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**GENERALIZED DEDEKIND ψ -FUNCTIONS WITH RESPECT
TO A POLYNOMIAL. II**

J. CHIDAMBARASWAMY

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For a given polynomial $f = f(x)$ of positive degree with integer coefficients and for given positive integers u, v , and t , the arithmetical function $\psi_{f,i}^{u,v}(n)$ is defined and some of its arithmetical properties are obtained in addition to its average order. $\psi_{x,1}^{k,1}(n)$ reduces to the function $\psi_{(k)}(n)$ studied recently by D. Suryanarayana and $\psi_{f,i}^{k,t}(n)$ to $\psi_{f,i}^{(k)}(n)$ studied more recently by the author.

Introduction. The Dedekind's ψ -function

$$(1.1) \quad \psi(n) = \sum_{d|n} \frac{d\phi(g)}{g}, \quad g = \left(d, \frac{n}{d}\right),$$

$\phi(n)$ being Euler's totient function is well known. He used this function in his study of elliptic modular functions [4]. As generalizations of this function, recently D. Suryanarayana [8] defined and studied the functions $\Psi_k(n)$, $\psi_k(n)$ and $\psi_{(k)}(n)$ all giving the function $\psi(n)$ for $k = 1$. The functions $\Psi_k(n)$ and $\psi_k(n)$ are defined respectively (see [8]) as the Dirichlet's convolution of a certain function with Klee's [6] totient function and as a sum similar to (1.1) using Cohen's [3] totient function, while $\psi_{(k)}(n)$ is defined as a multiplicative function whose values at prime powers p^α are given by

$$(1.2) \quad \psi_{(k)}(p^\alpha) = \sum_{j=0}^{\alpha} \binom{k-1}{j} \psi(p^{\alpha-j})$$

where for any nonnegative integers s and t

$$(1.3) \quad \binom{s}{t} = \frac{s(s-1)(s-t+1)}{1.2.3 \dots t}; \quad \binom{s}{0} \equiv 1.$$

We recall the Dirichlet convolution $(a*b)(n)$ of the arithmetical functions $a(n)$ and $b(n)$ is defined by

$$(1.4) \quad (a*b)(n) = \sum_{d|n} a(d)b\left(\frac{n}{d}\right).$$

In [2], using totient function $\Phi_{f,i}^{(k)}(n)$, (see [1]; the notation for $\Phi_{f,i}^{(k)}(n)$ is slightly different in [1]) $f = f(x)$ being a given polynomial of positive degree with integer coefficients, t and k being given positive integers, which includes as special cases when $f(x) = x$ and special values of k and t all the familiar totient functions, the

author defined and studied the functions $\Psi_{f,t}^{(k)}(n)$ and $\psi_{f,t}^{(k)}(n)$ as generalizations of $\Psi_k(n)$ and $\psi_k(n)$ respectively and among other things extended all the results in [8] regarding $\Psi_k(n)$ and $\psi_k(n)$ to $\Psi_{f,t}^{(k)}(n)$ and $\psi_{f,t}^{(k)}(n)$. In fact

$$(1.5) \quad \begin{aligned} & \text{i, } \Psi_{x,1}^{(k)}(n) = \Psi_k(n) , \\ & \text{ii, } \psi_{x,1}^{(k)}(n) = \psi_k(n) , \quad \text{and} \\ & \text{iii, } \Psi_{f,t}^{(k)}(n^k) = \psi_{f,t}^{(k)}(n) . \end{aligned}$$

In this paper, we define an arithmetical function $\psi_{f,t}^{u,v}(n)$ which includes as special cases not only the function $\psi_{(k)}(n)$ but also $\psi_{f,t}^{(k)}(n)$ (and hence also the function $\psi_k(n)$). In §2, the function $\psi_{f,t}^{u,v}(n)$ is defined and all the results in [8] concerning $\psi_{(k)}(n)$ are extended to this function and in §3 we obtain its average order subject to

$$(1.6) \quad N_f(n) = O(n^\varepsilon) , \quad 0 < \varepsilon < \frac{1}{u}$$

where $N_f(n)$ is the number of solutions (mod n) of

$$(1.7) \quad f(x) \equiv 0 \pmod{n} .$$

We note in passing that when $f(x) = x$, $N_f(n) = 1$ and that (1.6) is always satisfied if $f(x)$ is a primitive integral polynomial with discriminant $\neq 0$. (cf. Theorem 54 of [7]).

We need the following results about $\psi_{f,t}^{(k)}(n)$ which have been obtained in [2].

$$(1.8) \quad \begin{aligned} & \text{i, } \psi_{f,t}^{(k)}(n) \text{ is a multiplicative function of } n \\ & \text{ii, } \psi_{f,t}^{(k)}(p^\alpha) = p^{\alpha kt} \left\{ 1 + \frac{N_f^t(p^k)}{p^{kt}} \right\} \\ & \text{iii, } \psi_{f,t}^{(k)}(n) = n^{kt} \prod_{p|n} \left\{ 1 + \frac{N_f^t(p^k)}{p^{kt}} \right\} = n^{kt} \sum_{d|n} \frac{\mu^2(d) N_f^t(d^k)}{d^{kt}} , \end{aligned}$$

where $\mu(n)$ is the Mobius function and for any arithmetical function $g(n)$, $g^r(n) = (g(n))^r$.

We shall use the symbol $p^\alpha || n$ to mean that p^α is the highest power of p that divides n .

2. For a given polynomial f and for given positive integers u , v and t we define the arithmetical function $\psi_{f,t}^{u,v}(n)$ as a multiplicative function whose values at prime powers p^α are given by

$$(2.1) \quad \psi_{f,t}^{u,v}(p^\alpha) = \sum_{j=0}^{\alpha} \binom{u-1}{j} N_f^{jt}(p^v) \psi_{f,t}^{(v)}(p^{\alpha-j}) .$$

Clearly,

$$(2.2) \quad \psi_{f,i}^{1,k}(n) = \psi_{f,i}^{(k)}(n)$$

and from (ii) of (1.5) for $k = 1$ and (1.2)

$$(2.3) \quad \psi_{x,1}^{k,1}(n) = \psi_{(k)}(n).$$

Using (1.8), writing N for $N_f(p^v)$, and observing

$$\begin{aligned} \binom{s}{t} + \binom{s}{t+1} &= \binom{s+1}{t+1}, \quad \text{we get the r.h.s. of (2.1) is} \\ &= \sum_{j=0}^{\alpha-1} \binom{u-1}{j} N^{jt} \{p^{(\alpha-j)vt} + p^{(\alpha-j-1)vt} N^t\} + \binom{u-1}{\alpha} N^{\alpha t} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \left\{ \binom{u-1}{j-1} + \binom{u-1}{j} \right\} N^{jt} p^{(\alpha-j)vt} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \binom{u}{j} N^{jt} p^{(\alpha-j)vt}, \quad \text{for } \alpha > 0 \end{aligned}$$

and is 1 for $\alpha = 0$; consequently, we have since $\psi_{f,i}^{u,v}(n)$ is by definition multiplicative,

THEOREM 2.1.

$$\psi_{f,i}^{u,v}(n) = \prod_{p^\alpha | n} \left\{ \sum_{j=0}^{\alpha} \binom{u}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt} \right\}.$$

We observe that Theorem 2.1, (2.2), and the observations $\binom{s}{t} = 0$ for $t > s$ give (3 of (2.18) of [2])

$$(2.4) \quad \psi_{f,i}^{(k)}(n) = n^{kt} \prod_{p|n} \left\{ 1 + \frac{N_f^t(p^k)}{p^{kt}} \right\}$$

and Theorem 2.1 and (2.3) give (Theorem 3.3 of [8])

$$(2.5) \quad \psi_{(k)}(n) = \prod_{p^\alpha | n} \sum_{j=0}^{\alpha} \binom{k}{j} p^{\alpha-j}.$$

We define the function $\rho_{f,i}^{u,v}(n)$ as a multiplicative function whose values at prime powers p^α are given by

$$(2.6) \quad \rho_{f,i}^{u,v}(p^\alpha) = \binom{u}{\alpha} N_f^{\alpha t}(p^v),$$

so that,

$$(2.7) \quad \rho_{f,i}^{u,v}(n) = \prod_{p^\alpha | n} \binom{u}{\alpha} N_f^{\alpha t}(p^v).$$

We note that

$$(2.8) \quad \rho_{x,1}^{k,1}(n) = \prod_{p^{\alpha}|n} \binom{k}{\alpha} = \rho_{(k)}(n);$$

the function $\rho_{(k)}(n)$ is defined in [8]. Furthermore, it is easily seen that

$$(2.9) \quad \rho_{f,t}^{1,k}(n) = \prod_{p^{\alpha}|n} \binom{1}{\alpha} N_f^{\alpha t}(p^k) = \mu^2(n) N_f^t(n^k).$$

Since, by (2.6) and Theorem 2.1,

$$\sum_{d|p^{\alpha}} \rho_{f,t}^{u,v}(d) \left(\frac{n}{d}\right)^{vt} = \sum_{j=0}^{\alpha} \binom{u}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt} = \psi_{f,t}^{u,v}(p^{\alpha})$$

and since two multiplicative functions which agree at prime powers agree for all positive integers n , we have

THEOREM 2.2.

$$\psi_{f,t}^{u,v}(n) = \sum_{d|n} \rho_{f,t}^{u,v}(d) \left(\frac{n}{d}\right)^{vt} = (\rho_{f,t}^{u,v} * \lambda_{vt})(n)$$

where the arithmetical function $\lambda_r(n)$ is defined by

$$(2.10) \quad \lambda_r(n) = n^r.$$

We note that Theorem 2.2, (2.2), and (2.9) give (3, of (2.18) of [2])

$$(2.11) \quad \psi_{f,t}^{(k)}(n) = n^{kt} \sum_{d|n} \frac{\mu^2(d) N_f^t(d^k)}{d^{kt}}$$

and Theorem 2.2, (2.3) and (2.8) give (Theorem 3.9 of [8])

$$(2.12) \quad \psi_{(k)}(n) = n \sum_{d|n} \frac{\rho_{(k)}(d)}{d}.$$

THEOREM 2.3. For $u \geq 2$

$$\psi_{f,t}^{u,v}(n) = (\rho_{f,t}^{1,v} * \psi_{f,t}^{u-1,v})(n) = (\rho_{f,t}^{u-1,v} * \psi_{f,t}^{1,v})(n).$$

For the proof of Theorem 2.3, we need

LEMMA 2.1. For $u \geq 2$,

$$\rho_{f,t}^{u,v}(n) = (\rho_{f,t}^{1,v} * \rho_{f,t}^{u-1,v})(n) = (\rho_{f,t}^{u-1,v} * \rho_{f,t}^{1,v})(n).$$

Proof. The second equality is obvious since Dirichlet convolu-

tion is commutative. To prove the first equality it is enough to verify when $n = p^\alpha$, $\alpha \geq 0$, p a prime. If $\alpha = 0$, both sides are 1 and if $\alpha > 0$ by (2.6)

$$\begin{aligned} \sum_{d|p^\alpha} \rho_{f,t}^{1,v}(d) \rho_{f,t}^{u-1,v} \left(\frac{p^\alpha}{d} \right) &= \binom{u-1}{\alpha} N_f^{\alpha t}(p^v) + \binom{1}{1} N_f^t(p^v) \binom{u-1}{\alpha-1} N_f^{(\alpha-1)t}(p^v) \\ &= N_f^{\alpha t}(p^v) \left\{ \binom{u-1}{\alpha} + \binom{u-1}{\alpha-1} \right\} = \binom{u}{\alpha} N_f^{\alpha t}(p^v) = \rho_{f,t}^{u,v}(p^\alpha) \end{aligned}$$

and the proof of the lemma is complete.

We observe, Lemmas 2.1, 2.8, and (2.9) give (Theorem 3.12 of [8])

$$(2.13) \quad \rho_{(k)}(n) = \sum_{d|n} \mu^2(d) \rho_{(k-1)} \left(\frac{n}{d} \right), \quad k \geq 2.$$

Proof of Theorem 2.3. We first prove first equality. It is enough to verify this when $n = p^\alpha$, p a prime and $\alpha \geq 0$. If $\alpha = 0$, both sides are 1 while if $\alpha > 0$

$$\begin{aligned} \sum_{d|p^\alpha} \rho_{f,t}^{1,v}(d) \psi_{f,t}^{u-1,v} \left(\frac{p^\alpha}{d} \right) &\quad (\text{by (2.9) and Theorem 2.1}) \\ &= \rho_{f,t}^{1,v}(1) \psi_{f,t}^{u-1,v}(p^\alpha) + \rho_{f,t}^{1,v}(p) \psi_{f,t}^{u-1,v}(p^{\alpha-1}) \\ &= \sum_{j=0}^{\alpha} \binom{u-1}{j} N_f^{j t}(p^v) p^{(\alpha-j)vt} + N_f^t(p^v) \sum_{j=0}^{\alpha-1} \binom{u-1}{j} N_f^{j t}(p^v) p^{(\alpha-j-1)vt} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \left\{ \binom{u-1}{j-1} + \binom{u-1}{j} \right\} N_f^{j t}(p^v) p^{(\alpha-j)vt} \\ &= \sum_{j=0}^{\alpha} \binom{u}{j} N_f^{j t}(p^v) p^{(\alpha-j)vt} = \psi_{f,t}^{u,v}(p^\alpha) \end{aligned}$$

and the proof of the first equality is complete.

To complete the proof of the theorem, we have by Theorem 2.2 the associativity of Dirichlet convolution, and Lemma 2.1,

$$\begin{aligned} \rho_{f,t}^{u-1,v} * \psi_{f,t}^{1,v} &= \rho_{f,t}^{u-1,v} * (\rho_{f,t}^{1,v} * \lambda_{vt}) = (\rho_{f,t}^{u-1,v} * \rho_{f,t}^{1,v}) * \lambda_{vt} \\ &= \rho_{f,t}^{u,v} * \lambda_{vt} = \psi_{f,t}^{u,v}, \end{aligned}$$

and the proof is complete.

Theorem 2.3, (2.3), and (2.9) give Theorems 3.10 and 3.11 of [8]

$$(2.14) \quad \psi_{(k)}(n) = \sum_{d|n} \mu^2(d) \psi_{(k-1)} \left(\frac{n}{d} \right), \quad k \geq 2;$$

$$(2.15) \quad \psi_{(k)}(n) = \sum_{d|n} \rho_{(k-1)}(d) \psi \left(\frac{n}{d} \right), \quad k \geq 2.$$

3. We obtain in this section, the average order of $\rho_{f,i}^{u,v}(n)$ subject to (1.6).

LEMMA 3.1.

i, $\rho_{f,i}^{u,v}(n) = 0$ if n is not $(u+1)$ free

ii, $\rho_{f,i}^{u,v}(n) < 2^{uw(n)} N_f^{ut}(\gamma^v(n))$ if n is $u+1$ -free

where $w(n)$ is the number of distinct prime factors of n and $\gamma(n)$ is the largest square free divisor of n .

Proof. If n is not $u+1$ -free, there is a prime p such that $p^\alpha \parallel n$, $\alpha \geq u+1$ and so $\binom{u}{\alpha} = 0$ and hence (2.7) implies $\rho_{f,i}^{u,v}(n) = 0$.

If n is $u+1$ -free, then $p^\alpha \parallel n$ implies $\alpha \leq u$ and hence by (2.7), using the facts that $\binom{n}{\alpha} \leq 2^u$ and $N_f(n)$ is a multiplicative function of n , we have

$$\rho_{f,i}^{u,v}(n) = \prod_{p^\alpha \parallel n} \binom{u}{\alpha} N_f^{\alpha t}(p^v) \leq 2^{uw(n)} N_f^{ut}(\gamma^v(n))$$

and the proof of the lemma is complete.

We also need the following elementary estimates

$$(3.1) \quad \begin{aligned} \text{i, } & \sum_{n \leq x} n^r = \frac{x^{r+1}}{r+1} + O(x^r), \quad r > 0, \quad x \geq 1; \\ \text{ii, } & \sum_{n \leq x} \frac{1}{n^r} = O(x^{1-r}), \quad 0 < r < 1, \quad x \geq 1; \\ \text{iii, } & \sum_{n > x} \frac{1}{n^r} = O(x^{1-r}), \quad r > 1, \quad x \geq 1. \end{aligned}$$

LEMMA 3.2. Under the hypothesis (1.6), $\sum_{n=1}^{\infty} \rho_{f,i}^{u,v}(n)/n^{vt+1}$ converges and

$$(3.2) \quad c = \sum_{n=1}^{\infty} \frac{\rho_{f,i}^{u,v}(n)}{n^{vt+1}} = \left(\prod_p \left\{ 1 + \frac{N_f^t(p^v)}{n^{vt+1}} \right\} \right)^u.$$

Proof. If $d(n)$ is the number of divisors of n , we have (cf. Theorem 315 of [5]) $d(n) = O(n^\theta)$ for every $\theta > 0$ and hence

$$2^{uw(n)} = (2^{w(n)})^u \leq (d(n))^u = O(n^{u\theta}) \quad \text{for every } \theta > 0,$$

where the constant in the O -relation depends on u but not on n . Now, (1.6) and Lemma 3.1 give

$$(3.3) \quad \rho_{f,i}^{u,v}(n) = O(n^{uvt+u\theta}),$$

where the constant in the O -relation is independent of n . Hence

$$(3.4) \quad \frac{\rho_{f,i}^{u,v}(n)}{n^{vt+1}} = 0 \left(\frac{1}{n^1 + vt(1 - u\varepsilon) - u\theta} \right).$$

The first part of the lemma is clear since by (1.6) $1 - u\varepsilon > 0$ and we can choose θ so small that

$$(3.5) \quad vt(1 - u\varepsilon) - u\theta > 0.$$

Since $\rho_{f,i}^{u,v}(n)/n^{vt+1}$ is multiplicative we can express the sum of the series as an infinite product of Euler type and so we have

$$\sum_{n=1}^{\infty} \frac{\rho_{f,i}^{u,v}(n)}{n^{vt+1}} \prod_p \left\{ \sum_{m=0}^{\infty} \frac{\rho_{f,i}^{u,v}(p^m)}{(p^m)^{vt+1}} \right\}$$

and this by (2.6) and the fact $\binom{u}{\alpha} = 0$ for $\alpha > u$ is

$$\begin{aligned} &= \prod_p \left\{ \sum_{m=0}^u \frac{\binom{u}{m} N_f^{m,t}(p^v)}{(p^{vt+1})^m} \right\} \\ &= \prod_p \left\{ 1 + \frac{N_f^{1,t}(p^v)}{p^{vt+1}} \right\}^u \end{aligned}$$

and the proof of Lemma 3.2 is complete.

THEOREM 3.1. *Under the hypothesis (1.6),*

$$\sum_{n \leq x} \psi_{f,i}^{u,v}(n) = c \frac{x^{vt+1}}{vt+1} + E(x)$$

where

$$\begin{aligned} E(x) &= O(x^{vt}) \quad \text{if } vt(1 - u\varepsilon) > 1 \\ &= O(x^{1+u\theta+uvt\varepsilon}) \quad \text{for every } \theta < \frac{vt(1 - u\varepsilon)}{u} \end{aligned}$$

if $vt(1 - u\varepsilon) \leq 1$.

Proof. We have by Theorem 2.2,

$$\begin{aligned} \sum_{n \leq x} \psi_{f,i}^{u,v}(n) &= \sum_{n \leq x} \sum_{d \mid n} \rho_{f,i}^{u,v}(d) \delta^{vt} \\ &= \sum_{d \leq x} \rho_{f,i}^{u,v}(d) \delta^{vt} = \sum_{d \leq x} \rho_{f,i}^{u,v}(d) \sum_{\delta \leq x/d} \delta^{vt} \end{aligned}$$

and this by i , of (3.1) is

$$\sum_{d \leq x} \rho_{f,i}^{u,v}(d) \left\{ \frac{1}{vt+1} \left(\frac{x}{d} \right)^{vt+1} + o \left(\left(\frac{x}{d} \right)^{vt} \right) \right\}$$

which by Lemma 3.2 is equal to

$$(3.6) \quad c \frac{x^{vt+1}}{vt+1} + 0 \left(x^{vt+1} \sum_{n>x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}} \right) + 0 \left(x^{vt} \sum_{n \leq x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt}} \right).$$

Let $\theta > 0$ be so chosen that

$$(3.7) \quad \begin{cases} u\theta < vt(1-u\varepsilon) - 1, & \text{if } vt(1-u\varepsilon) > 1 \\ u\theta < vt(1-u\varepsilon), & \text{if } vt(1-u\varepsilon) \leq 1. \end{cases}$$

In any case $u\theta < vt(1-u\varepsilon)$. By (3.4) and (iii) of (3.1),

$$\begin{aligned} \sum_{n>x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}} &= 0 \left(\sum_{n>x} \frac{1}{n^{1+vt(1-u\varepsilon)-u\theta}} \right) \\ &= 0(x^{-vt(1-u\varepsilon)+u\theta}) \end{aligned}$$

and so, the second term in (3.6) is $0(x^{1+u\theta+uv\varepsilon})$.

Similarly,

$$\sum_{n \leq x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt}} = 0 \left(\sum_{n \leq x} \left(\frac{1}{n^{vt(1-u\varepsilon)-u\theta}} \right) \right),$$

and hence the third term in (3.6) is $0(x^{vt})$ or $0(x^{1+u\theta+uv\varepsilon})$ according as $vt(1-u\varepsilon) > 1$ or $vt(1-u\varepsilon) \leq 1$. Since $u\theta < vt(1-u\varepsilon) - 1$ implies $1 + u\theta + uv\varepsilon < vt$, the theorem is clear. Clearly, Theorem 3.1 can be stated as

THEOREM 3.1'. *Under the hypothesis (1.6), the average order of $\psi_{f,t}^{u,v}(n)$ is cn^{vt} , where c is given by (3.2).*

Since $\psi_{(k)}(n) = \psi_{x,1}^{k,1}(n)$, $N_x(n) = 1$, the r.h.s. of (3.2) in this case is

$$\left\{ \prod_p \left(1 + \frac{1}{p^2} \right) \right\}^k = \left\{ \prod_p \frac{(1+p^{-2})(1-p^{-2})}{1-p^{-2}} \right\}^k = \frac{\zeta^k(2)}{\zeta^k(4)}, \quad \zeta(s)$$

being the Riemann's ζ -function, and so from Theorem 3.1, we have

COROLLARY 3.1.1. (Theorem 4.4 of [7].)

The average order of $\psi_{(k)}(n)$ is $n(\zeta^k(2)/\zeta^k(4)) = n(15/\pi^2)^k$.

Similarly, Theorem 3.1, (2.2) and (2.9) give

COROLLARY 3.1.2. ((3.5) of [2].)

The average order of $\psi_{f,t}^{(k)}(n)$ is $\left\{ \sum_{n=1}^{\infty} (\mu^2(n) N_f^t(n^k) / n^{kt+1}) \right\} n^{kt}$.

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