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**THE DUNFORD-PETTIS PROPERTY FOR CERTAIN UNIFORM  
ALGEBRAS**

FREDDY DELBAEN

## THE DUNFORD-PETTIS PROPERTY FOR CERTAIN UNIFORM ALGEBRAS

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**A Banach space  $B$  has the Dunford-Pettis property if  $x_n^*(x_n) \rightarrow 0$  whenever  $x_n \rightarrow 0$  weakly and the sequence  $x_n^*$  tends to zero weakly in  $B^*$  (i.e.  $\sigma(B^*, B^{**})$ ). Suppose now that  $A$  is a uniform algebra on a compact space  $X$ . If  $\phi$  is a nonzero multiplicative linear functional on  $A$  then  $M_\phi$  is the set of positive representing measures of  $\phi$ . If  $A$  is such that a singular measure which is orthogonal to  $A$  must necessarily be zero and if all  $M_\phi$  are weakly compact sets then the algebra  $A$  as well as its dual have the Dunford-Pettis property.**

The idea of the proof is that  $A^*$  the dual of  $A$  can be decomposed into components for which the results of Chaumat [1] and Cnop-Delbaen [2] can be applied. The fact that an  $l_1$  sum of Dunford-Pettis spaces is also a Dunford-Pettis space then gives the result. In paragraph two some conditions ensuring the weak compactness of  $M_\phi$  are given. These conditions are related to those used in the definition of core and enveloping measures (see [6]).

**1. Notation and preliminaries.**  $X$  will be a compact space,  $A \subset \mathcal{C}(X)$  a closed subalgebra of the space of continuous complex-valued functions on  $X$ . The algebra  $A$  is supposed to contain the constants and to separate the points of  $X$ . The spectrum  $M_A$  is the set of all nonzero multiplicative linear functionals on  $A$ . If  $\phi \in M_A$  then  $M_\phi$  is the set of all positive measures on  $X$  representing  $\phi$ , i.e.

$$M_\phi = \left\{ \mu \in M(X) \mid \mu \geq 0 \text{ and } \forall f \in A \text{ we have } \phi(f) = \int f d\mu \right\}.$$

As well known  $M_\phi$  is a convex set, compact for the topology  $\sigma(M(X), \mathcal{C}(X))$ . We say that two multiplicative linear forms  $\phi$  and  $\psi$  belong to the same Gleason part if  $\|\phi - \psi\| < 2$  in  $A^*$ , the dual of  $A$ . It is well known that being in the same Gleason part is an equivalence relation and hence  $M_A = \bigcup_{\pi \in \Pi} \pi$  where  $\Pi$  is the set of all Gleason equivalence classes. For more details and any unexplained notion on uniform algebras we refer to [6].

If  $E$  is a Banach space then  $E$  has the Dunford-Pettis property if  $e_n^*(e_n) \rightarrow 0$  whenever  $e_n \rightarrow 0$  weakly and  $e_n^* \rightarrow 0$  weakly (i.e.  $\sigma(E^*, E^{**})$ ).

For more details and properties of such spaces see Grothendieck

[4] or [5], where it is also proved that  $L^1$  spaces and  $\mathcal{E}(X)$  spaces have the Dunford-Pettis property.

2. **Weak compactness of  $M_\phi$ .** We investigate under what conditions  $M_\phi$  is weakly compact. First we remark that if  $\psi$  and  $\phi$  are in the same Gleason part then there is an affine isomorphism linking  $M_\phi$  and  $M_\psi$ , see [6, p. 143]. It follows that  $M_\phi$  is weakly compact (i.e.  $\sigma(M(X), M(X)^*)$ ) if and only if  $M_\psi$  is weakly compact. Moreover if  $m_\phi$  is dominant in  $M_\phi$  and  $m_\psi$  is dominant in  $M_\psi$  then  $m_\phi$  is absolutely continuous with respect to  $m_\psi$ . (The existence of a dominant measure in  $M_\phi$  is given by [3, p. 307].)

LEMMA. *If  $\phi$  is an element of  $M_A$  then following are equivalent*

1.  $M_\phi$  is weakly compact.
2. If  $u_n$  is a sequence of continuous functions on  $X$  such that  $1 \geq u_n \geq 0$  and  $u_n \rightarrow 0$  pointwise then there is a subsequence  $n_k$  and functions  $v_k \in A$  such that  $\operatorname{Re} v_k \geq u_{n_k}$  and  $\phi(v_k) \rightarrow 0$ .
3. If  $u_n$  is a sequence of continuous functions on  $X$  such that  $1 \geq u_n \geq 0$  and  $u_n \rightarrow 0$  pointwise then there is a subsequence  $n_k$  and functions  $g_k \in A$  such that  $|g_k| \leq e^{-u_{n_k}}$  and  $\phi(g_k) \rightarrow 1$ .

*Proof.* (1)  $\Rightarrow$  (2) If  $M_\phi$  is weakly compact and  $u_n$  is a sequence as in (2) then  $\sup_{\mu \in M_\phi} \int u_n d\mu \rightarrow 0$  (see [4]). Hence if  $\varepsilon_n$  is a sequence of strictly positive numbers tending to zero then  $\exists v_n \in A$  such that  $\operatorname{Re} v_n \geq u_n$  and  $\phi(v_n) \leq \sup_{\mu \in M_\phi} \int u_n d\mu + \varepsilon_n$  (see [6 p. 82]). Clearly  $\phi(v_n) \rightarrow 0$ .

(2)  $\Rightarrow$  (3) Write  $g_k = e^{-v_k}$  and observe that  $|g_k| = e^{-\operatorname{Re} v_k} \leq e^{-u_{n_k}}$  and  $\phi(g_k) = e^{-\phi(v_k)} \rightarrow 1$ .

(3)  $\Rightarrow$  (1) If  $M_\phi$  is not weakly compact then following [4] there is a sequence of functions  $u_n \in \mathcal{E}(X)$  and a sequence of measures  $\mu_n \in M_\phi$  as well as  $\varepsilon > 0$  such that

(i)  $0 \leq u_n \leq 1$  and  $u_n \rightarrow 0$  pointwise

(ii)  $\int u_n d\mu_n > \varepsilon$ .

Let now  $g_k$  be as in (3) then

$$|\phi(g_k)| \leq \int |g_k| d\mu_{n_k} \leq \int e^{-u_{n_k}} d\mu_{n_k} \leq 1 - \frac{e-1}{e} \int u_{n_k} d\mu_{n_k} \leq 1 - \frac{e-1}{e} \varepsilon$$

and this contradicts  $\phi(g_k) \rightarrow 1$ .

REMARK. The conditions (2) and (3) are of course related to the conditions of being enveloped and being a core measure. The dif-

ference is that the sequence  $u_n$  is supposed to be uniformly bounded.

**COROLLARY.** *If  $A$  satisfies one of the following conditions then for all  $\phi \in M_A$ ,  $M_\phi$  is weakly compact.*

(1) *If  $1 \geq u_n \geq 0$ ;  $u_n \in \mathcal{C}(X)$  and  $u_n \rightarrow 0$  pointwise then there is a subsequence  $n_k$  and  $v_k \in A$  such that  $v_k$  are uniformly bounded,  $\operatorname{Re} v_k \geq u_{n_k}$  and  $v_k \rightarrow 0$  on  $X$ .*

(2) *If  $1 \geq u_n \geq 0$ ;  $u_n \in \mathcal{C}(X)$  and  $u_n \rightarrow 0$  pointwise then there is a subsequence  $n_k$  and  $g_k \in A$  such that  $|g_k| \leq e^{-u_{n_k}}$  and  $g_k \rightarrow 1$  on  $X$ .*

**3. The D.P. property for some uniform algebras.** In the following theorem we say that a measure  $\nu$  is singular to  $A$  if for all  $\phi$  and all  $\mu \in M_\phi$ , the measure  $\nu$  is singular with respect to  $\mu$ .

**THEOREM.**  *$A$  has the Dunford-Pettis property if*

(1) *for all  $\phi \in M_A$ , the set  $M_\phi$  is weakly compact,*

(2) *if  $\lambda$  is orthogonal to  $A$  and  $\lambda$  is singular to  $A$  then  $\lambda = 0$ .*

*Proof.* Of course we only have to prove that  $A^*$  has the D.P. property, since it follows from the definition that a Banach space is a Dunford-Pettis space as soon as its dual is a Dunford-Pettis space. We first prove the following lemma.

**LEMMA.** *If  $(E_\beta)_{\beta \in B}$  is a family of Banach spaces all having the D.P. property and if*

$$\left( \sum_{\beta} \oplus E_{\beta} \right)_{l_1} = E = \left\{ e = (e_{\beta})_{\beta \in B} \mid e_{\beta} \in E_{\beta}; \sum_{\beta} \|e_{\beta}\| = \|e\| < \infty \right\}$$

*then  $E$  has the D.P. property.*

*Proof.*  $\forall \beta$  let  $P_{\beta} : E \rightarrow E_{\beta}$  be the canonical projection.

Let  $e_n \in E$  such that  $e_n \rightarrow 0$  weakly and  $\|e_n\| \leq 1$ ;  $e_n^* \in E^*$  such that  $e_n^* \rightarrow 0$  weakly and  $\|e_n^*\| \leq 1$ ;  $P_{\beta} e_n = e_{n,\beta}$ ;  $P_{\beta}^* e_n^* = e_{n,\beta}^*$ ;  $t_{n,\beta} = e_{n,\beta}^*(e_{n,\beta})$ .

Only a denumerable part of the numbers  $t_{n,\beta}$  can be different from zero so we can take  $B = N$ . We first prove that the sum  $e_n^*(e_n) = \sum_{\beta} t_{n,\beta}$  converges uniformly in  $n$ , i.e.

(\*) for all  $\varepsilon > 0$  there is  $N$  such that  $\forall n$  we have  $\sum_{\beta > N} |t_{n,\beta}| < \varepsilon$ . If this is not the case then we start a well-known procedure. Let  $\varepsilon > 0$  be such that (\*) does not hold for this  $\varepsilon$ , take  $\delta_n > 0$  such that  $\sum_{n=1}^{\infty} \delta_n \leq \varepsilon/4$ . Let  $n_1 = 1$ ,  $N_0 = 0$ ,  $N_1$  such that  $\sum_{\beta > N_1} \|e_{n_1,\beta}\| \leq \delta_1$ .

Since  $e_{n_1,1}, \dots, e_{n_1,N_1} \rightarrow 0$  weakly we can find  $\bar{n}_2$  such that for all  $n \geq \bar{n}_2 \geq n_1$  we have  $\sum_{\beta=1}^{\infty} |e_{n,j}^*(e_{n_1,j})| \leq \delta_2$ . Let now  $n_2 \geq \bar{n}_2$  be such

that  $\sum_{\beta > N_1} |t_{n_2, \beta}| > \varepsilon$  and  $N_2 > N_1$  such that  $\sum_{\beta > N_2} \|e_{n_2, \beta}\| \leq \delta_2$ . Continuing this procedure we find two strictly increasing sequences  $(n_k, N_k)$  such that

- (1)  $\sum_{\beta > N_k} \|e_{n_k, \beta}\| \leq \delta_k$
- (2)  $\forall n \geq n_k$  the sum  $\sum_{\beta=1}^{N_{k-1}} |e_{n, j}^*(e_{n_{k-1}, \beta})| \leq \delta_k$
- (3)  $\sum_{\beta > N_{k-1}} |t_{n_k, \beta}| > \varepsilon$ .

Let now

$$e^* = (\gamma_1 e_{1,1}^*; \dots; \gamma_{N_1} e_{1, N_1}^*; \gamma_{N_1+1} e_{n_2, N_1+1}^*; \dots; \gamma_{N_2} e_{n_2, N_2}^*; \gamma_{N_2+1} e_{n_3, N_2+1}^*; \dots)$$

where  $\gamma_\beta$  is such that if  $N_{k-1} + 1 \leq \beta \leq N_k$  then  $\gamma_\beta e_{n_k, \beta}^*(e_{n_k, \beta}) = |t_{n_k, \beta}|$ . Clearly  $e^* \in E^*$  and  $\|e^*\| \leq 1$ . For all  $k \geq 2$

$$\begin{aligned} e^*(e_{n_k}) &= \sum_{j=1}^{k-1} \sum_{\beta=N_{j-1}+1}^{N_j} \gamma_\beta e_{n_j, \beta}^*(e_{n_j, \beta}) \\ &\quad + \sum_{\beta=N_{k-1}+1}^{N_k} |t_{n_k, \beta}| \\ &\quad + \sum_{\beta > N_k} \gamma_\beta e_\beta^*(e_{n_k, \beta}). \end{aligned}$$

So

$$\begin{aligned} |e^*(e_{n_k})| &\geq - \sum_{j=1}^{k-1} \delta_j + \sum_{\beta=N_{k-1}+1}^{N_k} |t_{n_k, \beta}| - \delta_k \\ &\geq - \sum_{j=1}^k \delta_j + \sum_{\beta > N_{k-1}} |t_{n_k, \beta}| - 2\delta_k \\ &\geq \varepsilon - 2 \sum_{j=1}^{\infty} \delta_j \geq \varepsilon/2. \end{aligned}$$

But this contradicts  $e_{n_k} \rightarrow 0$  weakly. This proves that (\*) is verified and hence  $\lim_{n \rightarrow \infty} \sum_{\beta} t_{n, \beta} = \sum_{\beta} \lim t_{n, \beta} = 0$ , since each of the  $E_\beta$  has the D.P. property.

REMARK. If  $E_n = l_2^n$  (i.e. the  $n$ -dimensional Hilbert space) then  $E = (\Sigma \oplus E_n)_{l_1}$  has the D.P. property but  $E^*$  has not, because as easily seen, the space  $E^*$  has a complemented subspace isometric to  $l_2$ , this contradicts D.P. (see [4]).

*Proof of the theorem.* For each  $\pi \in \Pi$  we select  $\phi_\pi \in \pi$  and  $m_\pi \in M_\phi$  dominant. By [6 p. 144] all  $m_\pi$  are mutually singular. Select now probability measures  $(m_\beta)_{\beta \in B}$  such that  $\{m_\pi \mid \pi \in \Pi\} \cup \{m_\beta \mid \beta \in B\}$  is a maximal family of mutually singular measures. (This can be done using Zorn's lemma.) An application of the Radon-Nikodym theorem yields:

$$M(X) = \mathcal{C}(X)^* = \left( \sum_{\alpha \in \Pi \cup B} \oplus L^1(m_\alpha) \right)_{l_1}.$$

For each  $\pi$  define  $N_\pi$  as the set  $\{\pi \in L^1(m_\pi) \mid \mu \perp A\}$ . The abstract F. and M. Riesz theorem [6] and hypothesis 2 give that

$$A^\perp = \left( \sum_{\pi \in I} \oplus N_\pi \right)_{l_1}$$

and hence

$$A^* = \left( \sum_{\pi \in I} \oplus L^1(m_\pi)/N_\pi \right)_{l_1} \oplus \left( \sum_{\beta \in B} \oplus L^1(m_\beta) \right)_{l_1}.$$

In [2] and [1] it is proved that the spaces  $L^1(m_\pi)/N_\pi$  have the Dunford-Pettis property. By the preceding lemma and Grothendieck's result that an  $L^1$  space is a Dunford-Pettis space we have that  $A^*$  has the D.P. property.

REMARK. (1) If  $D = \{z \mid |z| < 1\}$  and  $A$  is the so-called disc-algebra i.e.  $A = \{f \mid f \text{ analytic on } D, \text{ continuous on } \bar{D}\}$  then  $A$  satisfies all requirements hence  $A$  and  $A^*$  have the D.P. property.

(2) If  $K$  is a compact set which is finitely connected then by Wilken's theorem  $R(K)$  satisfies hypothesis 2 and by [6, p. 145, paragraph 3],  $R(K)$  also satisfies hypothesis 1. Consequently  $R(K)$  as well as  $R(K)^*$  have the Dunford-Pettis property.

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# Pacific Journal of Mathematics

Vol. 65, No. 1

September, 1976

David Lee Armacost, <i>Compactly cogenerated LCA groups</i> . . . . .	1
Sun Man Chang, <i>On continuous image averaging of probability measures</i> . . . . .	13
J. Chidambaraswamy, <i>Generalized Dedekind <math>\psi</math>-functions with respect to a polynomial. II</i> . . . . .	19
Freddy Delbaen, <i>The Dunford-Pettis property for certain uniform algebras</i> . . . . .	29
Robert Benjamin Feinberg, <i>Faithful distributive modules over incidence algebras</i> . . . . .	35
Paul Froeschl, <i>Chained rings</i> . . . . .	47
John Brady Garnett and Anthony G. O'Farrell, <i>Sobolev approximation by a sum of subalgebras on the circle</i> . . . . .	55
Hugh M. Hilden, José M. Montesinos and Thomas Lusk Thickstun, <i>Closed oriented 3-manifolds as 3-fold branched coverings of <math>S^3</math> of special type</i> . . . . .	65
Atsushi Inoue, <i>On a class of unbounded operator algebras</i> . . . . .	77
Peter Kleinschmidt, <i>On facets with non-arbitrary shapes</i> . . . . .	97
Narendrakumar Ramanlal Ladhawala, <i>Absolute summability of Walsh-Fourier series</i> . . . . .	103
Howard Wilson Lambert, <i>Links which are unknottable by maps</i> . . . . .	109
Kyung Bai Lee, <i>On certain <math>g</math>-first countable spaces</i> . . . . .	113
Richard Ira Loebel, <i>A Hahn decomposition for linear maps</i> . . . . .	119
Moshe Marcus and Victor Julius Mizel, <i>A characterization of non-linear functionals on <math>W_1^p</math> possessing autonomous kernels. I</i> . . . . .	135
James Miller, <i>Subordinating factor sequences and convex functions of several variables</i> . . . . .	159
Keith Pierce, <i>Amalgamated sums of abelian <math>l</math>-groups</i> . . . . .	167
Jonathan Rosenberg, <i>The <math>C^*</math>-algebras of some real and <math>p</math>-adic solvable groups</i> . . . . .	175
Hugo Rossi and Michele Vergne, <i>Group representations on Hilbert spaces defined in terms of <math>\partial_b</math>-cohomology on the Silov boundary of a Siegel domain</i> . . . . .	193
Mary Elizabeth Schaps, <i>Nonsingular deformations of a determinantal scheme</i> . . . . .	209
S. R. Singh, <i>Some convergence properties of the Bubnov-Galerkin method</i> . . . . .	217
Peggy Strait, <i>Level crossing probabilities for a multi-parameter Brownian process</i> . . . . .	223
Robert M. Tardiff, <i>Topologies for probabilistic metric spaces</i> . . . . .	233
Benjamin Baxter Wells, Jr., <i>Rearrangements of functions on the ring of integers of a <math>p</math>-series field</i> . . . . .	253
Robert Francis Wheeler, <i>Well-behaved and totally bounded approximate identities for <math>C_0(X)</math></i> . . . . .	261
Delores Arletta Williams, <i>Gauss sums and integral quadratic forms over local fields of characteristic 2</i> . . . . .	271
John Yuan, <i>On the construction of one-parameter semigroups in topological semigroups</i> . . . . .	285