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**GENERALIZED DEDEKIND  $\psi$ -FUNCTIONS WITH RESPECT  
TO A POLYNOMIAL. II**

J. CHIDAMBARASWAMY

## GENERALIZED DEDEKIND $\psi$ -FUNCTIONS WITH RESPECT TO A POLYNOMIAL II

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For a given polynomial  $f = f(x)$  of positive degree with integer coefficients and for given positive integers  $u, v$ , and  $t$ , the arithmetical function  $\psi_{f, u, v}^{(k)}(n)$  is defined and some of its arithmetical properties are obtained in addition to its average order.  $\psi_{f, u, v}^{(k)}(n)$  reduces to the function  $\psi_{(k)}(n)$  studied recently by D. Suryanarayana and  $\psi_{f, t}^{(k)}(n)$  to  $\psi_{f, t}^{(k)}(n)$  studied more recently by the author.

**Introduction.** The Dedekind's  $\psi$ -function

$$(1.1) \quad \psi(n) = \sum_{d|n} \frac{d\phi(d)}{n}, \quad g = \left(d, \frac{n}{d}\right),$$

$\phi(n)$  being Euler's totient function is well known. He used this function in his study of elliptic modular functions [4]. As generalizations of this function, recently D. Suryanarayana [8] defined and studied the functions  $\Psi_k(n)$ ,  $\psi_k(n)$  and  $\psi_{(k)}(n)$  all giving the function  $\psi(n)$  for  $k = 1$ . The functions  $\Psi_k(n)$  and  $\psi_k(n)$  are defined respectively (see [8]) as the Dirichlet's convolution of a certain function with Klee's [6] totient function and as a sum similar to (1.1) using Cohen's [3] totient function, while  $\psi_{(k)}(n)$  is defined as a multiplicative function whose values at prime powers  $p^\alpha$  are given by

$$(1.2) \quad \psi_{(k)}(p^\alpha) = \sum_{j=0}^{\alpha} \binom{k-1}{j} \psi(p^{\alpha-j})$$

where for any nonnegative integers  $s$  and  $t$

$$(1.3) \quad \binom{s}{t} = \frac{s(s-1)(s-2)\cdots(s-t+1)}{1.2.3\cdots t}; \quad \binom{s}{0} \equiv 1.$$

We recall the Dirichlet convolution  $(a*b)(n)$  of the arithmetical functions  $a(n)$  and  $b(n)$  is defined by

$$(1.4) \quad (a*b)(n) = \sum_{d|n} a(d)b\left(\frac{n}{d}\right).$$

In [2], using totient function  $\Phi_{f, t}^{(k)}(n)$ , (see [1]; the notation for  $\Phi_{f, t}^{(k)}(n)$  is slightly different in [1])  $f = f(x)$  being a given polynomial of positive degree with integer coefficients,  $t$  and  $k$  being given positive integers, which includes as special cases when  $f(x) = x$  and special values of  $k$  and  $t$  all the familiar totient functions, the

author defined and studied the functions  $\Psi_{f,i}^{(k)}(n)$  and  $\psi_{f,i}^{(k)}(n)$  as generalizations of  $\Psi_k(n)$  and  $\psi_k(n)$  respectively and among other things extended all the results in [8] regarding  $\Psi_k(n)$  and  $\psi_k(n)$  to  $\Psi_{f,i}^{(k)}(n)$  and  $\psi_{f,i}^{(k)}(n)$ . In fact

$$(1.5) \quad \begin{aligned} & \text{i, } \Psi_{x,1}^{(k)}(n) = \Psi_k(n), \\ & \text{ii, } \psi_{x,1}^{(k)}(n) = \psi_k(n), \quad \text{and} \\ & \text{iii, } \Psi_{f,i}^{(k)}(n^k) = \psi_{f,i}^{(k)}(n). \end{aligned}$$

In this paper, we define an arithmetical function  $\psi_{f,i}^{u,v}(n)$  which includes as special cases not only the function  $\psi_{(k)}(n)$  but also  $\psi_{f,i}^{(k)}(n)$  (and hence also the function  $\psi_k(n)$ ). In §2, the function  $\psi_{f,i}^{u,v}(n)$  is defined and all the results in [8] concerning  $\psi_{(k)}(n)$  are extended to this function and in §3 we obtain its average order subject to

$$(1.6) \quad N_f(n) = O(n^\varepsilon), \quad 0 < \varepsilon < \frac{1}{u}$$

where  $N_f(n)$  is the number of solutions (mod  $n$ ) of

$$(1.7) \quad f(x) \equiv 0 \pmod{n}.$$

We note in passing that when  $f(x) = x$ ,  $N_f(n) = 1$  and that (1.6) is always satisfied if  $f(x)$  is a primitive integral polynomial with discriminant  $\neq 0$ . (cf. Theorem 54 of [7]).

We need the following results about  $\psi_{f,i}^{(k)}(n)$  which have been obtained in [2].

$$(1.8) \quad \begin{aligned} & \text{i, } \psi_{f,i}^{(k)}(n) \text{ is a multiplicative function of } n \\ & \text{ii, } \psi_{f,i}^{(k)}(p^\alpha) = p^{\alpha kt} \left\{ 1 + \frac{N_f^t(p^k)}{p^{kt}} \right\} \\ & \text{iii, } \psi_{f,i}^{(k)}(n) = n^{kt} \prod_{p|n} \left\{ 1 + \frac{N_f^t(p^k)}{p^{kt}} \right\} = n^{kt} \sum_{d|n} \frac{\mu^t(d) N_f^t(d^k)}{d^{kt}}, \end{aligned}$$

where  $\mu(n)$  is the Mobius function and for any arithmetical function  $g(n)$ ,  $g^r(n) = (g(n))^r$ .

We shall use the symbol  $p^\alpha || n$  to mean that  $p^\alpha$  is the highest power of  $p$  that divides  $n$ .

2. For a given polynomial  $f$  and for given positive integers  $u$ ,  $v$  and  $t$  we define the arithmetical function  $\psi_{f,i}^{u,v}(n)$  as a multiplicative function whose values at prime powers  $p^\alpha$  are given by

$$(2.1) \quad \psi_{f,i}^{u,v}(p^\alpha) = \sum_{j=0}^{\alpha} \binom{u-1}{j} N_f^{jt}(p^v) \psi_{f,i}^{(v)}(p^{\alpha-j}).$$

Clearly,

$$(2.2) \quad \psi_{f,i}^{1,k}(n) = \psi_{f,i}^{(k)}(n)$$

and from (ii) of (1.5) for  $k = 1$  and (1.2)

$$(2.3) \quad \psi_{x,1}^{k,1}(n) = \psi_{(k)}(n).$$

Using (1.8), writing  $N$  for  $N_f(p^v)$ , and observing

$$\begin{aligned} \binom{s}{t} + \binom{s}{t+1} &= \binom{s+1}{t+1}, \quad \text{we get the r.h.s. of (2.1) is} \\ &= \sum_{j=0}^{\alpha-1} \binom{u-1}{j} N^{jt} \{p^{(\alpha-j)vt} + p^{(\alpha-j-1)vt} N\} + \binom{u-1}{\alpha} N^{\alpha t} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \left\{ \binom{u-1}{j-1} + \binom{u-1}{j} \right\} N^{jt} p^{(\alpha-j)vt} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \binom{u}{j} N^{jt} p^{(\alpha-j)vt}, \quad \text{for } \alpha > 0 \end{aligned}$$

and is 1 for  $\alpha = 0$ ; consequently, we have since  $\psi_{f,i}^{u,v}(n)$  is by definition multiplicative,

**THEOREM 2.1.**

$$\psi_{f,i}^{u,v}(n) = \prod_{p^{\alpha} | |n} \left\{ \sum_{j=0}^{\alpha} \binom{u}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt} \right\}.$$

We observe that Theorem 2.1, (2.2), and the observations  $\binom{s}{t} = 0$  for  $t > s$  give (3 of (2.18) of [2])

$$(2.4) \quad \psi_{f,i}^{(k)}(n) = n^{kt} \prod_{p|n} \left\{ 1 + \frac{N_f^t(p^k)}{p^{kt}} \right\}$$

and Theorem 2.1 and (2.3) give (Theorem 3.3 of [8])

$$(2.5) \quad \psi_{(k)}(n) = \prod_{p^{\alpha} | |n} \sum_{j=0}^{\alpha} \binom{k}{j} p^{\alpha-j}.$$

We define the function  $\rho_{f,i}^{u,v}(n)$  as a multiplicative function whose values at prime powers  $p^{\alpha}$  are given by

$$(2.6) \quad \rho_{f,i}^{u,v}(p^{\alpha}) = \binom{u}{\alpha} N_f^{\alpha t}(p^v),$$

so that,

$$(2.7) \quad \rho_{f,i}^{u,v}(n) = \prod_{p^{\alpha} | |n} \binom{u}{\alpha} N_f^{\alpha t}(p^v).$$

We note that

$$(2.8) \quad \rho_{x,1}^{k,1}(n) = \prod_{p^\alpha | n} \binom{k}{\alpha} = \rho_{(k)}(n) ;$$

the function  $\rho_{(k)}(n)$  is defined in [8]. Furthermore, it is easily seen that

$$(2.9) \quad \rho_{f,t}^{1,k}(n) = \prod_{p^\alpha | n} \binom{1}{\alpha} N_f^{\alpha t}(p^k) = \mu^t(n) N_f^t(n^k) .$$

Since, by (2.6) and Theorem 2.1,

$$\sum_{d|p^\alpha} \rho_{f,t}^{u,v}(d) \left( \frac{n}{d} \right)^{vt} = \sum_{j=0}^{\alpha} \binom{u}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt} = \psi_{f,t}^{u,v}(p^\alpha)$$

and since two multiplicative functions which agree at prime powers agree for all positive integers  $n$ , we have

**THEOREM 2.2.**

$$\psi_{f,t}^{u,v}(n) = \sum_{d|n} \rho_{f,t}^{u,v}(d) \left( \frac{n}{d} \right)^{vt} = (\rho_{f,t}^{u,v} * \lambda_{vt})(n)$$

where the arithmetical function  $\lambda_r(n)$  is defined by

$$(2.10) \quad \lambda_r(n) = n^r .$$

We note that Theorem 2.2, (2.2), and (2.9) give (3, of (2.18) of [2])

$$(2.11) \quad \psi_{f,t}^{(k)}(n) = n^{kt} \sum_{d|n} \frac{\mu^t(d) N_f^t(d^k)}{d^{kt}}$$

and Theorem 2.2, (2.3) and (2.8) give (Theorem 3.9 of [8])

$$(2.12) \quad \psi_{(k)}(n) = n \sum_{d|n} \frac{\rho_{(k)}(d)}{d} .$$

**THEOREM 2.3.** For  $u \geq 2$

$$\psi_{f,t}^{u,v}(n) = (\rho_{f,t}^{1,v} * \psi_{f,t}^{u-1,v})(n) = (\rho_{f,t}^{u-1,v} * \psi_{f,t}^{1,v})(n) .$$

For the proof of Theorem 2.3, we need

**LEMMA 2.1.** For  $u \geq 2$ ,

$$\rho_{f,t}^{u,v}(n) = (\rho_{f,t}^{1,v} * \rho_{f,t}^{u-1,v})(n) = (\rho_{f,t}^{u-1,v} * \rho_{f,t}^{1,v})(n) .$$

*Proof.* The second equality is obvious since Dirichlet convolu-

tion is commutative. To prove the first equality it is enough to verify when  $n = p^\alpha$ ,  $\alpha \geq 0$ ,  $p$  a prime. If  $\alpha = 0$ , both sides are 1 and if  $\alpha > 0$  by (2.6)

$$\begin{aligned} \sum_{d|p^\alpha} \rho_{f,t}^{1,v}(d) \rho_{f,t}^{u-1,v} \left( \frac{p^\alpha}{d} \right) &= \binom{u-1}{\alpha} N_f^{\alpha t}(p^v) + \binom{1}{1} N_f^t(p^v) \binom{u-1}{\alpha-1} N_f^{(\alpha-1)t}(p^v) \\ &= N_f^{\alpha t}(p^v) \left\{ \binom{u-1}{\alpha} + \binom{u-1}{\alpha-1} \right\} = \binom{u}{\alpha} N_f^{\alpha t}(p^v) = \rho_{f,t}^{u,v}(p^\alpha) \end{aligned}$$

and the proof of the lemma is complete.

We observe, Lemmas 2.1, 2.8, and (2.9) give (Theorem 3.12 of [8])

$$(2.13) \quad \rho_{(k)}(n) = \sum_{d|n} \mu^2(d) \rho_{(k-1)} \left( \frac{n}{d} \right), \quad k \geq 2.$$

*Proof of Theorem 2.3.* We first prove first equality. It is enough to verify this when  $n = p^\alpha$ ,  $p$  a prime and  $\alpha \geq 0$ . If  $\alpha = 0$ , both sides are 1 while if  $\alpha > 0$

$$\begin{aligned} \sum_{d|p^\alpha} \rho_{f,t}^{1,v}(d) \psi_{f,t}^{u-1,v} \left( \frac{p^\alpha}{d} \right) &\quad \text{(by (2.9) and Theorem 2.1)} \\ &= \rho_{f,t}^{1,v}(1) \psi_{f,t}^{u-1,v}(p^\alpha) + \rho_{f,t}^{1,v}(p) \psi_{f,t}^{u-1,v}(p^{\alpha-1}) \\ &= \sum_{j=0}^{\alpha} \binom{u-1}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt} + N_f^t(p^v) \sum_{j=0}^{\alpha-1} \binom{u-1}{j} N_f^{jt}(p^v) p^{(\alpha-j-1)vt} \\ &= p^{\alpha vt} + \sum_{j=1}^{\alpha} \left\{ \binom{u-1}{j-1} + \binom{u-1}{j} \right\} N_f^{jt}(p^v) p^{(\alpha-j)vt} \\ &= \sum_{j=0}^{\alpha} \binom{u}{j} N_f^{jt}(p^v) p^{(\alpha-j)vt} = \psi_{f,t}^{u,v}(p^\alpha) \end{aligned}$$

and the proof of the first equality is complete.

To complete the proof of the theorem, we have by Theorem 2.2 the associativity of Dirichlet convolution, and Lemma 2.1,

$$\begin{aligned} \rho_{f,t}^{u-1,v} * \rho_{f,t}^{1,v} &= \rho_{f,t}^{u-1,v} * (\rho_{f,t}^{1,v} * \lambda_{vt}) = (\rho_{f,t}^{u-1,v} * \rho_{f,t}^{1,v}) * \lambda_{vt} \\ &= \rho_{f,t}^{u,v} * \lambda_{vt} = \psi_{f,t}^{u,v}, \end{aligned}$$

and the proof is complete.

Theorem 2.3, (2.3), and (2.9) give Theorems 3.10 and 3.11 of [8]

$$(2.14) \quad \psi_{(k)}(n) = \sum_{d|n} \mu^2(d) \psi_{(k-1)} \left( \frac{n}{d} \right), \quad k \geq 2;$$

$$(2.15) \quad \psi_{(k)}(n) = \sum_{d|n} \rho_{(k-1)}(d) \psi \left( \frac{n}{d} \right), \quad k \geq 2.$$

3. We obtain in this section, the average order of  $\rho_{f,i}^{u,v}(n)$  subject to (1.6).

LEMMA 3.1.

i,  $\rho_{f,i}^{u,v}(n) = 0$  if  $n$  is not  $(u+1)$  free

ii,  $\rho_{f,i}^{u,v}(n) < 2^{uw(n)} N_f^{ut}(\gamma^v(n))$  if  $n$  is  $u+1$ -free

where  $w(n)$  is the number of distinct prime factors of  $n$  and  $\gamma(n)$  is the largest square free divisor of  $n$ .

*Proof.* If  $n$  is not  $u+1$ -free, there is a prime  $p$  such that  $p^\alpha \parallel n$ ,  $\alpha \geq u+1$  and so  $\binom{u}{\alpha} = 0$  and hence (2.7) implies  $\rho_{f,i}^{u,v}(n) = 0$ .

If  $n$  is  $u+1$ -free, then  $p^\alpha \parallel n$  implies  $\alpha \leq u$  and hence by (2.7), using the facts that  $\binom{n}{\alpha} \leq 2^u$  and  $N_f(n)$  is a multiplicative function of  $n$ , we have

$$\rho_{f,i}^{u,v}(n) = \prod_{p^\alpha \parallel n} \binom{u}{\alpha} N_f^{\alpha t}(p^v) \leq 2^{uw(n)} N_f^{ut}(\gamma^v(n))$$

and the proof of the lemma is complete.

We also need the following elementary estimates

$$(3.1) \quad \begin{aligned} \text{i, } & \sum_{n \leq x} n^r = \frac{x^{r+1}}{r+1} + O(x^r), \quad r > 0, \quad x \geq 1; \\ \text{ii, } & \sum_{n \leq x} \frac{1}{n^r} = O(x^{1-r}), \quad 0 < r < 1, \quad x \geq 1; \\ \text{iii, } & \sum_{n > x} \frac{1}{n^r} = O(x^{1-r}), \quad r > 1, \quad x \geq 1. \end{aligned}$$

LEMMA 3.2. Under the hypothesis (1.6),  $\sum_{n=1}^{\infty} \rho_{f,i}^{u,v}(n)/n^{vt+1}$  converges and

$$(3.2) \quad c = \sum_{n=1}^{\infty} \frac{\rho_{f,i}^{u,v}(n)}{n^{vt+1}} = \left( \prod_p \left\{ 1 + \frac{N_f^t(p^v)}{n^{vt+1}} \right\} \right)^u.$$

*Proof.* If  $d(n)$  is the number of divisors of  $n$ , we have (cf. Theorem 315 of [5])  $d(n) = O(n^\theta)$  for every  $\theta > 0$  and hence

$$2^{uw(n)} = (2^{w(n)})^u \leq (d(n))^u = O(n^{u\theta}) \quad \text{for every } \theta > 0,$$

where the constant in the 0-relation depends on  $u$  but not on  $n$ . Now, (1.6) and Lemma 3.1 give

$$(3.3) \quad \rho_{f,i}^{u,v}(n) = O(n^{uvt+u\theta}),$$

where the constant in the 0-relation is independent of  $n$ . Hence

$$(3.4) \quad \frac{\rho_{f,i}^{u,v}(n)}{n^{vt+1}} = 0 \left( \frac{1}{n^1 + vt(1 - u\varepsilon) - u\theta} \right).$$

The first part of the lemma is clear since by (1.6)  $1 - u\varepsilon > 0$  and we can choose  $\theta$  so small that

$$(3.5) \quad vt(1 - u\varepsilon) - u\theta > 0.$$

Since  $\rho_{f,i}^{u,v}(n)/n^{vt+1}$  is multiplicative we can express the sum of the series as an infinite product of Euler type and so we have

$$\sum_{n=1}^{\infty} \frac{\rho_{f,i}^{u,v}(n)}{n^{vt+1}} \prod_p \left\{ \sum_{m=0}^{\infty} \frac{\rho_{f,i}^{u,v}(p^m)}{(p^m)^{vt+1}} \right\}$$

and this by (2.6) and the fact  $\binom{u}{\alpha} = 0$  for  $\alpha > u$  is

$$\begin{aligned} &= \prod_p \left\{ \sum_{m=0}^u \frac{\binom{u}{m} N_f^m(p^v)}{(p^{vt+1})^m} \right\} \\ &= \prod_p \left\{ 1 + \frac{N_f^1(p^v)}{p^{vt+1}} \right\}^u \end{aligned}$$

and the proof of Lemma 3.2 is complete.

**THEOREM 3.1.** *Under the hypothesis (1.6),*

$$\sum_{n \leq x} \psi_{f,i}^{u,v}(n) = c \frac{x^{vt+1}}{vt+1} + E(x)$$

where

$$\begin{aligned} E(x) &= 0(x^{vt}) \quad \text{if } vt(1 - u\varepsilon) > 1 \\ &= 0(x^{1+u\theta+uvt\varepsilon}) \quad \text{for every } \theta < \frac{vt(1 - u\varepsilon)}{u} \end{aligned}$$

if  $vt(1 - u\varepsilon) \leq 1$ .

*Proof.* We have by Theorem 2.2,

$$\begin{aligned} \sum_{n \leq x} \psi_{f,i}^{u,v}(n) &= \sum_{n \leq x} \sum_{d \mid n} \rho_{f,i}^{u,v}(d) \delta^{vt} \\ &= \sum_{d \leq x} \rho_{f,i}^{u,v}(d) \delta^{vt} = \sum_{d \leq x} \rho_{f,i}^{u,v}(d) \sum_{\delta \leq x/d} \delta^{vt} \end{aligned}$$

and this by *i*, of (3.1) is

$$\sum_{d \leq x} \rho_{f,i}^{u,v}(d) \left\{ \frac{1}{vt+1} \left( \frac{x}{d} \right)^{vt+1} + 0 \left( \left( \frac{x}{d} \right)^{vt} \right) \right\}$$

which by Lemma 3.2 is equal to



$$(3.6) \quad \frac{x^{vt+1}}{vt+1} + O\left(x^{vt+1} \sum_{n>x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}}\right) + O\left(x^{vt} \sum_{n\leq x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt}}\right).$$

Let  $\theta > 0$  be so chosen that

$$(3.7) \quad \begin{cases} u\theta < vt(1-u\varepsilon) - 1, & \text{if } vt(1-u\varepsilon) > 1 \\ u\theta < vt(1-u\varepsilon), & \text{if } vt(1-u\varepsilon) \leq 1. \end{cases}$$

In any case  $u\theta < vt(1-u\varepsilon)$ . By (3.4) and (iii) of (3.1),

$$\begin{aligned} \sum_{n>x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt+1}} &= O\left(\sum_{n>x} \frac{1}{n^{1+vt(1-u\varepsilon)-u\theta}}\right) \\ &= O(x^{-vt(1-u\varepsilon)+u\theta}) \end{aligned}$$

and so, the second term in (3.6) is  $O(x^{1+u\theta+uvt\varepsilon})$ .

Similarly,

$$\sum_{n\leq x} \frac{\rho_{f,t}^{u,v}(n)}{n^{vt}} = O\left(\sum_{n\leq x} \left(\frac{1}{n^{vt(1-u\varepsilon)-u\theta}}\right)\right),$$

and hence the third term in (3.6) is  $O(x^{vt})$  or  $O(x^{1+u\theta+uvt\varepsilon})$  according as  $vt(1-u\varepsilon) > 1$  or  $vt(1-u\varepsilon) \leq 1$ . Since  $u\theta < vt(1-u\varepsilon) - 1$  implies  $1+u\theta+uvt\varepsilon < vt$ , the theorem is clear. Clearly, Theorem 3.1 can be stated as

**THEOREM 3.1'.** *Under the hypothesis (1.6), the average order of  $\psi_{f,t}^{u,v}(n)$  is  $cn^{vt}$ , where  $c$  is given by (3.2).*

Since  $\psi_{(k)}(n) = \psi_{x,1}^{k,1}(n)$ ,  $N_x(n) = 1$ , the r.h.s. of (3.2) in this case is

$$\left\{ \prod_p \left(1 + \frac{1}{p^2}\right) \right\}^k = \left\{ \prod_p \frac{(1+p^{-2})(1-p^{-2})}{1-p^{-2}} \right\}^k = \frac{\zeta^k(2)}{\zeta^k(4)}, \quad \zeta(s)$$

being the Riemann's  $\zeta$ -function, and so from Theorem 3.1, we have

**COROLLARY 3.1.1.** (Theorem 4.4 of [7].)

The average order of  $\psi_{(k)}(n)$  is  $n(\zeta^k(2)/\zeta^k(4)) = n(15/\pi^2)^k$ .

Similarly, Theorem 3.1, (2.2) and (2.9) give

**COROLLARY 3.1.2.** ((3.5) of [2].)

The average order of  $\psi_{f,t}^{(k)}(n)$  is  $\left\{ \sum_{n=1}^{\infty} (\mu^2(n) N_f^t(n^k)/n^{kt+1}) \right\} n^{kt}$ .

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