(t)(G_{ij}) - T(t)(G_{ij})| < \varepsilon_0$ . Therefore,  $S \in \mathcal{U}_T$ . This completes the proof of (A).

(B) Let  $X$  be a complete separable metric space. To show that the mapping  $\varphi_\lambda$  is open, it is equivalent to show the following: Let  $T \in C(M, P(X))$ , and  $\mu = \varphi_\lambda(T)$ . Let  $\mu_n$  be a sequence converging to  $\mu$  in  $P(X)$ . Then there is a sequence  $T_n \rightarrow T$  in  $C(M, P(X))$  such that  $\varphi_\lambda(T_n) = \mu_n$ .

For this purpose, we use Lemma 2.1 to pick a totally disconnected space  $G$ , continuous functions  $\varphi: G \rightarrow X$  and  $\tilde{\varphi}: P(X) \rightarrow P(G)$ , such

that  $\varphi\tilde{\varphi}(\mu) = \mu$ , and that  $\tilde{\varphi}$  is affine. Let  $\tilde{\mu}_n = \tilde{\varphi}\mu_n$ ,  $\tilde{\mu} = \tilde{\varphi}\mu$ . Then  $\tilde{\mu}_n \rightarrow \tilde{\mu}$  in  $P(G)$ . Let  $\tilde{T}(t) = \tilde{\varphi}T(t)$  for each  $t$ . Then  $\tilde{T} \in C(M, P(G))$ . It is easy to check that  $\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T)$ . In fact, this is obvious if there is a finite subset  $\{t_1, \dots, t_n\} \subseteq M$  with  $\lambda\{t_1, \dots, t_n\} = 1$ . In general, we may pick a net  $\lambda_\alpha \rightarrow \lambda$  in  $P(M)$  such that for each  $\alpha$ ,  $\lambda_\alpha(F_\alpha) = 1$  for some finite subset  $F_\alpha$  of  $M$ . Thus,  $\varphi_{\lambda_\alpha}(\tilde{T}) = \tilde{\varphi}\varphi_{\lambda_\alpha}(T)$ . Let  $\alpha \rightarrow \infty$ , then we obtain

$$\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T).$$

Hence  $\varphi_\lambda(\tilde{T}) = \tilde{\mu}$ . Since by (A), the function

$$\varphi_\lambda: C(M, P(G)) \longrightarrow P(G)$$

is open, hence, we may pick  $\tilde{T}_n \rightarrow \tilde{T}$  in  $C(M, P(G))$  such that  $\varphi_\lambda(\tilde{T}_n) = \tilde{\mu}_n$ . Let  $T_n(t) = \varphi\tilde{T}_n(t)$ . Then  $T_n \rightarrow \varphi\tilde{T} = T$  in  $C(M, P(X))$ , and the same argument in proving  $\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T)$  will give  $\varphi_\lambda(T_n) = \varphi\varphi_\lambda(\tilde{T}_n)$ . Therefore,

$$\varphi_\lambda(T_n) = \varphi\tilde{\mu}_n = \mu_n.$$

This proves (B), and so completes the proof of this theorem.

As a special case of Theorem 3.1, we let  $M = \{1, 2\}$  with the discrete topology. We obtain Eifler's result [2]:

**COROLLARY 3.2.** *Let  $X$  be a complete separable metric space, and let  $0 < \lambda < 1$ . Then the function*

$$\lambda: P(X) \times P(X) \longrightarrow P(X)$$

*defined by  $(\mu, \nu) \rightarrow \lambda\mu + (1 - \lambda)\nu$  is open.*

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