THE DUNFORD-PETTIS PROPERTY FOR CERTAIN UNIFORM ALGEBRAS

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A Banach space $B$ has the Dunford-Pettis property if $x^*_n(x_n) \to 0$ whenever $x_n \to 0$ weakly and the sequence $x^*_n$ tends to zero weakly in $B^*$ (i.e. $\sigma(B^*, B^{**})$). Suppose now that $A$ is a uniform algebra on a compact space $X$. If $\phi$ is a nonzero multiplicative linear functional on $A$ then $M_\phi$ is the set of positive representing measures of $\phi$. If $A$ is such that a singular measure which is orthogonal to $A$ must necessarily be zero and if all $M_\psi$ are weakly compact sets then the algebra $A$ as well as its dual have the Dunford-Pettis property.

The idea of the proof is that $A^*$ the dual of $A$ can be decomposed into components for which the results of Chaumat [1] and Cnop-Delbaen [2] can be applied. The fact that an $\ell_1$ sum of Dunford-Pettis spaces is also a Dunford-Pettis space then gives the result. In paragraph two some conditions ensuring the weak compactness of $M_\phi$ are given. These conditions are related to those used in the definition of core and enveloping measures (see [6]).

1. Notation and preliminaries. $X$ will be a compact space, $A \subset C(X)$ a closed subalgebra of the space of continuous complex-valued functions on $X$. The algebra $A$ is supposed to contain the constants and to separate the points of $X$. The spectrum $M_A$ is the set of all nonzero multiplicative linear functionals on $A$. If $\phi \in M_A$ then $M_\phi$ is the set of all positive measures on $X$ representing $\phi$, i.e.

$$M_\phi = \left\{ \mu \in M(X) \mid \mu \geq 0 \text{ and } \forall f \in A \text{ we have } \phi(f) = \int f d\mu \right\}.$$  

As well known $M_\phi$ is a convex set, compact for the topology $\sigma(M(X), C(X))$. We say that two multiplicative linear forms $\phi$ and $\psi$ belong to the same Gleason part if $\|\phi - \psi\| < 2$ in $A^*$, the dual of $A$. It is well known that being in the same Gleason part is an equivalence relation and hence $M_A = \bigcup_{\pi \in \Pi} \pi$ where $\Pi$ is the set of all Gleason equivalence classes. For more details and any unexplained notion on uniform algebras we refer to [6].

If $E$ is a Banach space then $E$ has the Dunford-Pettis property if $e_n^*(e_n) \to 0$ whenever $e_n \to 0$ weakly and $e_n^* \to 0$ weakly (i.e. $\sigma(E^*, E^{**})$).

For more details and properties of such spaces see Grothendieck.
[4] or [5], where it is also proved that $L^1$ spaces and $\mathcal{C}(X)$ spaces have the Dunford-Pettis property.

2. Weak compactness of $M_\phi$. We investigate under what conditions $M_\phi$ is weakly compact. First we remark that if $\psi$ and $\phi$ are in the same Gleason part then there is an affine isomorphism linking $M_\phi$ and $M_\psi$, see [6, p. 143]. It follows that $M_\phi$ is weakly compact (i.e. $\sigma(M(X), M(X)^*)$) if and only if $M_\psi$ is weakly compact. Moreover if $m_\phi$ is dominant in $M_\phi$ and $m_\psi$ is dominant in $M_\psi$ then $m_\phi$ is absolutely continuous with respect to $m_\psi$. (The existence of a dominant measure in $M_\phi$ is given by [3, p. 307].)

**Lemma.** If $\phi$ is an element of $M_A$ then following are equivalent
1. $M_\phi$ is weakly compact.
2. If $u_n$ is a sequence of continuous functions on $X$ such that $1 \geq u_n \geq 0$ and $u_n \to 0$ pointwise then there is a subsequence $n_k$ and functions $v_k \in A$ such that $\text{Re } v_k \geq u_{n_k}$ and $\phi(v_k) \to 0$.
3. If $u_n$ is a sequence of continuous functions on $X$ such that $1 \geq u_n \geq 0$ and $u_n \to 0$ pointwise then there is a subsequence $n_k$ and functions $g_k \in A$ such that $|g_k| \leq e^{-u_{n_k}}$ and $\phi(g_k) \to 1$.

**Proof.** (1) $\Rightarrow$ (2) If $M_\phi$ is weakly compact and $u_n$ is a sequence as in (2) then $\sup_{\mu \in M_\phi} \mu \to 0$ (see [4]). Hence if $\varepsilon_n$ is a sequence of strictly positive numbers tending to zero then $\exists v_n \in A$ such that $\text{Re } v_n \geq u_n$ and $\phi(v_n) \leq \sup_{\mu \in M_\phi} \mu + \varepsilon_n$ (see [6, p. 82]). Clearly $\phi(v_n) \to 0$.

(2) $\Rightarrow$ (3) Write $g_k = e^{-v_k}$ and observe that $|g_k| = e^{-\text{Re } v_k} \leq e^{-u_{n_k}}$ and $\phi(g_k) = e^{-\phi(v_k)} \to 1$.

(3) $\Rightarrow$ (1) If $M_\phi$ is not weakly compact then following [4] there is a sequence of functions $u_n \in \mathcal{C}(X)$ and a sequence of measures $\mu_n \in M_\phi$ as well as $\varepsilon > 0$ such that

(i) $0 \leq u_n \leq 1$ and $u_n \to 0$ pointwise

(ii) $\int u_n d\mu_n > \varepsilon$.

Let now $g_k$ be as in (3) then

$$|\phi(g_k)| \leq \int |g_k| d\mu_{n_k} \leq \int e^{-u_{n_k}} d\mu_{n_k} \leq 1 - \frac{e - 1}{e} \int u_{n_k} d\mu_{n_k} \leq 1 - \frac{e - 1}{e} \varepsilon$$

and this contradicts $\phi(g_k) \to 1$.

**Remark.** The conditions (2) and (3) are of course related to the conditions of being enveloped and being a core measure. The dif-
ference is that the sequence \( u_n \) is supposed to be uniformly bounded.

**Corollary.** If \( A \) satisfies one of the following conditions then for all \( \phi \in M_\alpha \), \( M_\phi \) is weakly compact.

1. If \( 1 \geq u_n \geq 0 \); \( u_n \in \mathcal{C}(X) \) and \( u_n \to 0 \) pointwise then there is a subsequence \( n_k \) and \( v_k \in A \) such that \( v_k \) are uniformly bounded, \( \Re v_k \geq u_{nk} \) and \( v_k \to 0 \) on \( X \).

2. If \( 1 \geq u_n \geq 0 \); \( u_n \in \mathcal{C}(X) \) and \( u_n \to 0 \) pointwise then there is a subsequence \( n_k \) and \( g_k \in A \) such that \( |g_k| \leq e^{-u_{nk}} \) and \( g_k \to 1 \) on \( X \).

3. The D.P. property for some uniform algebras. In the following theorem we say that a measure \( \nu \) is singular to \( A \) if for all \( \phi \) and all \( \mu \in M_\phi \), the measure \( \nu \) is singular with respect to \( \mu \).

**Theorem.** A has the Dunford-Pettis property if

1. for all \( \phi \in M_\alpha \), the set \( M_\phi \) is weakly compact,
2. if \( \lambda \) is orthogonal to \( A \) and \( \lambda \) is singular to \( A \) then \( \lambda = 0 \).

**Proof.** Of course we only have to prove that \( A^* \) has the D.P. property, since it follows from the definition that a Banach space is a Dunford-Pettis space as soon as its dual is a Dunford-Pettis space. We first prove the following lemma.

**Lemma.** If \( (E_\beta)_{\beta \in B} \) is a family of Banach spaces all having the D.P. property and if

\[
\left( \sum_\beta \oplus E_\beta \right)_{\text{top}} = E = \left\{ e = (e_\beta)_{\beta \in B} \mid e_\beta \in E_\beta; \sum_\beta \| e_\beta \| = \| e \| < \infty \right\}
\]

then \( E \) has the D.P. property.

**Proof.** \( \forall \beta \) let \( P_\beta : E \to E_\beta \) be the canonical projection.

Let \( e_n \in E \) such that \( e_n \to 0 \) weakly and \( \| e_n \| \leq 1 \); \( e_n^* \in E^* \) such that \( e_n^* \to 0 \) weakly and \( \| e_n^* \| \leq 1 \); \( P_\beta e_n = e_{n,\beta} \); \( P_\beta^* e_n^* = e_{n,\beta}^* \); \( t_{n,\beta} = e_{n,\beta}(e_{n,\beta}) \).

Only a denumerable part of the numbers \( t_{n,\beta} \) can be different from zero so we can take \( B = N \). We first prove that the sum \( e_n^*(e_n) = \sum_\beta t_{n,\beta} \) converges uniformly in \( n \), i.e.

\( (*) \) for all \( \varepsilon > 0 \) there is \( N \) such that \( \forall n \) we have \( \sum_{\beta > N} | t_{n,\beta} | < \varepsilon \). If this is not the case then we start a well-known procedure. Let \( \varepsilon > 0 \) be such that \( (*) \) does not hold for this \( \varepsilon \), take \( \delta_0 > 0 \) such that \( \sum_{n=1}^{\infty} \delta_n \leq \varepsilon / 4 \). Let \( n_1 = 1, N_0 = 0, N_1 \) such that \( \sum_{\beta > N_1} \| e_{n_1,\beta} \| \leq \delta_1 \).

Since \( e_{n_1,1}, \ldots, e_{n_1,N_1} \to 0 \) weakly we can find \( n_2 \) such that for all \( n \geq n_2 \) we have \( \sum_{\beta > N_1} | e_{n,\beta}(e_{n_1,\beta}) | \leq \delta_2 \). Let now \( n_3 \geq n_2 \) be such
that $\sum_{\beta>N_1} |t_{n_{\beta}, \beta}| > \varepsilon$ and $N_2 > N_1$ such that $\sum_{\beta>N_2} \|e_{n_{\beta}, \beta}\| \leq \delta_2$. Continuing this procedure we find two strictly increasing sequences $(n_k, N_k)$ such that

1. $\sum_{\beta>N_k} \|e_{n_k, \beta}\| \leq \delta_k$
2. $\forall n \geq n_k$ the sum $\sum_{\beta=N_k}^{N_{k+1}} |e_{n, \beta}(e_{n_{k-1}, \beta})| \leq \delta_k$
3. $\sum_{\beta>N_{k+1}} |t_{n_{k}, \beta}| > \varepsilon$.

Let now

$$e^* = (\gamma_1 e^*_{1,1}; \cdots; \gamma_{N_1} e^*_{N_1,N_1}; \cdots; \gamma_{N_2} e^*_{N_2,N_2}; \cdots)$$

where $\gamma_\beta$ is such that if $N_{k-1} + 1 \leq \beta \leq N_k$ then $\gamma_\beta e^*_{n_k, \beta}(e_{n_k, \beta}) = |t_{n_k, \beta}|$. Clearly $e^* \in E^*$ and $\|e^*\| \leq 1$. For all $k \geq 2$

$$e^*(e_{n_k}) = \sum_{j=1}^{k-1} \sum_{\beta=N_{j-1}+1}^{N_j} \gamma_\beta e^*_{\beta}(e_{n_j, \beta}) + \sum_{\beta=N_{k-1}+1}^{N_k} |t_{n_k, \beta}| + \sum_{\beta>N_k} \gamma_\beta e^*_{\beta}(e_{n_k, \beta}) .$$

So

$$|e^*(e_{n_k})| \geq -\sum_{j=1}^{k-1} \delta_j + \sum_{\beta=N_{k-1}+1}^{N_k} |t_{n_k, \beta}| - \delta_k$$

$$\geq -\sum_{j=1}^{k} \delta_j + \sum_{\beta>N_{k-1}} |t_{n_k, \beta}| - 2\delta_k$$

$$\geq \varepsilon - 2\sum_{j=1}^{\infty} \delta_j \geq \varepsilon/2 .$$

But this contradicts $e_{n_k} \rightharpoonup 0$ weakly. This proves that $(*)$ is verified and hence $\lim_{n \to \infty} \sum_{\beta} t_{n, \beta} = \sum_{\beta} \lim t_{n, \beta} = 0$, since each of the $E_\beta$ has the D.P. property.

**Remark.** If $E_n = l^2_n$ (i.e. the $n$-dimensional Hilbert space) then $E = (\Sigma \oplus E_n)_1$ has the D.P. property but $E^*$ has not, because as easily seen, the space $E^*$ has a complemented subspace isometric to $l^2$, this contradicts D.P. (see [4]).

**Proof of the theorem.** For each $\pi \in \Pi$ we select $\phi_\pi \in \pi$ and $m_\pi \in M_\phi$ dominant. By [6 p. 144] all $m_\pi$ are mutually singular. Select now probability measures $(m_\phi)_{\phi \in B}$ such that $\{m_\pi | \pi \in \Pi\} \cup \{m_\phi | _{\phi \in B}\}$ is a maximal family of mutually singular measures. (This can be done using Zorn's lemma.) An application of the Radon-Nikodym theorem yields:

$$M(X) = \mathcal{C}(X)^* = (\sum_{\pi \in \Pi} \phi \in B)_{l_1} .$$
For each \( \pi \) define \( N_\pi \) as the set \( \{ \pi \in L^1(\mu) \mid \mu \perp A \} \). The abstract F. and M. Riesz theorem [6] and hypothesis 2 give that 

\[
A^1 = \left( \sum_{\pi \in \mathcal{H}} \bigoplus N_\pi \right)_{1_1}
\]

and hence 

\[
A^* = \left( \sum_{\pi \in \mathcal{H}} \bigoplus L^1(\mu_\beta)/N_\pi \right)_{1_1} + \left( \sum_{\beta \in \mathcal{B}} \bigoplus L^1(\mu_\beta) \right)_{1_1}.
\]

In [2] and [1] it is proved that the spaces \( L^1(\mu_\beta)/N_\pi \) have the Dunford-Pettis property. By the preceding lemma and Grothendieck's result that an \( L^1 \) space is a Dunford-Pettis space we have that \( A^* \) has the D.P. property.

**Remark.** (1) If \( D = \{ z \mid |z| < 1 \} \) and \( A \) is the so-called disc-algebra i.e. \( A = \{ f \mid f \text{ analytic on } D, \text{ continuous on } \overline{D} \} \) then \( A \) satisfies all requirements hence \( A \) and \( A^* \) have the D.P. property.

(2) If \( K \) is a compact set which is finitely connected then by Wilken's theorem \( R(K) \) satisfies hypothesis 2 and by [6, p. 145, paragraph 3], \( R(K) \) also satisfies hypothesis 1. Consequently \( R(K) \) as well as \( R(K)^* \) have the Dunford-Pettis property.

**References**


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