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CLOSED ORIENTED 3-MANIFOLDS AS 3-FOLD BRANCHED COVERINGS OF S^3 OF SPECIAL TYPE

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CLOSED ORIENTED 3-MANIFOLDS AS 3-FOLD BRANCHED COVERINGS OF S^3 OF SPECIAL TYPE

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It has been shown by Hilden and Montesinos independently that any closed oriented 3-manifold M is a 3-fold irregular branched covering of S^3, $p: M \to S^3$. The purpose of this paper is to show that the branch covering space map can be chosen in such a way that the set of points at which p fails to be a local homeomorphism is the boundary of a disc in M. One application of this result is a new proof that a closed oriented 3-manifold is parallelizable.

1. Introduction. Let $p: X \to Y$ be a nondegenerate simplical map between compact triangulated combinatorial n-manifolds such that $p(\partial X) \subset \partial Y$. If the restriction of p to a map of the complements of the $n-2$ skeletons is a covering space map, then $p: X \to Y$ is called a *branched covering space*. A point $x \in X$ is called an ordinary point if p maps some neighborhood of x homeomorphically into Y. The complement of the set of ordinary points is called the *branch cover*. The image of the branch cover is called the *branch set*.

A *represented link* is a polygonal link L in S^3 together with a transitive representation ω of the link group $\pi(S^3 - L)$ into Σ_d, the symmetric group on d-symbols. The representation is called *simple* if each meridian of L is represented by a transposition.

Given a represented link (L, ω), there is a unique closed orientable 3-manifold, $M(L, \omega)$, determined by the representation. The manifold $M(L, \omega)$ is a d-fold branched covering space of S^3 branched over L. A detailed description of how to construct $M(L, \omega)$ is given in [9].

If M is a compact orientable 3-manifold and $p: M \to S^3$ is a branched covering space such that the branch set is a link L, then $M = M(L, \omega)$ where $\omega(\gamma)$ is the permutation γ induces on the left cosets of $p_*\pi_1(M - p^{-1}(L))$ by left multiplication.

Alexander's theorem ([1]) states that any closed orientable 3-manifold is an $M(L, \omega)$ for some link L and some presentation ω. Two of the co-authors of this paper showed, in [6], [7], [4] and [5], using different methods, that any closed orientable 3-manifold is an $M(L, \omega)$, where ω is a *simple* representation of $\pi(S^3 - L)$ onto Σ_3 and L is a knot. Thus any closed orientable 3-manifold is an irregular

3-fold branched covering space of S^3 branched over a knot.

The purpose of this paper is to prove the following strengthened version of the preceding theorem:

THEOREM. *If M is a closed orientable 3-manifold then there is a simple 3-fold branched covering space $p: M \rightarrow S^3$ such that the branch cover is the boundary of a disc in M.*

An application of this theorem and some remarks relating this result to the Poincaré conjecture and the homeomorphism problem are given in the final sections.

2. Notation and conventions. Throughout the paper we shall work in the PL category. Thus all curves, surfaces, 3-manifolds, and maps will be assumed piecewise linear without it being explicitly stated. Given a branched covering space $p: M \rightarrow N$, the branch cover will be denoted $B(p)$, or just by B if it is clear from the context which map we mean. The set $p^{-1}(p(B)) - B$ will be denoted by $B'(p)$ or B' and will be called the *pseudo branch cover*. The homology class of a closed curve, or union of closed curves, L, in $H_1(X; Z_2)$ will be denoted by $[L]$.

We shall be particularly concerned with three-fold branched covering space maps $p: M \rightarrow S^3(p: E \rightarrow D^3)$ such that the branch set is a link (resp. union of proper arcs) and such that the associated representation $\omega: \pi(S^3\text{-branch set}) \rightarrow \Sigma_3(\text{resp. } \omega: \pi(D^3\text{-branch set}) \rightarrow \Sigma_3)$ is *simple*. Such maps occur so frequently we shall simply call them 3-*maps*. If the branch set is a link in S^3 (union of proper arcs in D^3), it follows from the method for constructing $M(L, \omega)$ given in [9] that necessary and sufficient conditions for the branch covering space map $p: M \rightarrow S^3(\text{resp. } p: E \rightarrow D^3)$ to be simple is for the inverse image of every point in the branch set to consist of one point in the branch cover and one point in the pseudo branch cover.

3. Various useful lemmas. In this section we prove several lemmas that will be useful in the sequel.

LEMMA 1. *Let M be a closed orientable 3-manifold. There is a 3-map $p: M \rightarrow S^3$ such that the branch cover bounds a surface in M.*

Proof. From Theorem 4 of [7], it follows that M can be represented as $M(L, \omega)$ for a special, simple, represented link (L, ω) onto Σ_3. A typical example of (L, ω) is illustrated in Figure 1. In Figure 1, each overpass x' of L has been labeled with the transposition $\omega(x)$, where x is the Wirtinger generator associated to the over-

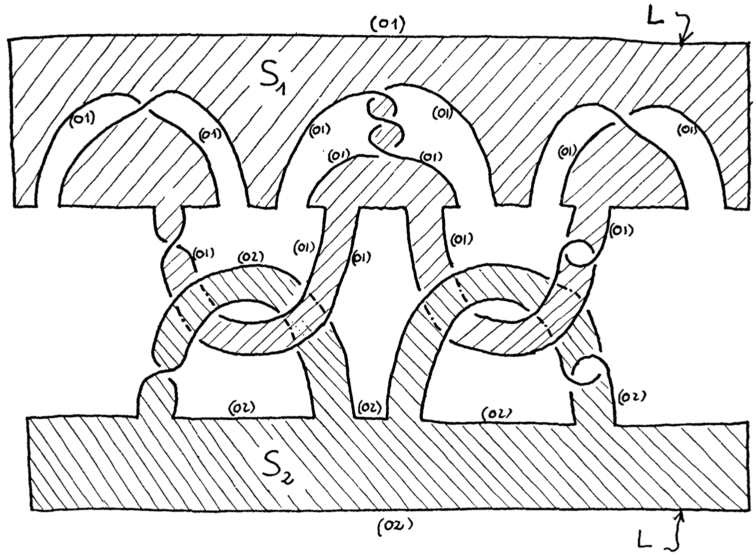

Figure 1

pass x'. The link L, as it is constructed in [7], always bounds a surface S with two components S_1 and S_2, where S_1 and S_2 are "disks with twisted ribbons attached" (see Fig. 1). The ribbons from the upper disc intertwine with the ribbons from the lower disc. Next we construct $M(L, \omega)$ by the procedure given in [9], Chapter III. First, we choose a splitting complex containing S as is illustrated in Figure 2. The representation ω gives rise to an

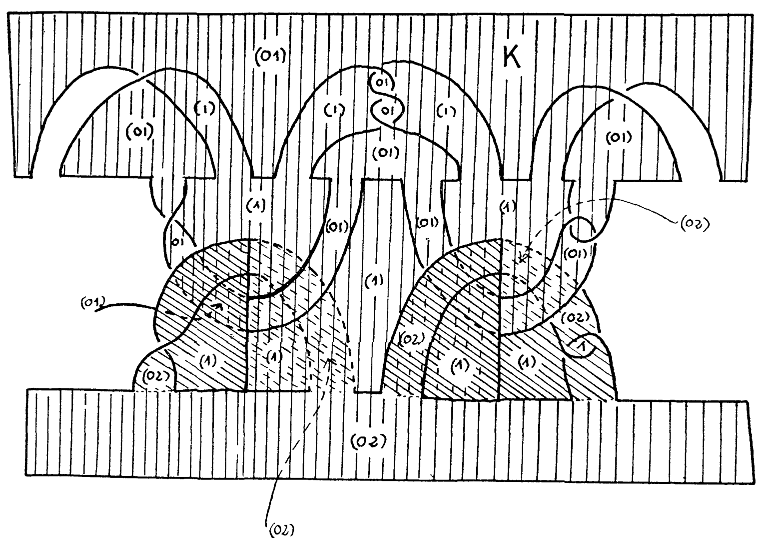

Figure 2

assignment of permutations to the 2-cells of K with ordered coboundary. In order to construct $M(L, \omega)$, it suffices to consider the part of K endowed with permutations different from the identity. Since this part is just S, we obtain $M(L, \omega)$ by taking three disjoint copies of S^3 split along S, X_0, X_1, and X_2, and pasting their boundaries together along S according to the assignment of permutations determined by ω. The permutation assigned to each 2-cell of S_i is $(0i)$ (for $i = 1, 2$). The branch cover B, is thus contained in the copy X_0 and it is the boundary of the union of any one of the two copies of S_1 in X_0 and any one of the two copies of S_2 in X_0. Hence, since $X_0 \subset M(L, \omega)$, B bounds a surface in M.

LEMMA 2. *Let M be a closed 3-manifold and let $p: M \to S^3$ be a 3-map. Let l be a simple arc in M intersecting $B(p)$ exactly in its endpoints P_0 and P_1. There is a homeomorphism ψ of M such that ψ is the identity on $B(p)$ and $p^{-1}(p(\psi(l)))$ is an arc whose endpoints lie in $B'(p)$.*

Proof. The reader should refer to Figure 3 in the following proof. By a general position argument there is an isotopy ψ_1, fixed on B, such that p embeds $\psi_1(l)$. Thus we may assume that p embedded l in the first place. There are two lifts of the curve $p(l)$ that begin at P_0. Call the other lift $\hat{l}$. If the other endpoint of $\hat{l}$

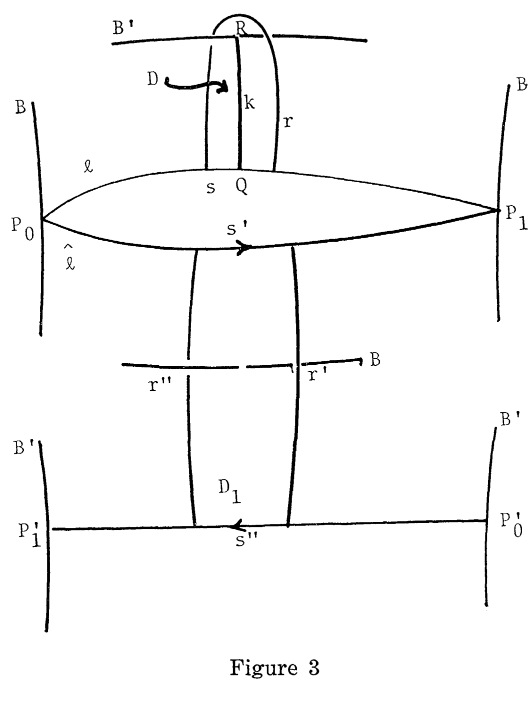

Figure 3

lies on $B'(p)$, then $p^{-1}(l)$ consists of three arcs, l, $\hat{l}$, and say m, such that $\hat{l} \cap m = \varnothing$, $l \cap \hat{l} = \{P_0\}$, $l \cap m = \{P_1\}$ and we are done. Suppose the other endpoint of $\hat{l}$ is P_1.

Let Q be any point in the interior of l and let k be an arc from Q to a point R on B' such that p embeds k and k intersects $p^{-1}(p(l))$ only in Q. Let D be a 2-disc containing k such that p embeds D, D intersects B' exactly once, transversally at R, $D \cap B = \varnothing$, and D is bounded by two arcs r and s where $s = D \cap l$ is a subarc of the interior of l and $r = \overline{\partial D - s}$. (See Fig. 3.) Since $D \cap (B \cup B') = \{R\}$, $p(D)$ intersects the branch set once and $p^{-1}(p(D))$ consists of two components D and D_1. The disk D is mapped homeomorphically onto $p(D)$ by p. The map p, restricted to D_1 defines a two-fold branched covering of the disc $p(D)$ with single branch point $p(R)$. Thus D_1 is also a disc. The boundary of D_1 is composed of four arcs s', s'' and r', r'' where $s' \cup s'' = p^{-1}(p(s)) - s$, s' lies on $\hat{l}$, and $r' \cup r'' = p^{-1}(p(r)) - r$. Consider the arc $(l - s) \cup r$. The set $p^{-1}(p((l - s) \cup r))$ consists of an arc from P_1' to P_0, an arc from P_0 to P_1 and an arc from P_1 to P_0' where $\{P_0'\} = p^{-1}(p(P_0)) - \{P_0\}$ and $\{P_1'\} = p^{-1}(p(P_1)) - \{P_1\}$. Now let ψ be a homeomorphism of M supported on a ball containing D and not intersecting B such that $\psi(s) = r$ and $\psi(l - s) = l - s$. Since $p^{-1}(p(\psi(l)))$ is an arc whose endpoints lie on B' we are done.

LEMMA 3. *Let $p\colon M \to S^3$ be a 3-map. Let D be a disc in M such that $D \cap B(p)$ consists of two disjoint subarcs, a_1 and a_2, of ∂D, such that p embeds D, and such that $p^{-1}(p(D))$ is also a disc.*

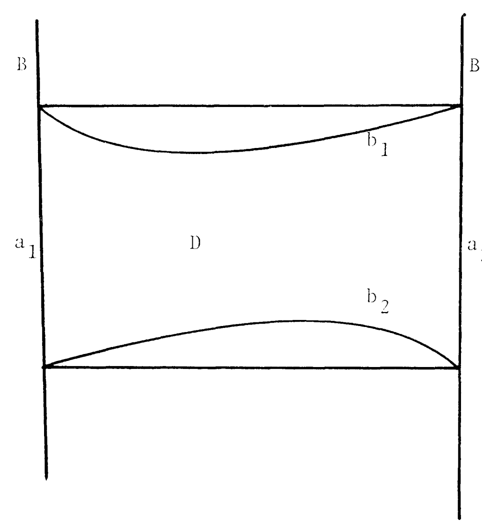

Figure 4

Then there is a 3-map $q: M \to S^3$ *such that* $B(q) = [B(p) - (a_1 \cup a_2)] \cup [b_1 \cup b_2)]$ *where* b_1 *and* b_2 *are proper disjoint subarcs of* D *such that* $b_1 \cup b_2 \cup a_1 \cup a_2$ *is the boundary of a subdisc of* D (see Fig. 4).

Proof. In the proof of this lemma we shall refer to Figures 4, 5, 6, and 7. Let N be a ball in S^3 containing $p(D)$ such that $N \cap p(B) = p(a_1 \cup a_2)$, such that $p(\partial D - a_1 \cup a_2)$ lies in ∂N and such that $p(a_1)$ and $p(a_2)$ are unknotted and unlinked in N. (Intuitively N is just the disc $p(D)$ thickened a little, but in such a way that the top and bottom of $p(D)$ still lie in ∂N.) Let $E = p^{-1}(N)$. Then E is a 3-fold irregular covering of the ball N branched over two unknotted, unlinked arcs. The fundamental group of a ball minus two unknotted, unlinked arcs is free on two generators. There are two ways to map this group homeomorphically into Σ_3, up to equivalence, such that meridians are mapped into transpositions. Either the two meridians are mapped onto the same transposition, in which case the associated branched covering space is disconnected and consists of the union of a ball and a solid torus, or they are mapped into different transpositions, in which case the associated branched covering space is a ball. Thus the 3-manifold E could be either a ball or the disjoint union of a ball and a solid torus. The latter is impossible since each component of E contains points of the disc $p^{-1}(D)$. Thus E is a ball and $E \cap B(p) = a_1 \cup a_2$. Since $p(a_1)$ and $p(a_2)$ are unknotted and unlinked in N, it follows that a_1 and a_2 are unknotted and unlinked in E.

We may represent M as $\overline{M - E} \cup_i E$, S^3 as $\overline{S^3 - N} \cup_j N$ and p as $p_1 \cup p_2$ where p_1 and p_2 are the restrictions of p to $\overline{M - E}$ and E respectively, and i and j are the natural identifications $i: \partial(\overline{M - E}) \to \partial E$ and $j: \partial(\overline{S^3 - N}) \to \partial N$. Thus $p_1 \cup p_2: \overline{M - E} \cup_i E \to \overline{S^3 - N} \cup_j N$.

Now suppose that φ and ψ are homeomorphisms of ∂E and ∂N respectively such that $p_2\varphi i = \psi j p_1$. The condition $p_2 \varphi i = \psi j p_1$ forces ψ to carry the branch set to the branch set and φ to carry the branch cover to the branch cover and it causes the map $p_1 \cup p_2: \overline{M - E} \cup_{\varphi i} E \to \overline{S^3 - N} \cup_{\psi j} N$ to be well defined. Moreover the map $p_1 \cup p_2$ is a branched covering space map and it follows from the remarks at the end of §2 that it is a 3-map. The new branch cover is $(\overline{M - E} \cap B(p)) \cup_{\varphi i} (a_1 \cup a_2)$. Suppose further that $\bar{\varphi}$ is an extension of φ to a homeomorphism of E. We next define a homeomorphism $h = \mathrm{id} \cup \bar{\varphi}: \overline{M - E} \cup_i E \to \overline{M - E} \cup_{\varphi i} E$, and a 3-map $q = ph$. The branch cover of q, $B(q) = h^{-1}(B(p)) = [\mathrm{id} \cup \bar{\varphi}]^{-1}(\overline{M - E} \cap B(p) \cup_{\varphi i} (a_1 \cup a_2)) = \overline{M - E} \cap B(p) \cup_i \bar{\varphi}^{-1}(a_1 \cup a_2)$. All that remains to be done to finish the proof is to show that we can find homeomorphisms φ, ψ, and $\bar{\varphi}$ of ∂E, ∂N, and E respectively such that $p_2 \varphi i = \psi j p_1$, $\bar{\varphi}$ extends φ, and such that $\bar{\varphi}^{-1}(a_1 \cup a_2)$ lies

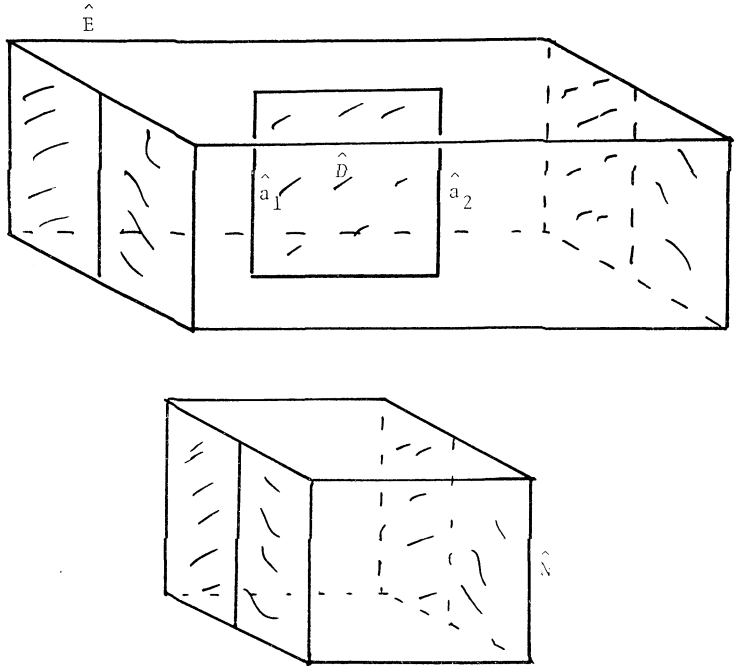

Figure 5

in the disc D and $a_1 \cup a_2 \cup \bar{\varphi}^{-1}(a_1 \cup a_2)$ is the boundary of a subdisc of D. To do this we shall explicitly define a 3-map of a ball $\hat{E}$ onto a ball $\hat{N}$. In what follows we shall denote a subset X of E^3 by $X = \{x, y, z$ satisfying property $Q\}$ rather than $X = \{(x, y, z)\} \mid (x, y, z)$ satisfies property $Q\}$.

Let $\hat{E}$ be the cube $\{-1 \leqq x \leqq 2, -1 \leqq y \leqq 1, 0 \leqq z \leqq 1\}$ with identifications $(-1, y, z) = (1, -y, z)$ and $(2, y, z) = (2, -y, z)$ (see Figure 5) and let $\hat{N}$ be the cube $\{-1 \leqq x \leqq 0, -1 \leqq y \leqq 1, 0 \leqq z \leqq 1\}$ with identifications $(-1, y, z) = (-1, -y, z)$ and $(0, y, z) = (0, -y, z)$. Let $\pi: \hat{E} \to \hat{N}$ be defined as follows: $\pi(x, y, z) = (x, y, z)$ if $-1 \leqq x \leqq 0$, $\pi(x, y, z) = (-x, -y, z)$ if $0 \leqq x \leqq 1$ and $\pi(x, y, z) = (x - 2, y, z)$ if $1 \leqq x \leqq 2$. The map π is a 3-map such that $B(\pi) = \hat{a}_1 \cup \hat{a}_2$ where $\hat{a}_1 = \{x = 0, y = 0\}$ and $\hat{a}_2 = \{x = 1, y = 0\}$. Let $\hat{D}$ be the disc $\{0 \leqq x \leqq 1, y = 0, 0 \leqq z \leqq 1\}$.

Now let σ be a homeomorphism of N onto $\hat{N}$ such that $\sigma(p(a_1)) = \pi(\hat{a}_1)$, $\sigma(p(a_2)) = \pi(\hat{a}_2)$, and $\sigma(p(D))$ is the disk $\{-1 \leqq x \leqq 0, y = 0\}$. The map σp is a 3-map of $\hat{E}$ onto N. The 3-fold irregular branched coverings of a 3-manifold X with branch set the 1-complex with no free ends K are classified by equivalence classes of transitive representations $\rho: \pi_1(X - K) \to \Sigma_3$ (the permutation group on three letters), such that meridians are mapped into transpositions (see [2]). Since $\pi_1(\hat{N} - p(\hat{a}_1 \cup \hat{a}_2))$ is free on two generators, both meridians,

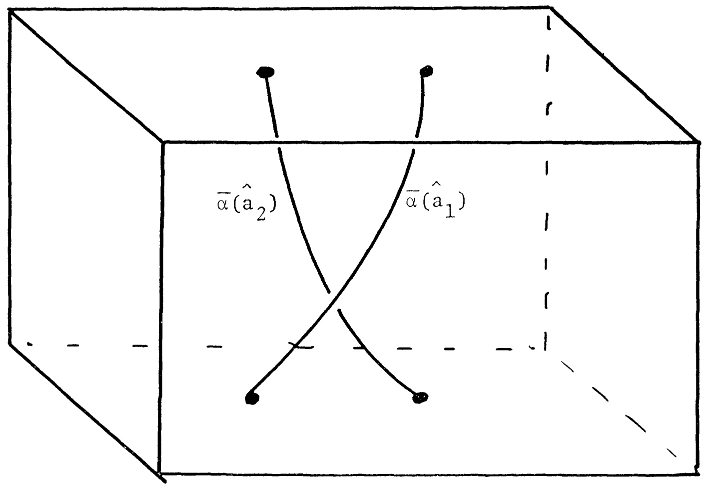

Figure 6

there is only one equivalence class. Hence there is a homeomorphism $\tau: E \rightarrow \hat{E}$ such that $\pi\tau = \sigma p$. Thus $\sigma(D) = \hat{D}$.

We shall define two homeomorphisms, α and β, of $\partial\hat{E}$ that preserve the fibres of π. Let α be the identity on $\{z = 0\} \cap \partial\hat{E}$, be $180°$ rotation about $\{x = 1/2, y = 0\}$ on $\{z = 1\} \cap \partial\hat{E}$, map the circle $\{z = t\} \cap \partial\hat{E}$ into itself in such a way that arclength is preserved, and map the line $\{x = -1, y = 1\}$ onto the arc $[\{y = 1, x = 3z - 1\}]$. One can check directly that α preserves fibres and so projects to a homeomorphism α_1 of $\partial\hat{N}$. Intuitively, if we think of $\hat{E}$ as a cylinder with axis parallel to the z-axis, the homeomorphism α rotates the discs perpendicular to the axis by an amount increasing from 0 to 180 degrees as we go from bottom to top. We may extend α to a homeomorphism $\bar{\alpha}$ of $\hat{E}$ that doesn't preserve the fibres of π. The two arcs $\bar{\alpha}(\hat{a}_1 \cup \hat{a}_2)$ form a "half spiral" like that shown in Figure 6.

Now we shall define the homeomorphism β. Let $\gamma(t) = (t, y(t), z(t))$ be a curve from $(0, 1, 0)$ to $(1, -1, 1)$ such that the projection of γ on the $y - z$ plane, $P_{yz}\gamma(t)$ injective and travels from $(1, 0)$ to $(-1, 1)$ around the point $(0, 1/2)$ in a direction that would appear counter clockwise to an observer $\{x > 0\}$. Now, if $0 \leqq x \leqq 1$, let the homeomorphism $\beta^*(x, y, z)$ be defined in such a way that $\{\beta^*(x, 1, 0), 0 \leqq x \leqq 1\} = \{\gamma(t), 0 \leqq t \leqq 1\}$, the circles $\partial\hat{E} \cap \{x = c\}$ are left invariant, and arclength in these circles is preserved. If $P = (x, y, z)$ belongs to $\partial\hat{E}$ and if $-1 \leqq x \leqq 0$ define $\beta(P) = R_1\beta^*R_1(P)$ where R_1 is $180°$ rotation about $\{x = 0, y = 0\}$; if $0 \leqq x \leqq 1$ let $\beta(P) = \beta^*(P)$, and if $1 \leqq x \leqq 2$ let $\beta(P) = R_2\beta^*R_2(P)$ where R_2 is $180°$ rotation about $\{x = 1, y = 0\}$. Again one can check directly that β preserves the fibres of π and projects to a homeomorphism β_1 of $\partial\hat{N}$. Intuitively, if we think of $\hat{E}$ as a "cylinder" with axis parallel

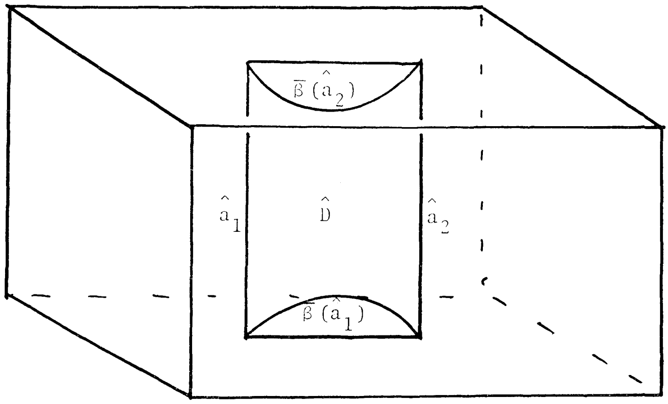

Figure 7

to the x-axis, then β rotates each disc perpendicular to the axis by an amount that increases from 0 to 360 degrees as x travels from 0 to 2, and decreases from 0 to -180 degrees as x travels from 0 to -1. We can easily extend β to a homeomorphism $\bar\beta$ of $\hat E$ such that $\bar\beta$ leaves the discs $\hat E \cap \{x = \text{constant}\}$ invariant and such that $\bar\beta(\bar\alpha(\hat a_1 \cup \hat a_2)) \subset \hat D$. The effect of $\bar\beta$ on $\bar\alpha(\hat a_1 \cup \hat a_2)$ is is depicted in Figure 7. If we define $\varphi = \sigma^{-1}(\beta\alpha)^{-1}\sigma$, $\varphi = \tau^{-1}(\beta_1\alpha_1)^{-1}\tau$ and $\bar\varphi = \sigma^{-1}(\bar\beta\bar\alpha)^{-1}\sigma$, then $p_2\varphi i = \psi j p_1$, $\bar\varphi$ extends φ, and $a_1 \cup a_2 \cup \varphi^{-1}(a_1 \cup a_2)$ is the boundary of a subdisc of D. The proof of Lemma 3 is complete.

4. The main theorem. This section is devoted to stating and proving the main theorem and giving an application.

THEOREM 4. *Let M be a closed oriented 3-manifold. There is a simple, 3-fold irregular branched covering $p: M \to S^3$ such that the branch cover $B(p)$ is the boundary of a disc in M.*

Proof. By Lemma 1 we may assume $B(p)$ bounds a surface F in M. By running tubes between various components of F, if necessary, we may assume F is connected. If F is any bounded surface other than a disc, there is a proper arc l in F such that $F - l$ is connected. By Lemma 2 we may assume that $p^{-1}(p(l))$ is an arc with endpoints in $B'(p)$. We can find a closed disc neighborhood D of l in F (see Figure 8) such that ∂D is the union of two disjoint arcs, a_1 anp a_2, in ∂F, and two disjoint proper arcs, c_1 and c_2, such that p embeds D. The inverse image, $p^{-1}(D)$ consists of three discs, glued along two pairs of arcs. Thus $p^{-1}(p(D))$ is either a disc or it is disconnected. The latter is impossible since any component of $p^{-1}(p(D))$ contains points in the connected arc $p^{-1}(p(l))$. By Lemma 3 there is a 3-map $q_1: M \to S^3$ such that

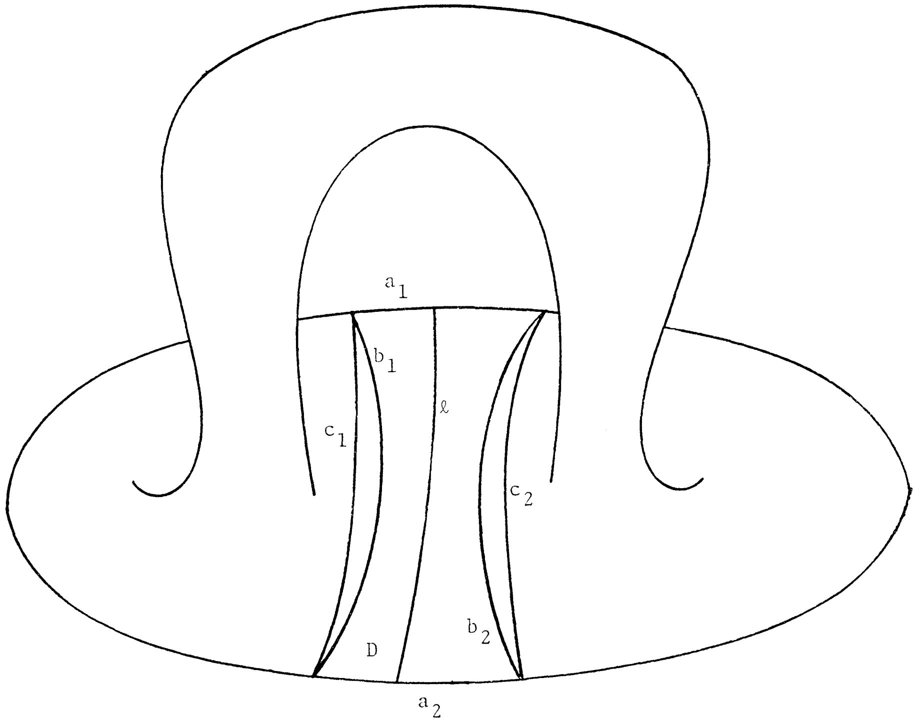

Figure 8

$B(q_1) = [B(p) - (a_1 \cup a_2)] \cup (b_1 \cup b_2)$ where the arcs b_1 and b_2 are proper subarcs of D and $a_1 \cup a_2 \cup b_1 \cup b_2$ bounds subdisc D' of D. Thus $B(q_1)$ bounds the surface $\overline{F - D'}$. The Euler characteristic of F, $\chi(F)$, equals the number of vertices plus faces of F minus the number of edges of F for any given triangulation. An easy count of vertices, edges and faces shows that $\chi(\overline{F - D'}) = \chi(F) + 1$. The Euler characteristic is $+1$ for the disc and nonpositive for every other bounded surface. Thus, if we iterate this process $|\chi(F)| + 1$ times we obtain a 3-map $q: M \rightarrow S^3$ such that $B(q)$ is the boundary of a disc in M.

Next, as an application of this theorem we show that every oriented 3-manifold has trivial tangent bundle. This can be shown using the Hirsch-Smale immersion theory, or by an obstruction theory argument using cohomology operations; but the proof that follows is more elementary than these.

THEOREM 5. *A closed orientable smooth* 3*-manifold is parallelizable.*

Proof. We may smooth the map $p: M \rightarrow S^3$, whose existence is guaranteed by Theorem 4, so that it is a regular immersion except at points of the branch cover. Let P be a point of the branch set and let l be an arc connecting the two points in the inverse image of P such that l intersects the disc D that the branch cover bounds only in one endpoint. Let the 3-ball B in M be some regular neighborhood of $D \cup l$. Restricting p to M-interior B we have a regular immersion of M-interior B in E^3 that is covered by a bundle map of the tangent bundles. Since $\tau(E^3)$, the tangent bundle of E^3 is trivial; it follows that $\tau(M) \mid (M\text{-interior } B)$ is trivial. $\tau(M) \mid B$ is also trivial. $\tau(M)$ is obtained by gluing these two bundles together along $S^2 = \partial B$ via a map of $S^2 \to GL(3, R)$. Since $\pi_2 GL(3, R) = 0$ this map is homotopic to a constant and $\tau(M)$ is trivial.

5. Concluding remarks. It is instructive to compare the method of proof of Lemma 3 (which is crucial for Theorems 4 and 5) with the operations D_1, D_2, and D_3 of p. 183 of [8]. The methods are quite similar although there the emphasis is on changes made in the range of the branched covering space projection while in the proof of Lemma 3 the emphasis is on changes made in the domain of the projection.

One possible approach to the Poincaré conjecture is to begin with a simply connected closed 3-manifold M, remove and open ball U from M and imerse $M - U$ in S^3. Then perform "geometrical operations" in order to simplify the immersion. If the immersion can be simplified to the point where it is an embedding then $M - U$ must be a ball and $M = S^3$. Theorem 4 implies (by removing a small neighborhood of the disc) that we can start with an immersion that is at most three to one and that has very special behavior on $\partial(M - U)$.

The homeomorphism problem is this: Find an algorithm to determine if a "given" 3-manifold is S^3. The methods used in this paper to prove Theorem 4 are constructive so that if we are "given" a 3-manifold (say, as a Heegaard splitting) there is an algorithm which finds a represented link (L, ω) and a simple 3-fold branched covering space $p: M = M(L, \omega) \to S^3$ such that the branch cover bounds a disc.

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