Pacific Journal of Mathematics

FAITHFUL DISTRIBUTIVE MODULES OVER INCIDENCE ALGEBRAS

ROBERT BENJAMIN FEINBERG

Vol. 65, No. 1 September 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 1, 1976

FAITHFUL DISTRIBUTIVE MODULES OVER INCIDENCE ALGEBRAS

ROBERT B. FEINBERG

Let Q be a lower finite quasi-ordered set and let $I(Q)$ be the incidence algebra of Q over a field K. In this paper we determine all faithful distributive modules over $I(Q)$ and relate the result to the structure of the outer automorphism group of the algebra. In the case when Q is finite we also determine all left ideals L of $I(Q)$ such that ${}_{I(Q)}I(Q)/L$ is a faithful distributive module over $I(Q)$.

In the abstract characterization of incidence algebras it is necessary to consider the existence and structure of faithful distributive modules over them. In §1 we determine up to isomorphism all faithful distributive modules over an incidence algebra. In §2 we relate this result to the structure of the outer automorphism group of the algebra. In §3 we consider the incidence algebra of a finite quasi-ordered set and determine for it all left ideals L with the property that the quotient module determined by L is a faithful distributive module over the algebra.

Let Q be a locally finite quasi-ordered (q.o.) set, i.e., Q has a relation r which is reflexive and transitive and for which every segment $[x, y] = \{z \in Q; xrzry\}$ is finite. The *incidence algebra* $I(Q)$ of Q over a field K is the (associative unital) algebra of functions $f: Q \times Q \to K$ with the property that $f(x, y) \neq 0 \Rightarrow xry$, under the product

$$f^*g(x, y) = \sum_{xrzry} f(x, z)g(z, y) .$$

The unit δ of $I(Q)$ is defined by

$$\delta(x, y) = 1 \quad \text{if} \quad x = y$$
$$= 0 \quad \text{if} \quad x \neq y .$$

For $xry \in Q$, define $\delta_{xy} \in I(Q)$ by

$$\delta_{xy}(u, v) = 1 \quad \text{if} \quad x = u, y = v$$
$$= 0 \quad \text{otherwise.}$$

and set $e_x = \delta_{xx}$.

The *standard topology* on $I(Q)$ is defined by stipulating that a net $\{f_\lambda\}_{\lambda \in \theta} \subseteq I(Q)$ converges to $f \in I(Q)$ iff $\forall xry \in Q$, $f_\lambda(x, y) = f(x, y)$ eventually, i.e., $\exists \mu \in \theta$ such that $\forall \mu' \geqq \mu$, $f_{\mu'}(x, y) = f(x, y)$. It is easy to verify that $I(Q)$ equipped with the above topology is a topological algebra, when the field K has the discrete topology. When Q is finite, the standard topology on $I(Q)$ reduces to the discrete topology.

We now introduce the following equivalence relation on Q: $x \sim y$ iff xry and yrx. Let $\bar{x}$ denote the $\sim$ equivalence class of $x \in Q$ and let $\hat{Q}$ denote the collection of all equivalence classes. Then $\hat{Q}$ is a partially ordered (p.o.) set under the relation $\leqq$ defined by $\bar{x} \leqq \bar{y}$ iff xry. Using the these notions it is convenient at this time to present the following two technical results on $I(Q)$, in the case when Q is a finite q.o. set.

LEMMA 0.1 ([2, § 2], [5, § 1]). The maximal ideals of $I(Q)$ are all subspaces of the form $J_{\bar{x}} = \{f \in I(Q);\ f(x_1, x_2) = 0\ \ \forall x_1, x_2 \in Q$ with $\bar{x}_1 = \bar{x}_2 = \bar{x}\}$, where $\bar{x}$ is some element of $\hat{Q}$. Further, assuming $\bar{x} = \{x_i\}_{i=1}^n$ and letting K_n denote the (simple) algebra of $n \times n$ matrices over the field K, we have that the map $\pi_{\bar{x}}: I(Q) \to K_n$ defined by $(\pi_{\bar{x}} f)_{uv} = f(x_u, x_v)$, $1 \leqq u, v \leqq n$, is an epimorphism with kernel $J_{\bar{x}}$, so that $I(Q)/J_{\bar{x}} \simeq K_n$.

We will use $|S|$ to denote the cardinality of a set S.

LEMMA 0.2 ([2, § 1]). *An element f of $I(Q)$ is invertible in $I(Q)$ iff $\pi_{\bar{x}}(f)$ is invertible in $K_{|\bar{x}|}$, $\forall \bar{x} \in Q$.*

Now let A be a topological algebra and let M be a (unital left) module over A. If the action of A on M is continuous in A when M has the discrete topology, we shall say that M is *topologically compatible* (with A). If M has a distributive lattice of submodules, and is faithful and topologically compatible, we shall say that M is a *faithful distributive module* (over A). The following three results on faithful distributive modules over incidence algebras are from [5, § 2].

LEMMA 0.3. *Let Q be locally finite. Then $I(Q)$ has a faithful distributive module iff Q is lower finite.*

We will explicitly construct such modules in the course of proving Theorem 1.1, to follow.

LEMMA 0.4. *Let M be a topologically compatible, faithful module over $I(Q)$. Then M is faithful distributive iff $e_x \cdot M$ is 1-dimensional $\forall x \in Q$. In particular, when Q is a finite quasi-ordered set of cardinality n, a faithful module M is faithful distributive iff* *dim* $M = n$.

LEMMA 0.5. *Let M be a faithful distributive module over $I(Q)$. For each $z \in Q$, chose $m_z \in e_z \cdot M \backslash \{0\}$. Then $B = \{m_z\}_{z \in Q}$ is a basis for M, and for $wrx \in Q$, $\delta_{wx} \cdot m_x = \alpha(w, x) m_w$, for some $\alpha(w, x) \in K \backslash \{0\}$.*

1. Faithful distributive modules. In this section we determine up to isomorphism all faithful distributive mondules over an incidence algebra. We begin with some new notions. An element $f \in I(Q)$ is *multiplicative* if (i) $f(x, y) \neq 0 \forall xry \in Q$ (ii) $f(x, y)f(y, z) = f(x, z) \forall xryrz \in Q$. It is easy to show that the collection of multiplicative elements forms an abelian group under the pointwise product, $fg(x, y) = f(x, y)g(x, y)$. We shall denote this group by $Z(Q)$. The identity ζ is given by $\zeta(x, y) = 1$, $\forall xry$ in Q and the inverse $\tilde{f}$ of $f \in Z(Q)$ is given by $\tilde{f}(x, y) = [f(x, y)]^{-1}$. It follows from (i) and (ii) above that for all $f \in Z(Q)$, $f(x, x) = 1 \forall x \in Q$.

Given any $\alpha: Q \rightarrow K \backslash \{0\}$, we define $d_\alpha \in Z(Q)$ by $d_\alpha(x, y) = \alpha(x)\alpha(y)^{-1}$. Let $B(Q) = \{d_\alpha \in Z(Q)\}$. Then $B(Q)$ is a (necessarily normal) subgroup of $Z(Q)$, and thus we may construct the quotient group $H(Q) = Z(Q)/B(Q)$. It may be shown, for example that $H(Q) \simeq K$ for the quasi-ordered set $\{x_1, x_2, x_3, x_4\}$, where $x_1 r x_3$, $x_1 r x_4$, $x_2 r x_3$, $x_2 r x_4$. On the other hand, $H(Q) = 0$ if Q has an element z comparable to all others. For if $\sigma \in Z(Q)$, we may define $\alpha: Q \rightarrow K \backslash \{0\}$ by

$$\begin{aligned} \alpha(w) = 1\ , &\quad \text{if} \quad w = z \\ \sigma(z, w)\ , &\quad \text{if} \quad zrw \\ \sigma(w, z)^{-1}\ , &\quad \text{if} \quad wrz\ . \end{aligned}$$

Then α is well-defined and $\sigma = d_\alpha$ as may be verified, so that $\sigma \in B(Q)$.

We can now present our main result.

THEOREM 1.1. *If Q is a lower finite* q.o. *set, then there is a bijective correspondence between the elements of $H(Q)$ and the isomorphism classes of faithful distributive modules over $I(Q)$.*

Proof. For any $\nu \in Z(Q)$, let M_ν be the K-vector space with basis $\{z\}_{z \in Q}$. For $f \in I(Q)$ and $x \in Q$ (considered as a basis element of M_ν), set

$$f \cdot_\nu x = \sum_{wrx} f(w, x)\nu(w, x) w\ .$$

Then $\cdot_\nu$ is well-defined, by lower finiteness of Q, and the fact that ν is multiplicative implies that for all $g \in I(Q)$, $(f * g) \cdot_\nu x = f \cdot_\nu (g \cdot_\nu x)$. Now extend by linearity to define an action $\cdot_\nu$ of $I(Q)$ on M_ν. It is readily verified that this action makes M_ν a faithful module over $I(Q)$. Further, for each $x \in Q$, $e_x \cdot_\nu M_\nu = Kx$, so that $e_x \cdot_\nu M_\nu$ is 1-dimensional. It is easy to see that M_ν is topologically compatible with $I(Q)$. Hence by Lemma 0.4, M_ν is a faithful distributive module over $I(Q)$.

In the case when $\nu = \zeta$ and Q is finite, of cardinality n, M_ν affords the canonical representation of $I(Q)$ as an algebra of $n \times n$ matrices over K. See [5, §2] or [6] for details.

Suppose now that $\nu, \tau \in Z(Q)$ and that $\tau B(Q) = \nu B(Q)$. We show that M_ν and M_τ are isomorphic $I(Q)$-modules. Note that $\tau\bar{\nu} \in B(Q)$, so that $\exists \alpha\colon Q \to K \backslash \{0\}$ such that $\tau\bar{\nu} = d_\alpha$, i.e., for all $xry \in Q$, $\tau(x, y)\nu(x, y)^{-1} = d_\alpha(x, y) = \alpha(x)\alpha(y)^{-1}$, or $\tau(x, y)\alpha(y) = \alpha(x)\nu(x, y)$. Define $\varphi\colon M_\nu \to M_\tau$ by setting $\varphi(x) = \alpha(x)x$, for $x \in Q$, and extend by linearity to a map on M. Then φ is a linear bijection. Note also that for $f \in I(Q)$ and $x \in M_\nu$,

$$\begin{aligned}\varphi(f_\nu \cdot x) &= \varphi \sum_{wrx} f(w, x)\nu(w, x)w \\ &= \sum_{wrx} f(w, x)\nu(w, x)\alpha(w)w \;,\end{aligned}$$

while

$$f \cdot_\tau \varphi(x) = f \cdot_\tau \alpha(x)x = \sum_{wrx} f(w, x)\tau(w, x)\alpha(x)w \;.$$

It follows that $\varphi(f \cdot_\nu x) = f \cdot_\tau \varphi(x)$, so that φ preserves module action as well. Hence φ is a module isomorphism.

Conversely, suppose that $\nu, \tau \in Z(Q)$ and that M_ν and M_τ are isomorphic $I(Q)$-modules. We show that $\tau B(Q) = \nu B(Q)$. Let φ: $M_\nu \to M_\tau$ be a module isomorphism. Since ν is multiplicative, $\nu(x, x) = 1$, $\forall x \in Q$. Hence $\varphi(x) = \varphi[\nu(x, x)x] = \varphi(e_x \cdot_\tau x) = e_x \cdot_\tau \varphi(x)$. This implies that $\varphi(x) = \alpha(x)x$, for some $\alpha(x) \in K \backslash \{0\}$. Now for $wrx \in Q$, $\varphi(\delta_{wx} \cdot_\nu x) = \varphi[\nu(w, x)w] = \nu(w, x)\varphi(w) = \nu(w, x)\alpha(w)w$, while also

$$\varphi(\delta_{wx} \cdot_\nu x) = \delta_{wx} \cdot_\tau \varphi(x) = \delta_{wx} \cdot_\tau \alpha(x)x = \tau(w, x)\alpha(x)w \;.$$

This implies that $\nu(w, x)\alpha(w) = \tau(w, x)\alpha(x)$, or equivalently, $\tau\bar{\nu}(w, x) = d_\alpha(w, x)$. Hence $\tau\bar{\nu} = d_\alpha \in B(Q)$, so that $\tau B(Q) = \nu B(Q)$.

Now define $\xi\colon H(Q) \to$ {isomorphism classes of faithful distributive $I(Q)$-modules} by $\xi[\nu B(Q)] =$ isomorphism class of M_ν. The above results show that ξ is well-defined and 1-1. We show that ξ is onto. Let M be any faithful distributive module over $I(Q)$. For each $z \in Q$, choose $m_z \in e_z \cdot M \backslash \{0\}$. Then by Lemma 0.5, $B = \{m_z\}_{z \in Q}$ is a basis for M, and for wrx in Q, $\delta_{wx} \cdot m_x = \alpha(w, x)m_w$, some $\alpha(w, x) \in K \backslash \{0\}$. For $wrxry \in Q$ we have $\alpha(w, y)m_w = \delta_{wy} \cdot m_y = \delta_{wx} * \delta_{xy} \cdot m_y = \delta_{wx}\alpha(x, y)m_x = \alpha(w, x)\alpha(x, y)m_w$. Hence, α is multiplicative, considered as an element of $I(Q)$. It follows readily that M and M_α are isomorphic as $I(Q)$-modules, via the map sending $m_x \in M$ to $x \in M_\alpha$. Thus $\xi[\alpha B(Q)]$ is the module isomorphism class of M, showing that ξ is onto.

COROLLARY 1.1.1. *Let Q be a lower finite* q.o. *set. Then $I(Q)$ has a unique isomorphism class of faithful distributive modules iff $H(Q) = 0$.*

These results may be dualized in a straightforward way to the consideration of isomorphism classes of *cofaithful right comodules* over the *incidence coalgebra* $C(Q)$. See [5] for more details on these notions.

As a simple application of the results in this section, let Q_1 be the quasi-ordered set $\{x_i\}_{i=1}^n$, where $x_i r x_j$, $1 \leqq i \leqq j \leqq n$. Then $I(Q_1)$ is isomorphic to T_n, the algebra of $n \times n$ upper triangular matrices over K, and the faithful distributive modules over $I(Q)$ are exactly the faithful modules of dimension n, by Lemma 0.4. It then follows directly from Corollary 1.1.1 that $I(Q_1) \simeq T_n$ has a unique isomorphism class of faithful modules of dimension n. Note that each such module affords the canonical n-dimensional representation of T_n. As another application, let Q_2 be the quasi-ordered set $\{y_i\}_{i=1}^n$, where $y_i r y_j$, $1 \leqq i, j \leqq n$. Then $I(Q_2)$ is isomorphic to K_n, the algebra of $n \times n$ matrices over K, and the faithful distributive modules over $I(Q_2)$ are exactly the faithful irreducible modules. It then follows directly from Corollary 1.1.1 that $I(Q_2) \simeq K_n$ has a unique isomorphism class of faithful irreducible (= primitive) modules.

2. **Automorphisms.** In this section we study the automorphisms of an incidence algebra, relating our results to those of the last section. We restrict ourselves to topological automorphisms of $I(Q)$, i.e., automorphisms which preserve topological structure, and we let Aut $I(Q)$ denote the group of all such automorphisms of $I(Q)$, where Q is a locally finite q.o. set.

We begin by discussing three classes of (topological) automorphisms of $I(Q)$.

1. For $f \in I(Q)$, f invertible, the *inner automorphism* of $I(Q)$ with respect to f is given by $I_f(g) = f^* g^* f^{-1}$. Let Inn $I(Q)$ denote the group of all such automorphisms. Then Inn $I(Q)$ is a normal subgroup of Aut I(Q).

2. For $\nu \in I(Q)$, ν multiplicative, there is an automorphism L_ν of $I(Q)$ defined by $L_\nu(f) = \nu f$. Let $\mathscr{L}[I(Q)]$ denote the group of all such automorphisms. Then Inn $I(Q) \cap \mathscr{L}[I(Q)] = \{L_\alpha \in [I(Q)]; \alpha \in B(Q)\}$

as is shown in [1].

3. Let $\operatorname{Aut} Q$ denote the automorphism group of Q, i.e., the collection of order-preserving bijections. For $\kappa \in \operatorname{Aut} Q$ there is an automorphism C_κ of $I(Q)$ defined by $C_\kappa f(x, y) = f(\kappa x, \kappa y)$.

We now examine the structure of $\operatorname{Aut} Q$ more closely. For $x \in Q$, let $\bar{x}$ be its $\sim$ equivalence class, as defined in § 0, and let $\hat{Q}$ denote the p.o. set of equivalence classes of Q. It is easy to see that there is a canonical embedding of $\operatorname{Aut} Q$ into $\operatorname{Aut} \hat{Q}$ given by $\varphi \to \hat{\varphi}$, where $\hat{\varphi}(\bar{x}) = \overline{\varphi x}$. Further, the image of $\operatorname{Aut} Q$ under this embedding, denoted $\overline{\operatorname{Aut} Q}$, consists of all $\psi \in \operatorname{Aut} \hat{Q}$ such that $|\psi(\bar{x})| = |\bar{x}|$, $\forall \bar{x} \in Q$.

We can now state the following result, which completely describes the structure of Aut I(Q).

THEOREM 2.1. *Assume* $\theta \in \operatorname{Aut} I(Q)$, *where* Q *is locally finite. Then* $\theta = I_f \circ L_\nu \circ C_\kappa$, *for suitably chosen* $I_f \in \operatorname{Inn} I(Q)$, $L_\nu \in \mathscr{L}[(Q)]$, *and* $\kappa \in \operatorname{Aut} Q$. *Further,* κ *is uniquely determined up to its image in* $\operatorname{Aut} \hat{Q}$.

This theorem is proven by Baclawski [1], assuming that Q is a p.o. set, and consequently without topological restrictions on $\operatorname{Aut} I(Q)$. The proof for the more general case is a straightforward generalization of Baclawski's proof, and utilizes knowledge about the maximal closed ideals of $I(Q)$ as well as the fact that every automorphism of K_n is inner. See [4., Ch. 7] for details.

Now let $\operatorname{Out} I(Q) = \operatorname{Aut} I(Q)/\operatorname{Inn} I(Q)$, the *outer automorphism group* of $I(Q)$. The following corollary is an easy consequence of Theorem 2.1, along with our knowledge of $\operatorname{Inn} I(Q) \cap \mathscr{L}[I(Q)]$.

COROLLARY 2.1.1. $\operatorname{Out} I(Q) \simeq \overline{\operatorname{Aut}(Q)}$ *iff* $H(Q) = 0$.

In particular, $\operatorname{Out} I(Q) \simeq \overline{\operatorname{Aut}(Q)}$ in the case when Q is connected and has an element above or below all other elements. This latter result is obtained by different means in [7].

The concluding corollary of this section utilizes Corollary 2.1.1, and provides a purely algebraic relation between the results of the last section and of this one.

COROLLARY 2.1.2. *$I(Q)$ has a unique isomorphism class of faithful distributive modules iff* $\operatorname{Out} I(Q) \simeq \overline{\operatorname{Aut}(Q)}$.

3. Faithful distributive left ideals. In this section we study left ideals L of $I(Q)$ such that ${}_{I(Q)}I(Q)/L$ is a faithful distributive module over $I(Q)$. Such left ideals occur only when Q is finite, and we begin by working towards a proof of this fact.

The first result needed is readily established.

LEMMA 3.1. *Let R be associative ring with unit and let M be a module over R. Then M is cyclic iff $\exists$ a left ideal L of R such that $M \simeq {}_R R/L$, where ${}_R R$ denotes R regarded as a left module over itself and $\simeq$ denotes left R-module isomorphism. Further, in the case when M is cyclic with generator m, we may take L to be $\{\alpha \in A; \alpha \cdot m = 0\}$.*

Before continuing, recall that if Q is a locally finite q.o. set, then $I(Q)$ has a faithful distributive module iff Q is lower finite (Lemma 0.3).

LEMMA 3.2. *Let Q be a lower finite* q.o. *set and let M be any faithful distributive module over $I(Q)$ which is cyclic, with generator m. If*

$$m = \sum_{i=1}^{n} a_i m_{z_i},\ m_{z_i} \in e_{z_i} \cdot M \backslash \{0\},\ a_i \in K \backslash \{0\}\ ,$$

then every maximal element x of Q must be $\sim$ equivalent to at least one z_i, $1 \leqq i \leqq n$.

Proof. Suppose there is a maximal element x of Q such that $\bar{x} \neq \bar{z}_i$, $1 \leqq i \leqq n$. Since m is a generator of M, $m_x = f \cdot m$, for some $f \in I(Q)$. Hence $m_x = e_x \cdot m_x = (e_x * f) \cdot m$. Now since x is a maximal element of Q,

$$e_x * f = \sum_{x' \in \bar{x}} f(x, x') \delta_{xx'}\ .$$

It follows then that $(e_x * f) \cdot m = 0$, since for all $x' \in \bar{x}$, $x' \notin \{z_i\}_{i=1}^{n}$. From this contradiction we obtain that any maximal element x of Q is $\sim$ equivalent to at least one of $z_1, \cdots, z_n$.

COROLLARY 3.2.1. *Let Q be lower finite. If $I(Q)$ has a faithful distributive module which is cyclic, then Q is finite.*

Proof. By Lemma 3.2 there is a finite set $\{z_i\}_{i=1}^{n} \subseteqq Q$ such that every maximal element is $\sim$ equivalent to at least one z_i, $1 \leqq i \leqq n$. Since Q is locally finite, the $\sim$ equivalence class of any element is finite. Thus Q has only finitely many maximal elements, say $\{x_j\}_{j=1}^{d}$.

Now $Q = \bigcup_{j=1}^{d} \{y; yrx_j\}$, and since Q is lower finite, each set $\{y; yrx_j\}$ is finite. Therefore Q is finite.

LEMMA 3.3. *Let Q be a finite quasi-ordered set. Let M be a faithful distributive module over $I(Q)$ and let $B = \{m_z\}_{z \in Q}$ be a basis for M, as described in Lemma 0.5. Set*

$$m = \sum_{j=1}^{d} m_{x_j} ,$$

where $\{x_j\}_{j=1}^{d}$ is the collection of maximal elements of Q. Then m is a generator of M.

Proof. For every $y \in Q \exists$ maximal element x_i such that yrx_i. We then have that

$$\delta_{yx_i} \cdot \sum_{j=1}^{d} m_{x_j} = \delta_{yx_i} \cdot m_{x_i} = \alpha_{yx_i} m_y ,$$

for some $\alpha_{yx_i} \in K \backslash \{0\}$. This shows that $I(Q) \cdot m = M$, so that m is a generator of M.

We can now establish the fact cited at the beginning of the section.

LEMMA 3.4. *Let Q be a lower finite quasi-ordered set. The following are equivalent.*

(1) *Q is finite.*

(2) *$I(Q)$ has a left ideal L such that ${}_{I(Q)}I(Q)/L$ is a faithful distributive module over $I(Q)$.*

(3) *$I(Q)$ has a faithful distributive module which is cyclic.*

(4) *Every faithful distributive module over $I(Q)$ is cyclic.*

Proof. (1) $\Rightarrow$ (4) Lemma 3.3.
(4) $\Rightarrow$ (3) Immediate.
(3) $\Leftrightarrow$ (2) Lemma 3.1.
(3) $\Rightarrow$ (1) Corollary 3.2.1.

We shall assume that all q.o. sets Q are finite in the remainder of this section. Our object of study will be *faithful distributive left ideals* of $I(Q)$, left ideals L with the property that ${}_{I(Q)}I(Q)/L$ is a faithful distributive module over $I(Q)$. It is easy to show that an arbitrary left ideal L' is faithful distributive iff (i) L' contains no nonzero two sided ideal $I(Q)$, and (ii) the lattice of left ideals of $I(Q)$ containing L' is distributive.

We now present the main result of this section, which is essentially a determination of all faithful distributive left ideals of $I(Q)$.

THEOREM 3.5. *Let Q be a finite quasi-ordered set. Then given any isomorphism class $\mathscr{M}$ of faithful distributive modules over $I(Q)$, there is a unique isomorphism class $\mathscr{L}$ of left ideals of $I(Q)$ such that for all $L \in \mathscr{L}$ and $M \in \mathscr{M}$, ${}_{I(Q)}I(Q)/L$ and M are isomorphic $I(Q)$-modules.*

Proof. Let $\{\bar{x}_i\}_{i=1}^n$ be the collection of maximal elements of $\hat{Q}$, where $\bar{x}_i = \{x_{ij}\}_{j=1}^{n_i} \subseteqq Q$. It then follows from Lemmas 3.3 and 3.1 that

$$m = \sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{x_{ij}} ,$$

is a generator of M, that $L = \{f \in I(Q); f \cdot m = 0\}$ is a left ideal of $I(Q)$, and that $M \simeq {}_{I(Q)}I(Q)/L$.

Now suppose L' is any other left ideal of $I(Q)$ such that ${}_{I(Q)}I(Q)/L' \simeq M$, via an isomorphism ψ. Then $m' = \psi(\delta + L')$ is a generator of M. Further, for all $f \in I(Q)$, $\psi(f + L') = \psi(f[\delta + L']) = f \cdot \psi(\delta + L') = f \cdot m'$. Hence $f \in L' \Leftrightarrow \psi(f + L') = 0 \Leftrightarrow f \cdot m' = 0$, so that $L' = \{f \in I(Q); f \cdot m' = 0\}$.

Assume

$$m' = \sum_{i=1}^{n} \sum_{j=1}^{n_i} a_{ij} m_{x_{ij}} + \sum_{k=1}^{h} b_k m_{y_k} ,$$

where

$$\{a_{ij}\}_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq n_i}} , \{b_k\}_{1 \leqq k \leqq h} \subseteqq K$$

and

$$\{y_k\}_{k=1}^h$$

are the nonmaximal elements of Q. We know by Lemma 3.2 that for each $i \leqq n$ there is at least one $j \leqq n_i$ such that $a_{ij} \neq 0$.

For $1 \leqq i \leqq n$, $1 \leqq j \leqq n_i$, let

$$\begin{aligned} a'_{ij} &= a_{ij} \quad \text{if} \quad a_{ij} \neq 0 \\ &= 1 \quad \text{if} \quad a_{ij} = 0 \end{aligned}$$

and let

$$\begin{aligned} b_{ij} &= 0 \quad \text{if} \quad a_{ij} \neq 0 \\ &= -1 \quad \text{if} \quad a_{ij} = 0 . \end{aligned}$$

For each y_k, choose y'_k maximal in Q such that $y_k r y'_k$. For $1 \leqq i \leqq n$, $1 \leqq j \leqq n_i - 1$, define $c_{ij} \in K$ by

$$\delta_{x_{ij} x_{i,j+1}} \cdot m_{x_{i,j+1}} = c_{ij} m_{x_{ij}} .$$

For $1 \leqq i \leqq n$, define $c_{i,n_i} \in K$ by

$$\delta_{x_{i,n_i}x_{i1}} \cdot m_{x_{i1}} = c_{i,n_i} m_{x_{i,n_i}} .$$

Finally, for $1 \leqq k \leqq h$ define $c_k \in K$ by

$$\delta_{y_k y'_k} \cdot m_{y'_k} = c_k m_{y_k} .$$

Define $g \in I(Q)$ by

$$g = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n_i} a'_{ij} e_{x_{ij}} + \sum_{j=1}^{n_i - 1} \frac{b_{ij}}{c_{ij}} \delta_{x_{ij}x_{i,j+1}} + \frac{b_{i,n_i}}{c_{i,n_i}} \delta_{x_{i,n_i}x_{ij}} \right\} + \sum_{k=1}^{h} \frac{b_k}{c_k} \delta_{y_k y'_k} .$$

It is easy to verify that $g \cdot m = m'$. Note also that for each $\bar{z}_i \in Q$, $\pi_{\bar{z}_i}(g)$ is an invertible element of $K_{|\bar{z}_i|}$, since its determinant is $\prod_{j=1}^{|\bar{z}_i|} a'_{ij} \neq 0$. Hence by Lemma 0.2, g is invertible in $I(Q)$.

Define $\rho: L \to L'$ by $\rho(f) = f^* g^{-1}$, for $f \in L$. Then ρ is an isomorphism between L and L' as $I(Q)$-modules, as is readily verified. It follows that there is a unique isomorphism class $\mathscr{L}$ of left ideals of $I(Q)$ such that for all $L^* \in \mathscr{L}$, ${}_{I(Q)}I(Q)/L^* \simeq M$. The theorem now follows immediately.

As a simple application of the results in this section, let $Q_1 = \{x_i\}_{i=1}^{n}$, where $x_i r x_j$, $1 \leqq i \leqq j \leqq n$. Then $I(Q) \simeq T_n$, the algebra of $n \times n$ upper triangular matrices over K, as noted at the end of § 1. Since $I(Q_1) \simeq T_n$ has a unique isomorphism class of faithful distributive modules, it follows directly from Theorem 3.5 that T_n has a unique isomorphism class of faithful distributive left ideals. Note that the collection of $n \times n$ upper triangular matrices over K with O's in the last column is an example of such an ideal. As another application, let $Q_2 = \{y_i\}_{i=1}^{n}$, where $y_i r y_j$, $1 \leqq i, j \leqq n$. Then $I(Q_2) \simeq K_n$, the algebra of $n \times n$ matrices over K. Since $I(Q_2) \simeq K_n$ has a unique isomorphism class of faithful distributive modules (which are exactly the primitive modules), it follows directly from Theorem 3.5 that K_n has a unique isomorphism class of faithful distributive left ideals (which are exactly the maximal left ideals.) Note that the collection of $n \times n$ matrices over K with O's in a specified column is an example of such an ideal.

REFERENCES

1. K. Baclawski, *Automorphisms and derivations of incidence algebras*, Proc. Amer, Math. Soc., **38** (1972), 351-356.

2. W. R. Belding, *Incidence rings of pre-ordered sets*, Notre Dame J. Formal Logic,

14 (1973), 481-509.
3. P. Doubilet, G.-C. Rota and R. Stanley, *The idea of generating function,* Proc. Sixth Berkeley Sympos. on Math. Statistics and Probability, Vol. II, Univ. of California Press, Berkeley, Calif., 1972, 267-318.
4. R. B. Feinberg, *Characterization of Incidence Algebras*, Dissertation, Univ. of Wisconsin, 1974.
5. ———, *Characterization of incidence algebras*, to appear, Discrete Math.
6. ———, *Polynomial identities of incidence algebras*, Proc. Amer. Math. Soc., **55** (1976), 25-28.
7. W. Scharlau, *Automorphisms and Involutions of Incidence Algebras*, preprint.

Received November 20, 1975. This paper was prepared while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234. The results presented are contained in the author's doctoral dissertation written at the University of Wisconsin. The author's research was sponsored in part by the United States Army under Contract No. DA-31-124-ARO-D-462.

CLARKSON COLLEGE OF TECHNOLOGY

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 65, No. 1 September, 1976