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For a given polynomial $f=f(x)$ of positive degree with integer coefficients and for given positive integers u, v, and t, the arithmetical function $\psi_{f,t}^{u;v}(n)$ is defined and some of its arithmetical properties are obtained in addition to its average order. $\psi_{x,1}^{k;1}(n)$ reduces to the function $\psi_{(k)}(n)$ studied recently by D. Suryanarayana and $\psi_{f,t}^{1;k}(n)$ to $\psi_{f,t}^{(k)}(n)$ studied more recently by the author.

Introduction. The Dedekind's ψ-function

$$\psi(n)=\sum_{d\mid n}\frac{d\phi(g)}{g}\,,\qquad g=\left(d,\frac{n}{d}\right), \tag{1.1}$$

$\phi(n)$ being Euler's totient function is well known. He used this function in his study of elliptic modular functions [4]. As generalizations of this function, recently D. Suryanarayana [8] defined and studied the functions $\Psi_k(n)$, $\psi_k(n)$ and $\psi_{(k)}(n)$ all giving the function $\psi(n)$ for $k=1$. The functions $\Psi_k(n)$ and $\psi_k(n)$ are defined respectively (see [8]) as the Dirichlet's convolution of a certain function with Klee's [6] totient function and as a sum similar to (1.1) using Cohen's [3] totient function, while $\psi_{(k)}(n)$ is defined as a multiplicative function whose values at prime powers p^α are given by

$$\psi_{(k)}(p^\alpha)=\sum_{j=0}^{\alpha}\binom{k-1}{j}\psi(p^{\alpha-j}) \tag{1.2}$$

where for any nonnegative integers s and t

$$\binom{s}{t}=\frac{s(s-1)(s-t+1)}{1.2.3\cdots t}\,;\ \binom{s}{0}\equiv 1\,. \tag{1.3}$$

We recall the Dirichlet convolution $(a*b)(n)$ of the arithmetical functions $a(n)$ and $b(n)$ is defined by

$$(a*b)(n)=\sum_{d\mid n}a(d)b\left(\frac{n}{d}\right). \tag{1.4}$$

In [2], using totient function $\Phi_{f,t}^{(k)}(n)$, (see [1]; the notation for $\Phi_{f,t}^{(k)}(n)$ is slightly different in [1]) $f=f(x)$ being a given polynomial of positive degree with integer coefficients, t and k being given positive integers, which includes as special cases when $f(x)=x$ and special values of k and t all the familiar totient functions, the author defined and studied the functions $\Psi_{f,t}^{(k)}(n)$ and $\psi_{f,t}^{(k)}(n)$ as generalizations of $\Psi_k(n)$ and $\psi_k(n)$ respectively and among other things extended all the results in [8] regarding $\Psi_k(n)$ and $\psi_k(n)$ to $\Psi_{f,t}^{(k)}(n)$ and $\psi_{f,t}^{(k)}(n)$. In fact

$$\text{(1.5)}\qquad \begin{aligned} &\text{i,} \quad \Psi_{x,1}^{(k)}(n) = \Psi_k(n)\,, \\ &\text{ii,} \quad \psi_{x,1}^{(k)}(n) = \psi_k(n)\,, \quad \text{and} \\ &\text{iii,} \quad \Psi_{f,t}^{(k)}(n^k) = \psi_{f,t}^{(k)}(n)\,. \end{aligned}$$

In this paper, we define an arithmetical function $\psi_{f,t}^{u,v}(n)$ which includes as special cases not only the function $\psi_{(k)}(n)$ but also $\psi_{f,t}^{(k)}(n)$ (and hence also the function $\psi_k(n)$). In §2, the function $\psi_{f,t}^{u,v}(n)$ is defined and all the results in [8] concerning $\psi_{(k)}(n)$ are extended to this function and in § 3 we obtain its average order subject to

$$\text{(1.6)}\qquad N_f(n) = 0(n^\varepsilon)\,, \qquad 0 < \varepsilon < \frac{1}{u}$$

where $N_f(n)$ is the number of solutions (mod n) of

$$\text{(1.7)}\qquad f(x) \equiv 0 \pmod{n}\,.$$

We note in passing that when $f(x) = x$, $N_f(n) = 1$ and that (1.6) is always satisfied if $f(x)$ is a primitive integral polynomial with descriminant $\neq 0$. (cf. Theorem 54 of [7]).

We need the following results about $\psi_{f,t}^{(k)}(n)$ which have been obtained in [2].

$$\text{(1.8)}\qquad \begin{aligned} &\text{i,} \quad \psi_{f,t}^{(k)}(n) \ \text{is a multiplicative function of } n \\ &\text{ii,} \quad \psi_{f,t}^{(k)}(p^\alpha) = p^{\alpha kt}\left\{1 + \frac{N_f^t(p^k)}{p^{kt}}\right\} \\ &\text{iii,} \quad \psi_{f,t}^{(k)}(n) = n^{kt}\prod_{p\mid n}\left\{1 + \frac{N_f^t(p^k)}{p^{kt}}\right\} = n^{kt}\sum_{d\mid n}\frac{\mu^2(d)N_f^t(d^k)}{d^{kt}}\,, \end{aligned}$$

where $\mu(n)$ is the Mobius function and for any arithmetical function $g(n)$, $g^r(n) = (g(n))^r$.

We shall use the symbol $p^\alpha \| n$ to mean that p^α is the highest power of p that divides n.

2. For a given polynomial f and for given positive integers u, v and t we define the arithmetical function $\psi_{f,t}^{u,v}(n)$ as a multiplicative function whose values at prime powers p^α are given by

$$\text{(2.1)}\qquad \psi_{f,t}^{u,v}(p^\alpha) = \sum_{j=0}^{\alpha}\binom{u-1}{j}N_f^{jt}(p^v)\psi_{f,t}^{(v)}(p^{\alpha-j})\,.$$

Clearly,

$$\psi_{f,t}^{1,k}(n) = \psi_{f,t}^{(k)}(n) \tag{2.2}$$

and from (ii) of (1.5) for $k = 1$ and (1.2)

$$\psi_{x,1}^{k,1}(n) = \psi_{(k)}(n) . \tag{2.3}$$

Using (1.8), writing N for $N_f(p^v)$, and observing

$\binom{s}{t} + \binom{s}{t+1} = \binom{s+1}{t+1}$, we get the r.h.s. of (2.1) is

$$\begin{aligned} &= \sum_{j=0}^{\alpha-1} \binom{u-1}{j} N^{jt}\{p^{(\alpha-j)vt} + p^{(\alpha-j-1)vt}N^t\} + \binom{u-1}{\alpha} N^{\alpha t} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \left\{\binom{u-1}{j-1} + \binom{u-1}{j}\right\} N^{jt} p^{(\alpha-j)vt} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \binom{u}{j} N^{jt} p^{(\alpha-j)vt} , \quad \text{for} \quad \alpha > 0 \end{aligned}$$

and is 1 for $\alpha = 0$; consequently, we have since $\psi_{f,t}^{u,v}(n)$ is by definition multiplicative,

THEOREM 2.1.

$$\psi_{f,t}^{u,v}(n) = \prod_{p^\alpha \| n} \left\{\sum_{j=0}^{\alpha} \binom{u}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt}\right\} .$$

We observe that Theorem 2.1, (2.2), and the observations $\binom{s}{t} = 0$ for $t > s$ give (3 of (2.18) of [2])

$$\psi_{f,t}^{(k)}(n) = n^{kt} \prod_{p|n} \left\{1 + \frac{N_f^t(p^k)}{p^{kt}}\right\} \tag{2.4}$$

and Theorem 2.1 and (2.3) give (Theorem 3.3 of [8])

$$\psi_{(k)}(n) = \prod_{p^\alpha \| n} \sum_{j=0}^{\alpha} \binom{k}{j} p^{\alpha-j} . \tag{2.5}$$

We define the function $\rho_{f,t}^{u,v}(n)$ as a multiplicative function whose values at prime powers p^α are given by

$$\rho_{f,t}^{u,v}(p^\alpha) = \binom{u}{\alpha} N_f^{\alpha t}(p^v) , \tag{2.6}$$

so that,

$$\rho_{f,t}^{u,v}(n) = \prod_{p^\alpha \| n} \binom{u}{\alpha} N_f^{\alpha t}(p^v) . \tag{2.7}$$

We note that

$$\rho_{x,1}^{k,1}(n) = \prod_{p^\alpha \| n} \binom{k}{\alpha} = \rho_{(k)}(n) ; \tag{2.8}$$

the function $\rho_{(k)}(n)$ is defined in [8]. Furthermore, it is easily seen that

$$\rho_{f,t}^{1,k}(n) = \prod_{p^\alpha \| n} \binom{1}{\alpha} N_f^{\alpha t}(p^k) = \mu^2(n) N_f^t(n^k) . \tag{2.9}$$

Since, by (2.6) and Theorem 2.1,

$$\sum_{d | p^\alpha} \rho_{f,t}^{u,v}(d) \left(\frac{n}{d}\right)^{vt} = \sum_{j=0}^{\alpha} \binom{u}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt} = \psi_{f,t}^{u,v}(p^\alpha)$$

and since two multiplicative functions which agree at prime powers agree for all positive integers n, we have

THEOREM 2.2.

$$\psi_{f,t}^{u,v}(n) = \sum_{d | n} \rho_{f,t}^{u,v}(d) \left(\frac{n}{d}\right)^{vt} = (\rho_{f,t}^{u,v} * \lambda_{vt})(n)$$

where the arithmetical function $\lambda_r(n)$ *is defined by*

$$\lambda_r(n) = n^r . \tag{2.10}$$

We note that Theorem 2.2, (2.2), and (2.9) give (3, of (2.18) of [2])

$$\psi_{f,t}^{(k)}(n) = n^{kt} \sum_{d | n} \frac{\mu^2(d) N_f^t(d^k)}{d^{kt}} \tag{2.11}$$

and Theorem 2.2, (2.3) and (2.8) give (Theorem 3.9 of [8])

$$\psi_{(k)}(n) = n \sum_{d | n} \frac{\rho_{(k)}(d)}{d} . \tag{2.12}$$

THEOREM 2.3. *For* $u \geqq 2$

$$\psi_{f,t}^{u,v}(n) = (\rho_{f,t}^{1,v} * \psi_{f,t}^{u-1,v})(n) = (\rho_{f,t}^{u-1,v} * \psi_{f,t}^{1,v})(n) .$$

For the proof of Theorem 2.3, *we need*

LEMMA 2.1. *For* $u \geqq 2$,

$$\rho_{f,t}^{u,v}(n) = (\rho_{f,t}^{1,v} * \rho_{f,t}^{u-1,v})(n) = (\rho_{f,t}^{u-1,v} * \rho_{f,t}^{1,v})(n) .$$

Proof. The second equality is obvious since Dirichlet convolution is commutative. To prove the first equality it is enough to verify when $n = p^\alpha$, $\alpha \geqq 0$, p a prime. If $\alpha = 0$, both sides are 1 and if $\alpha > 0$ by (2.6)

$$\sum_{d \mid p^\alpha} \rho_{f,t}^{1,v}(d)\rho_{f,t}^{u-1,v}\left(\frac{p^\alpha}{d}\right) = \binom{u-1}{\alpha} N_f^{\alpha t}(p^v) + \binom{1}{1} N_f^t(p^v)\binom{u-1}{\alpha-1} N_f^{(\alpha-1)t}(p^v)$$
$$= N_f^{\alpha t}(p^v)\left\{\binom{u-1}{\alpha} + \binom{u-1}{\alpha-1}\right\} = \binom{u}{\alpha} N_f^{\alpha t}(p^v) = \rho_{f,t}^{u,v}(p^\alpha)$$

and the proof of the lemma is complete.

We observe, Lemmas 2.1, 2.8, and (2.9) give (Theorem 3.12 of [8])

$$\rho_{(k)}(n) = \sum_{d \mid n} \mu^2(d)\rho_{(k-1)}\left(\frac{n}{d}\right), \qquad k \geqq 2 . \tag{2.13}$$

Proof of Theorem 2.3. We first prove first equality. It is enough to verify this when $n = p^\alpha$, p a prime and $\alpha \geqq 0$. If $\alpha = 0$, both sides are 1 while if $\alpha > 0$

$$\sum_{d \mid p^\alpha} \rho_{f,t}^{1,v}(d)\psi_{f,t}^{u-1,v}\left(\frac{p^\alpha}{d}\right) \qquad \text{(by (2.9) and Theorem 2.1)}$$
$$= \rho_{f,t}^{1,v}(1)\psi_{f,t}^{u-1,v}(p^\alpha) + \rho_{f,t}^{1,v}(p)\psi_{f,t}^{u-1,v}(p^{\alpha-1})$$
$$= \sum_{j=0}^{\alpha} \binom{u-1}{j} N_f^{jt}(p^v)p^{(\alpha-j)vt} + N_f^t(p^v)\sum_{j=0}^{\alpha-1}\binom{u-1}{j} N_f^{jt}(p^v)p^{(\alpha-j-1)vt}$$
$$= p^{\alpha vt} + \sum_{j=1}^{\alpha}\left\{\binom{u-1}{j-1} + \binom{u-1}{j}\right\} N_f^{jt}(p^v)p^{(\alpha-j)vt}$$
$$= \sum_{j=0}^{\alpha}\binom{u}{j} N_f^{jt}(p^v)p^{(\alpha-j)vt} = \psi_{f,t}^{u,v}(p^\alpha)$$

and the proof of the first equality is complete.

To complete the proof of the theorem, we have by Theorem 2.2 the associativity of Dirichlet convolution, and Lemma 2.1,

$$\rho_{f,t}^{u-1,v} * \psi_{f,t}^{1,v} = \rho_{f,t}^{u-1,v} * (\rho_{f,t}^{1,v} * \lambda_{vt}) = (\rho_{f,t}^{u-1,v} * \rho_{f,t}^{1,v}) * \lambda_{vt}$$
$$= \rho_{f,t}^{u,v} * \lambda_{vt} = \psi_{f,t}^{u,v} ,$$

and the proof is complete.

Theorem 2.3, (2.3), and (2.9) give Theorems 3.10 and 3.11 of [8]

$$\psi_{(k)}(n) = \sum_{d \mid n} \mu^2(d)\psi_{(k-1)}\left(\frac{n}{d}\right), \qquad k \geqq 2 ; \tag{2.14}$$

$$\psi_{(k)}(n) = \sum_{d \mid n} \rho_{(k-1)}(d)\psi\left(\frac{n}{d}\right), \qquad k \geqq 2 . \tag{2.15}$$

3. We obtain in this section, the average order of $\psi_{f,t}^{u,v}(n)$ subject to (1.6).

LEMMA 3.1.
i, $\rho_{f,t}^{u,v}(n) = 0$ *if n is not* $(u+1)$ *free*
ii, $\rho_{f,t}^{u,v}(n) < 2^{uw(n)} N_f^{ut}(\gamma^v(n))$ *if n is* $u+1$*-free*
where $w(n)$ *is the number of distinct prime factors of* n *and* $\gamma(n)$ *is the largest square free divisor of* n.

Proof. If n is not $u+1$-free, there is a prime p such that $p^\alpha \| n$, $\alpha \geqq u+1$ and so $\binom{u}{\alpha} = 0$ and hence (2.7) implies $\rho_{f,t}^{u,v}(n) = 0$.
If n is $u+1$-free, then $p^\alpha \| n$ implies $\alpha \leqq u$ and hence by (2.7), using the facts that $\binom{n}{\alpha} \leqq 2^u$ and $N_f(n)$ is a multiplicative function of n, we have

$$\rho_{f,t}^{u,v}(n) = \prod_{p^\alpha \| n} \binom{u}{\alpha} N_f^{\alpha t}(p^v) \leqq 2^{uw(n)} N_f^{ut}(\gamma^v(n))$$

and the proof of the lemma is complete.
We also need the following elementary estimates

$$\text{(3.1)} \quad \begin{aligned} &\text{i,} \quad \sum_{n\leqq x} n^r = \frac{x^{r+1}}{r+1} + 0(x^r)\,, \quad r>0\,, \quad x \geqq 1\,; \\ &\text{ii,} \quad \sum_{n\leqq x} \frac{1}{n^r} = 0(x^{1-r})\,, \quad 0<r<1\,, \quad x\geqq 1\,; \\ &\text{iii,} \quad \sum_{n>x} \frac{1}{n^r} = 0(x^{1-r})\,, \quad r>1\,, \quad x \geqq 1\,. \end{aligned}$$

LEMMA 3.2. *Under the hypothesis* (1.6), $\sum_{n=1}^\infty \rho_{f,t}^{u,v}(n)/n^{vt+1}$ *converges and*

$$c = \sum_{n=1}^{\infty} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}} = \left(\prod_p \left\{ 1 + \frac{N_f^t(p^v)}{n^{vt+1}} \right\} \right)^u . \tag{3.2}$$

Proof. If $d(n)$ is the number of divisors of n, we have (cf. Theorem 315 of [5]) $d(n) = 0(n^\theta)$ for every $\theta > 0$ and hence

$$2^{uw(n)} = (2^{w(n)})^u \leqq (d(n))^u = 0(n^{u\theta}) \quad \text{for every} \quad \theta > 0\,,$$

where the constant in the 0-relation depends on u but not on n. Now, (1.6) and Lemma 3.1 give

$$\rho_{f,t}^{u,v}(n) = 0(n^{uvt\varepsilon + u\theta})\,, \tag{3.3}$$

where the constant in the 0-relation is independent of n. Hence

$$\frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}} = 0\left(\frac{1}{n^{1} + vt(1 - u\varepsilon) - u\theta}\right). \tag{3.4}$$

The first part of the lemma is clear since by (1.6) $1 - u\varepsilon > 0$ and we can choose θ so small that

$$vt(1 - u\varepsilon) - u\theta > 0\,. \tag{3.5}$$

Since $\rho_{f,t}^{u,v}(n)/n^{vt+1}$ is multiplicative we can express the sum of the series as an infinite product of Euler type and so we have

$$\sum_{n=1}^{\infty} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}} \prod_{p} \left\{ \sum_{m=0}^{\infty} \frac{\rho_{f,t}^{u,v}(p^m)}{(p^m)^{vt+1}} \right\}$$

and this by (2.6) and the fact $\binom{u}{\alpha} = 0$ for $\alpha > u$ is

$$= \prod_{p} \left\{ \sum_{m=0}^{u} \frac{\binom{u}{m} N_f^{mt}(p^v)}{(p^{vt+1})^m} \right\}$$

$$= \prod_{p} \left\{ 1 + \frac{N_f^t(p^v)}{p^{vt+1}} \right\}^u$$

and the proof of Lemma 3.2 is complete.

THEOREM 3.1. *Under the hypothesis* (1.6),

$$\sum_{n \leqq x} \psi_{f,t}^{u,v}(n) = c\frac{x^{vt+1}}{vt+1} + E(x)$$

where

$$E(x) = 0(x^{vt}) \quad \text{if} \quad vt(1 - u\varepsilon) > 1$$
$$= 0(x^{1+u\theta+uvt\varepsilon}) \quad \text{for every} \quad \theta < \frac{vt(1 - u\varepsilon)}{u}$$

if $vt(1 - u\varepsilon) \leqq 1$.

Proof. We have by Theorem 2.2,

$$\sum_{n \leqq x} \psi_{f,t}^{u,v}(n) = \sum_{n \leqq x} \sum_{d\delta = n} \rho_{f,t}^{u,v}(d)\delta^{vt}$$
$$= \sum_{d\delta \leqq x} \rho_{f,t}^{u,v}(d)\delta^{vt} = \sum_{d \leqq x} \rho_{f,t}^{u,v}(d) \sum_{\delta \leqq x/d} \delta^{vt}$$

and this by i, of (3.1) is

$$\sum_{d \leqq x} \rho_{f,t}^{u,v}(d) \left\{ \frac{1}{vt+1} \left(\frac{x}{d}\right)^{vt+1} + 0\left(\left(\frac{x}{d}\right)^{vt}\right) \right\}$$

which by Lemma 3.2 is equal to

$$(3.6)\qquad c\frac{x^{vt+1}}{vt+1}+0\Big(x^{vt+1}\sum_{n>x}\frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}}\Big)+0\Big(x^{vt}\sum_{n\leqq x}\frac{\rho_{f,t}^{u,v}(n)}{n^{vt}}\Big).$$

Let $\theta>0$ be so chosen that

$$(3.7)\qquad \begin{cases} u\theta<vt(1-u\varepsilon)-1\,, & \text{if } vt(1-u\varepsilon)>1\\ u\theta<vt(1-u\varepsilon)\,, & \text{if } vt(1-u\varepsilon)\leqq 1\,.\end{cases}$$

In any case $u\theta<vt(1-u\varepsilon)$. By (3.4) and (iii) of (3.1),

$$\sum_{n>x}\frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}}=0\Big(\sum_{n>x}\frac{1}{n^{1+vt(1-u\varepsilon)-u\theta}}\Big)$$
$$=0(x^{-vt(1-u\varepsilon)+u\theta})$$

and so, the second term in (3.6) is $0(x^{1+u\theta+uvt\varepsilon})$.

Similarly,

$$\sum_{n\leqq x}\frac{\rho_{f,t}^{u,v}(n)}{n^{vt}}=0\Big(\sum_{n\leqq x}\Big(\frac{1}{n^{vt(1-u\varepsilon)-u\theta}}\Big)\Big),$$

and hence the third term in (3.6) is $0(x^{vt})$ or $0(x^{1+u\theta+uvt\varepsilon})$ according as $vt(1-u\varepsilon)>1$ or $vt(1-u\varepsilon)\leqq 1$. Since $u\theta<vt(1-u\varepsilon)-1$ implies $1+u\theta+uvt\varepsilon<vt$, the theorem is clear. Clearly, Theorem 3.1 can be stated as

THEOREM 3.1′. *Under the hypothesis* (1.6), *the average order of* $\psi_{f,t}^{u,v}(n)$ *is* cn^{vt}, *where* c *is given by* (3.2).

Since $\psi_{(k)}(n)=\psi_{x,1}^{k,1}(n)$, $N_x(n)=1$, the r.h.s. of (3.2) in this case is

$$\Big\{\prod_p\Big(1+\frac{1}{p^2}\Big)\Big\}^k=\Big\{\prod_p\frac{(1+p^{-2})(1-p^{-2})}{1-p^{-2}}\Big\}^k=\frac{\zeta^k(2)}{\zeta^k(4)},\qquad \zeta(s)$$

being the Riemann's ζ-function, and so from Theorem 3.1, we have

COROLLARY 3.1.1. (Theorem 4.4 of [7].)

The average order of $\psi_{(k)}(n)$ is $n(\zeta^k(2)/\zeta^k(4))=n(15/\pi^2)^k$.

Similarly, Theorem 3.1, (2.2) and (2.9) give

COROLLARY 3.1.2. ((3.5) of [2].)

The average order of $\psi_{f,t}^{(k)}(n)$ is $\{\sum_{n=1}^{\infty}(\mu^2(n)N_f^t(n^k)/n^{kt+1})\}n^{kt}$.

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