GENERALIZED DEDEKIND $\psi$-FUNCTIONS WITH RESPECT TO A POLYNOMIAL. II

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For a given polynomial $f = f(x)$ of positive degree with integer coefficients and for given positive integers $u, v,$ and $t,$ the arithmetical function $\psi_{f, t}^u(n)$ is defined and some of its arithmetical properties are obtained in addition to its average order. $\psi_{f, t}^u(n)$ reduces to the function $\psi_{\alpha}(n)$ studied recently by D. Suryanarayana and $\psi_{f, t}^u(n)$ to $\psi^g(n)$ studied more recently by the author.

Introduction. The Dedekind’s $\psi$-function

\begin{equation}
\psi(n) = \sum_{d|n} \frac{d \phi(g)}{g}, \quad g = \left( d, \frac{n}{d} \right),
\end{equation}

$\phi(n)$ being Euler’s totient function is well known. He used this function in his study of elliptic modular functions [4]. As generalizations of this function, recently D. Suryanarayana [8] defined and studied the functions $\Psi_k(n), \psi_k(n)$ and $\psi_{t,k}(n)$ all giving the function $\psi(n)$ for $k = 1.$ The functions $\Psi_k(n)$ and $\psi_k(n)$ are defined respectively (see [8]) as the Dirichlet’s convolution of a certain function with Klee’s [6] totient function and as a sum similar to (1.1) using Cohen’s [3] totient function, while $\psi_{t,k}(n)$ is defined as a multiplicative function whose values at prime powers $p^s$ are given by

\begin{equation}
\psi_{t,k}(p^s) = \sum_{j=0}^{s} \binom{k - 1}{j} \psi(p^{s-j})
\end{equation}

where for any nonnegative integers $s$ and $t$

\begin{equation}
\binom{s}{t} = \frac{s(s - 1)(s - t + 1)}{1.2.3 \cdots t}; \binom{s}{0} \equiv 1.
\end{equation}

We recall the Dirichlet convolution $(a*b)(n)$ of the arithmetical functions $a(n)$ and $b(n)$ is defined by

\begin{equation}
(a*b)(n) = \sum_{d|n} a(d)b\left( \frac{n}{d} \right).
\end{equation}

In [2], using totient function $\Phi_{f, t}^k(n),$ (see [1]; the notation for $\Phi_{f, t}^k(n)$ is slightly different in [1]) $f = f(x)$ being a given polynomial of positive degree with integer coefficients, $t$ and $k$ being given positive integers, which includes as special cases when $f(x) = x$ and special values of $k$ and $t$ all the familiar totient functions, the
author defined and studied the functions $\Psi^{(k)}_{f,i}(n)$ and $\psi^{(k)}_{f,i}(n)$ as generalizations of $\Psi_k(n)$ and $\psi_k(n)$ respectively and among other things extended all the results in [8] regarding $\Psi_k(n)$ and $\psi_k(n)$ to $\Psi^{(k)}_{f,i}(n)$ and $\psi^{(k)}_{f,i}(n)$. In fact

\begin{align}
\text{i, } \Psi^{(k)}_{x,i}(n) &= \Psi_k(n), \\
\text{ii, } \psi^{(k)}_{x,i}(n) &= \psi_k(n), \text{ and} \\
\text{iii, } \Psi^{(k)}_{f,i}(n^k) &= \psi^{(k)}_{f,i}(n).
\end{align}

In this paper, we define an arithmetical function $\psi^{(k)}_{n}(n)$ which includes as special cases not only the function $\psi_{(k)}(n)$ but also $\psi^{(k)}_{f,i}(n)$ (and hence also the function $\psi_k(n)$). In §2, the function $\psi^{(k)}_{n}(n)$ is defined and all the results in [8] concerning $\psi_{(k)}(n)$ are extended to this function and in §3 we obtain its average order subject to

\begin{equation}
N_f(n) = O(n^\varepsilon), \quad 0 < \varepsilon < \frac{1}{u}
\end{equation}

where $N_f(n)$ is the number of solutions (mod $n$) of

\begin{equation}
f(x) \equiv 0 \pmod{n}.
\end{equation}

We note in passing that when $f(x) = x$, $N_f(n) = 1$ and that (1.6) is always satisfied if $f(x)$ is a primitive integral polynomial with discriminant $\neq 0$. (cf. Theorem 54 of [7]).

We need the following results about $\psi^{(k)}_{n}(n)$ which have been obtained in [2].

\begin{enumerate}
\item $\psi^{(k)}_{n}(n)$ is a multiplicative function of $n$
\item $\psi^{(k)}_{n}(p^a) = p^{ak} \left\{ 1 + \frac{N^{(k)}_f(p^b)}{p^{kt}} \right\}$
\item $\psi^{(k)}_{n}(n) = n^{kt} \prod_{p \mid n} \left\{ 1 + \frac{N^{(k)}_f(p^b)}{p^{kt}} \right\} = n^{kt} \sum_{d \mid n} \mu(d) N^{(k)}_f(d^b)\frac{d^{kt}}{d^{kt}}$,
\end{enumerate}

where $\mu(n)$ is the Mobius function and for any arithmetical function $g(n)$, $g^*(n) = (g(n))^r$.

We shall use the symbol $p^{a} \mid n$ to mean that $p^a$ is the highest power of $p$ that divides $n$.

2. For a given polynomial $f$ and for given positive integers $u$, $v$ and $t$ we define the arithmetical function $\psi^{(k)}_{f,i}(n)$ as a multiplicative function whose values at prime powers $p^a$ are given by

\begin{equation}
\psi^{(k)}_{f,i}(p^a) = \sum_{j=0}^{a} \binom{u-1}{j} N^{(k)}_f(p^j) \psi^{(k)}_{f,i}(p^{a-j}).
\end{equation}

Clearly,
and from (ii) of (1.5) for \( k = 1 \) and (1.2)

(2.3) \[ \psi_{s;1}^k(n) = \psi_{(k)}(n). \]

Using (1.8), writing \( N \) for \( N_j(p^s) \), and observing

\[
\left( \frac{s}{t} \right) + \left( \frac{s}{t + 1} \right) = \left( \frac{s + 1}{t + 1} \right),
\]

we get the r.h.s. of (2.1) is

\[
= \sum_{j=0}^{a-1} \binom{u-1}{j} N_j^t(p^{(a-j)t} + p^{(a-j-1)t} N_j^t) + \binom{u-1}{\alpha} N_\alpha^t
\]

\[
= p^{at} + \sum_{j=1}^{a} \left( \binom{u-1}{j} + \binom{u-1}{j-1} \right) N_j^t p^{(a-j)t}
\]

\[
= p^{at} + \sum_{j=1}^{a} \binom{u}{j} N_j^t p^{(a-j)t}, \text{ for } \alpha > 0
\]

and is 1 for \( \alpha = 0 \); consequently, we have since \( \psi_{s;1}^k(n) \) is by definition multiplicative,

**Theorem 2.1.**

\[ \psi_{s;1}^\alpha(n) = \prod_{p^\alpha \mid n} \left\{ \sum_{j=0}^{a} \binom{u}{j} N_j^t(p^\alpha) p^{(a-j)t} \right\}. \]

We observe that Theorem 2.1, (2.2), and the observations

\( \left( \frac{s}{t} \right) = 0 \) for \( t > s \) give (3 of (2.18) of [2])

(2.4) \[ \psi_{s;1}^k(n) = n^{bt} \prod_{p^\alpha \mid n} \left\{ 1 + \frac{N_j^t(p^k)}{p^{kt}} \right\} \]

and Theorem 2.1 and (2.3) give (Theorem 3.3 of [8])

(2.5) \[ \psi_{(k)}(n) = \prod_{p^\alpha \mid n} \sum_{j=0}^{a} \binom{k}{j} p^{a-j}. \]

We define the function \( \rho_{s;1}^\alpha(n) \) as a multiplicative function whose values at prime powers \( p^\alpha \) are given by

(2.6) \[ \rho_{s;1}^\alpha(p^\alpha) = \binom{u}{\alpha} N_j^t(p^\alpha), \]

so that,

(2.7) \[ \rho_{s;1}^\alpha(n) = \prod_{p^\alpha \mid n} \binom{u}{\alpha} N_j^t(p^\alpha). \]
We note that

\begin{equation}
\rho_{t;1}(n) = \prod_{p} \left( \frac{k}{\alpha} \right) = \rho_{(k)}(n);
\end{equation}

the function \( \rho_{(k)}(n) \) is defined in [8]. Furthermore, it is easily seen that

\begin{equation}
\rho_{t;1}^{(1)}(n) = \prod_{p} \left( \frac{1}{\alpha} \right) N_{t}^{\pi}(p^{k}) = \mu(n) N_{t}^{\pi}(n^{k}).
\end{equation}

Since, by (2.6) and Theorem 2.1,

\[
\sum_{d \mid n} \rho_{t;1}^{(1)}(d) \left( \frac{n}{d} \right) = \sum_{j=0}^{\alpha} \left( \frac{u}{j} \right) N_{t}^{\pi}(p^{j}) p^{(\alpha-j)\pi} = \psi_{t;1}^{(1)}(p^{\alpha})
\]

and since two multiplicative functions which agree at prime powers agree for all positive integers \( n \), we have

**Theorem 2.2.**

\[
\psi_{t;1}^{(1)}(n) = \sum_{d \mid n} \rho_{t;1}^{(1)}(d) \left( \frac{n}{d} \right) = (\rho_{t;1}^{(1)} * \lambda)(n)
\]

where the arithmetical function \( \lambda_{r}(n) \) is defined by

\begin{equation}
\lambda_{r}(n) = n^{r}.
\end{equation}

We note that Theorem 2.2, (2.2), and (2.9) give (3, of (2.18) of [2])

\begin{equation}
\psi_{(k)}(n) = n^{k} \sum_{d \mid n} \frac{\mu(d) N_{t}^{\pi}(d^{k})}{d^{k}}
\end{equation}

and Theorem 2.2, (2.3) and (2.8) give (Theorem 3.9 of [8])

\begin{equation}
\psi_{(k)}(n) = n \sum_{d \mid n} \frac{\rho_{(k)}(d)}{d}.
\end{equation}

**Theorem 2.3.** \( \text{For} \ u \geq 2 \)

\[
\psi_{t;1}^{(1)}(n) = (\rho_{t;1}^{(1)} * \psi_{t;1}^{(1)})(n) = (\rho_{t;1}^{(1)} * \psi_{t;1}^{(1)}) (n).
\]

*For the proof of Theorem 2.3, we need*

**Lemma 2.1.** \( \text{For} \ u \geq 2, \)

\[
\rho_{t;1}^{(1)}(n) = (\rho_{t;1}^{(1)} * \rho_{t;1}^{(1)})(n) = (\rho_{t;1}^{(1)} * \rho_{t;1}^{(1)})(n).
\]

*Proof. The second equality is obvious since Dirichlet convolu-
tion is commutative. To prove the first equality it is enough to verify when \( n = p^\alpha, \alpha \geq 0, p \) a prime. If \( \alpha = 0 \), both sides are 1 and if \( \alpha > 0 \) by (2.6)

\[
\sum_{d \mid p^\alpha} \rho_{j,t}^{\nu,v}(d) \rho_{j,t}^{\nu^*-1,v^*}\left(\frac{p^\alpha}{d}\right) = \left(\frac{u - 1}{\alpha}\right) N_f^{\nu}(p^\alpha) + \left(\frac{1}{1}\right) N_f^{\nu}(p^\alpha)\left(\frac{u - 1}{\alpha - 1}\right) N_f^{\nu^*-1,v^*}(p^\alpha)
\]

\[
= N_f^{\nu}(p^\alpha) \left(\frac{u - 1}{\alpha}\right) + \left(\frac{u - 1}{\alpha - 1}\right) = \left(\frac{u}{\alpha}\right) N_f^{\nu}(p^\alpha) = \rho_{j,t}^{\nu,v}(p^\alpha)
\]

and the proof of the lemma is complete.

We observe, Lemmas 2.1, 2.8, and (2.9) give (Theorem 3.12 of [8])

\[
(2.13) \quad \rho_{(k)}(n) = \sum_{d \mid n} \mu^t(d) \rho_{(k-1)}\left(\frac{n}{d}\right), \quad k \geq 2.
\]

Proof of Theorem 2.3. We first prove first equality. It is enough to verify this when \( n = p^\alpha, p \) a prime and \( \alpha \geq 0 \). If \( \alpha = 0 \), both sides are 1 while if \( \alpha > 0 \),

\[
\sum_{d \mid p^\alpha} \rho_{j,t}^{\nu,v}(d) \psi_{j,t}^{\nu^*-1,v^*}\left(\frac{p^\alpha}{d}\right) = \rho_{j,t}^{\nu,v}(1) \psi_{j,t}^{\nu^*-1,v^*}(p^\alpha) + \rho_{j,t}^{\nu,v}(p) \psi_{j,t}^{\nu^*-1,v^*}(p^{\alpha-1})
\]

\[
= \sum_{j=0}^\alpha \left(\frac{u - 1}{j}\right) N_f^{\nu}(p^\alpha) p^{(\alpha-j)v} + N_f^{\nu}(p^\alpha) \sum_{j=0}^{\alpha-1} \left(\frac{u - 1}{j}\right) N_f^{\nu}(p^\alpha) p^{(\alpha-j-1)v}
\]

\[
= p^{\alpha v} + \sum_{j=1}^\alpha \left(\frac{u - 1}{j}\right) + \left(\frac{u - 1}{j}\right) \right) N_f^{\nu}(p^\alpha) p^{(\alpha-j)v} = \psi_{j,t}^{\nu,v}(p^\alpha)
\]

and the proof of the first equality is complete.

To complete the proof of the theorem, we have by Theorem 2.2 the associativity of Dirichlet convolution, and Lemma 2.1,

\[
\rho_{j,t}^{\nu,v} \psi_{j,t}^{\nu^*} = (\rho_{j,t}^{\nu,v} \ast \rho_{j,t}^{\nu^*} \ast \lambda_{vt} = \rho_{j,t}^{\nu,v} \ast \lambda_{vt} = \psi_{j,t}^{\nu,v}
\]

and the proof is complete.

Theorem 2.3, (2.3), and (2.9) give Theorems 3.10 and 3.11 of [8]

\[
(2.14) \quad \psi_{(k)}(n) = \sum_{d \mid n} \mu^t(d) \psi_{(k-1)}\left(\frac{n}{d}\right), \quad k \geq 2;
\]

\[
(2.15) \quad \psi_{(k)}(n) = \sum_{d \mid n} \rho_{(k-1)}(d) \psi\left(\frac{n}{d}\right), \quad k \geq 2.
\]
3. We obtain in this section, the average order of $\psi_f',v_t(n)$ subject to (1.6).

**Lemma 3.1.**

i, $\rho_{f';i}(n) = 0$ if $n$ is not $(u+1)$ free

ii, $\rho_{f';i}(n) < 2^w(n)N_f'(\gamma(n))$ if $n$ is $u+1$-free

where $w(n)$ is the number of distinct prime factors of $n$ and $\gamma(n)$ is the largest square free divisor of $n$.

**Proof.** If $n$ is not $u+1$-free, there is a prime $p$ such that $p^\alpha || n$, $\alpha \geq u + 1$ and so $\left(\frac{u}{\alpha}\right) = 0$ and hence (2.7) implies $\rho_{f';i}(n) = 0$.

If $n$ is $u+1$-free, then $p^\alpha || n$ implies $\alpha \leq u$ and hence by (2.7), using the facts that $\left(\frac{n}{\alpha}\right) \leq 2^u$ and $N_f(n)$ is a multiplicative function of $n$, we have

$$\rho_{f';i}(n) = \prod_{p^\alpha || n} \left(\frac{u}{\alpha}\right) N_f'(p^\alpha) \leq 2^w(n)N_f'(\gamma(n))$$

and the proof of the lemma is complete.

We also need the following elementary estimates

\begin{enumerate}
  \item $\sum_{n \geq x} n^r = \frac{x^{r+1}}{r+1} + O(x^r), \quad r > 0, \quad x \geq 1;
  \item $\sum_{n \geq x} \frac{1}{n^r} = 0(x^{-r}), \quad 0 < r < 1, \quad x \geq 1;
  \item $\sum_{n > x} \frac{1}{n^r} = 0(x^{-r}), \quad r > 1, \quad x \geq 1$.
\end{enumerate}

**Lemma 3.2.** Under the hypothesis (1.6), $\sum_{n=1}^\infty \rho_{f';i}(n)/n^{s+l}+1$ converges and

$$c = \sum_{n=1}^\infty \frac{\rho_{f';i}(n)}{n^{s+l+1}} = \left(\prod_p \left(1 + \frac{N_{f'}(p^s)}{n^{s+l}}\right)\right)^u.$$

**Proof.** If $d(n)$ is the number of divisors of $n$, we have (cf. Theorem 315 of [5]) $d(n) = 0(n^\theta)$ for every $\theta > 0$ and hence

$$2^n w(n)^s \leq (d(n))^s = 0(n^{s\theta})$$

for every $\theta > 0$, where the constant in the $0$-relation depends on $u$ but not on $n$. Now, (1.6) and Lemma 3.1 give

$$\rho_{f';i}(n) = 0(n^{u+u\theta})$$

where the constant in the $0$-relation is independent of $n$. Hence
\[ \rho_{f_i}^*(n) = 0 \left( \frac{1}{n^1 + vt(1 - u\varepsilon) - u\theta} \right). \]

The first part of the lemma is clear since by (1.6) \( 1 - u\varepsilon > 0 \) and we can choose \( \theta \) so small that

\[ vt(1 - u\varepsilon) - u\theta > 0. \]

Since \( \rho_{f_i}^*(n)/n^{vt+1} \) is multiplicative we can express the sum of the series as an infinite product of Euler type and so we have

\[ \sum_{n=1}^{\infty} \frac{\rho_{f_i}^*(n)}{n^{vt+1}} \prod_{p} \left\{ \sum_{m=0}^{\infty} \frac{\rho_{f_i}^*(p^m)}{(p^m)^{vt+1}} \right\} \]

and this by (2.6) and the fact \( \left( \frac{u}{\alpha} \right) = 0 \) for \( \alpha > u \) is

\[ = \prod_{p} \left\{ \sum_{m=0}^{\infty} \left( \frac{u}{m} \right) \frac{N_f^m t(p^v)}{(p^v t^{vt+1})^m} \right\} \]

\[ = \prod_{p} \left\{ 1 + \frac{N_f^v(p^v)}{p^{vt+1}} \right\} \]

and the proof of Lemma 3.2 is complete.

**Theorem 3.1.** Under the hypothesis (1.6),

\[ \sum_{n \leq x} \psi_{f_i}^*(n) = e^{\frac{x^{vt+1}}{vt + 1}} + E(x) \]

where

\[ E(x) = 0(x^{vt}) \text{ if } vt(1 - u\varepsilon) > 1 \]

\[ = 0(x^{1+u\theta + uvt}) \text{ for every } \theta < \frac{vt(1 - u\varepsilon)}{u} \]

if \( vt(1 - u\varepsilon) \leq 1. \)

**Proof.** We have by Theorem 2.2,

\[ \sum_{n \leq x} \psi_{f_i}^*(n) = \sum_{n \leq x} \sum_{d \mid n} \rho_{f_i}^*(d) \delta^{vt} \]

\[ = \sum_{d \leq x} \rho_{f_i}^*(d) \delta^{vt} = \sum_{d \leq x} \rho_{f_i}^*(d) \sum_{\delta \mid x/d} \delta^{vt} \]

and this by \( i \), of (3.1) is

\[ \sum_{d \leq x} \rho_{f_i}^*(d) \left\{ \frac{1}{vt + 1} \left( \frac{x}{d} \right)^{vt+1} + O\left( \left( \frac{x}{d} \right)^{vt} \right) \right\} \]

which by Lemma 3.2 is equal to
Let $\theta > 0$ be so chosen that

\[
\left\{ \begin{array}{ll}
u \theta < vt(1 - u\varepsilon) - 1, & \text{if } vt(1 - u\varepsilon) > 1 \\
u \theta < vt(1 - u\varepsilon), & \text{if } vt(1 - u\varepsilon) \leq 1. 
\end{array} \right.
\]

(3.7)

In any case $u \theta < vt(1 - u\varepsilon)$. By (3.4) and (iii) of (3.1),

\[
\sum_{n > x} \frac{\rho_{\nu; \varepsilon}^o(n)}{n^v t^{1 + u\varepsilon}} = 0 \left( \sum_{n > x} \frac{1}{n^{v t(1 - u\varepsilon) - u \theta}} \right) = 0(x^{-vt(1-u\varepsilon)+u\theta})
\]

and so, the second term in (3.6) is $0(x^{vt} + u\theta + uvt\varepsilon)$.

Similarly,

\[
\sum_{n \geq x} \frac{\rho_{\nu; \varepsilon}^o(n)}{n^v t} = 0 \left( \sum_{n \geq x} \left( \frac{1}{n^{v t(1 - u\varepsilon) - u \theta}} \right) \right),
\]

and hence the third term in (3.6) is $0(x^{vt})$ or $0(x^{1+u\theta+uvt\varepsilon})$ according as $vt(1 - u\varepsilon) > 1$ or $vt(1 - u\varepsilon) \leq 1$. Since $u \theta < vt(1 - u\varepsilon) - 1$ implies $1 + u \theta + uvt\varepsilon < vt$, the theorem is clear. Clearly, Theorem 3.1 can be stated as

**THEOREM 3.1'.** Under the hypothesis (1.6), the average order of $\psi_{\nu; \varepsilon}^o(n)$ is $c n^{vt}$, where $c$ is given by (3.2).

Since $\psi_{(k)}(n) = \psi_{x,1}^{k,1}(n), N_z(n) = 1$, the r.h.s. of (3.2) in this case is

\[
\left\{ \prod_p \left( 1 + \frac{1}{p^2} \right) \right\}^k = \left\{ \prod_p \left( \frac{1 + p^{-2}(1 - p^{-2})}{1 - p^{-2}} \right) \right\}^k = \frac{\zeta(k)(2)}{\zeta(k)(4)}, \quad \zeta(s)
\]

being the Riemann's $\zeta$-function, and so from Theorem 3.1, we have

**COROLLARY 3.1.1.** (Theorem 4.4 of [7].)

The average order of $\psi_{(k)}(n)$ is $n(\zeta^k(2)/\zeta^k(4)) = n(15/\pi^2)^k$.

Similarly, Theorem 3.1, (2.2) and (2.9) give

**COROLLARY 3.1.2.** ((3.5) of [2].)

The average order of $\psi_{(k)}^{(i)}(n)$ is $\left( \sum_{n=1}^{\infty} (t^2(n)N_i^{(k)}(n^k)/n^{kt+1}) \right) n^{kt}$. 
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