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The primary purpose of this paper is to investigate the structures of functionals and homomorphisms of unbounded operator algebras called symmetric $\sharp$-algebras, $EC^{\sharp}$-algebras and $EW^{\sharp}$-algebras. First, we give the definitions and the fundamental properties of such algebras. In particular, we define several locally convex topologies on such algebras; a weak topology, a strong topology, a σ-weak topology and a σ-strong topology. In the next section, we study the elementary operations on $EW^{\sharp}$-algebras. We can define induced and reduced $EW^{\sharp}$-algebras, the product of $EW^{\sharp}$-algebras and homomorphisms called an induction and an amplification. In the final two sections, we obtain the main results (Theorem 4.8 and 5.5) which are described here. It is shown that a linear functional f on a closed $EW^{\sharp}$-algebra $\mathfrak{A}$ on $\mathfrak{D}$ is weakly continuous (resp. σ-weakly continuous) if and only if $f(A) = \sum_{i=1}^{n}(A\xi_i \mid \eta_i)$, $A \in \mathfrak{A}$; ξ_i, $\eta_i \in \mathfrak{D}\,(i = 1, 2, \cdots, n)$ (resp. $f(A) = \sum_{n=1}^{\infty}(A\xi_n \mid \eta_n)$; ξ_n, $\eta_n \in \mathfrak{D}(n = 1, 2, \cdots)$ and $\sum_{n=1}^{\infty}\|T\xi_n\|^2 < \infty$, $\sum_{n=1}^{\infty}\|T\eta_n\|^2 < \infty$ for all $T \in \mathfrak{A}$). Also, it is shown that a σ-weakly continuous homomorphism of a closed $EW^{\sharp}$-algebra $\mathfrak{A}$ onto a closed $EW^{\sharp}$-algebra $\mathfrak{B}$ is decomposed in the following three types; a spatial isomorphism, an induction and an amplification.

1. Introduction. In [2], G. R. Allan defined a class of locally convex involution algebras called GB^*-algebras, and proved that, in the commutative case, a GB^*-algebra is algebraically isomorphic to an algebra of extended-complex-valued continuous functions on a compact Hausdorff space. After that, in [4], P. G. Dixon considered the noncommutative case and characterized GB^*-algebras as a certain class of algebras of closed operators on a Hilbert space. And so, it seems that we should study representations onto algebras of closed operators on Hilbert spaces as those of locally convex $*$-algebras. Hence, in the previous paper [9] the author studied representations of locally convex $*$-algebras onto algebras of closed operators on Hilbert spaces. In order to investigate such representations in detail, it seems that we should begin by studying a class of algebras of closed operators on Hilbert spaces. In this paper we study unbounded operator algebras called symmetric $\sharp$-algebras, $EC^{\sharp}$-algebras and $EW^{\sharp}$-algebras. The author would like to thank Professors R. T. Powers and P. G. Dixon for giving him the basic ideas in [4, 5, 9].

2. Definitions and fundamental properties. For the definitions and the basic properties concerning unbounded representations (resp. locally convex $*$-algebras) the reader is referred to [9, 11] (resp. [2, 4]).

Let π be a closed $*$-representation on a Hilbert space $\mathfrak{H}$ of a pseudo-complete symmetric locally convex $*$-algebra A. Then $\pi(A)$ is an algebra of linear operators all defined on a common dense domain $\mathfrak{D}(\pi)$ in $\mathfrak{H}$ and we have

$$(\pi(x)\xi \mid \eta) = (\xi \mid \pi(x^*)\eta)$$

for all $\xi, \eta \in \mathfrak{D}(\pi)$ and $x \in A$, and $(I + \pi(x^*)\pi(x))^{-1}$ exists and lies in $\pi(A)$, where I is an identity operator on $\mathfrak{D}(\pi)$. On the basis of $\pi(A)$ we define a certain unbounded operator algebra.

Let $\mathfrak{D}$ be a pre-Hilbert space with inner product $(\mid)$ and let $\mathfrak{H}$ be the completion of $\mathfrak{D}$. We denote the set of all linear operators on $\mathfrak{D}$ by $\mathscr{L}(\mathfrak{D})$.

DEFINITION 2.1. Let $\mathfrak{A}$ be a subalgebra of $\mathscr{L}(\mathfrak{D})$ with an identity operator I. $\mathfrak{A}$ is called a symmetric $\sharp$-algebra on $\mathfrak{D}$ if the following conditions (1) and (2) are satisfied;

(1) There exists an involution on $\mathfrak{A}$; $A \to A^\sharp$ such that

$$(A\xi \mid \eta) = (\xi \mid A^\sharp\eta)$$

for all $A \in \mathfrak{A}$ and $\xi, \eta \in \mathfrak{D}$,

(2) $(I + A^\sharp A)^{-1}$ exists and lies in $\mathfrak{A}_b$ for all $A \in \mathfrak{A}$, where let $\mathfrak{A}_b$ be the set of all bounded operators in $\mathfrak{A}$.

Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. Each A in $\mathfrak{A}$ is a closable operator on $\mathfrak{H}$ and hence we denote the closure of A by $\bar{A}$ and put $\bar{\mathfrak{A}} = \{\bar{A};\ A \in \mathfrak{A}\}$.

DEFINITION 2.2. Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. If $\bar{\mathfrak{A}}_b$ is a C^*-algebra (resp. W^*-algebra), then $\mathfrak{A}$ is said to be an $EC^\sharp$-algebra (resp. $EW^\sharp$-algebra).

REMARK. If $\mathfrak{A}$ is an $EC^\sharp$-algebra (resp. $EW^\sharp$-algebra) on $\mathfrak{D}$, then $\bar{\mathfrak{A}}$ becomes an EC^*-algebra (resp. EW^*-algebra) defined by P. G. Dixon [5].

Let S, T be closed operators on a Hilbert space $\mathfrak{H}$. If $S + T$ is closable, then $\overline{S + T}$ is called the strong sum of S and T, and is denoted $S \dotplus T$. The strong product is likewise defined to be $\overline{ST}$, if it exists, and is denoted $S \cdot T$. The strong scalar multiplication of $\lambda \in \mathfrak{C}$ ($\mathfrak{C}$; the field of complex numbers) and T is defined by $\lambda \cdot T = \lambda T$ if $\lambda \neq 0$ and $\lambda \cdot T = 0$, if $\lambda = 0$.

THEOREM 2.3. *Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. Then we have*

$$\bar{A} + \bar{B} = \overline{A + B},\ \bar{A}\cdot\bar{B} = \overline{AB},\ \lambda\cdot\bar{A} = \overline{\lambda A},\ \bar{A}^* = \bar{A}^\sharp ,$$

for all $A, B \in \mathfrak{A}$ and $\lambda \in \mathfrak{C}$. Therefore $\bar{\mathfrak{A}}$ is a $*$-algebra of closed operators under the operations of strong sum, strong product, adjoint and strong scalar multiplication and furthermore $(\bar{I}+\bar{A}^*\bar{A})^{-1}$ exists and lies in $\bar{\mathfrak{A}}_b$ for all $A \in \mathfrak{A}$.

Proof. We shall show that $\bar{A}^* = \bar{A}^\sharp$ for every $A \in \mathfrak{A}$. Suppose $A^\sharp = A$. Then $(I + A^2)^{-1} \in \mathfrak{A}_b$ and we have

$$A^2(I + A^2)^{-2} = ((I + A^2) - I)(I + A^2)^{-2} = (I + A^2)^{-1} - (I + A^2)^{-2}$$

and hence $A^2(I + A^2)^{-2} \in \mathfrak{A}_b$. For each $\xi \in \mathfrak{D}$ we get $\| A(I + A^2)^{-1}\xi \|^2 \leqq \| \overline{A^2(I + A^2)^{-2}} \| \, \| \xi \|^2$, and so $A(I + A^2)^{-1} \in \mathfrak{A}_b$. Furthermore we have

$$\begin{aligned}&(iI - A)(-iI - A)(I + A^2)^{-1}\\ &\quad = (iI - A)\{-i(I + A^2)^{-1} - A(I + A^2)^{-1}\} = I\end{aligned}$$

and

$$\{-i(I + A^2)^{-1} - A(I + A^2)^{-1}\}(iI - A) = I .$$

Therefore $(iI - A)^{-1}$ exists and lies in $\mathfrak{A}_b$. For each $\gamma = \alpha + \beta i \in \mathfrak{C} - \mathfrak{R}$ ($\mathfrak{R}$; the field of real numbers) we have

$$(\lambda I - A) = \beta\left\{iI - \frac{1}{\beta}(A - \alpha I)\right\}$$

and therefore $(\lambda I - A)^{-1}$ exists and lies in $\mathfrak{A}_b$. Therefore $\overline{(\lambda I - A)^{-1}} = (\lambda\bar{I} - \bar{A})^{-1}$ is bounded for all $\lambda \in \mathfrak{C} - \mathfrak{R}$, i.e., $\bar{A}$ has a real spectrum. Furthermore, since $A^* \supset A^\sharp = A$, $\bar{A}$ is hermitian. Therefore $\bar{A}$ is selfadjoint, i.e., we have $\bar{A}^* = \bar{A} = \bar{A}^\sharp$.

For each $A \in \mathfrak{A}$ we show that $\bar{A}^* = \bar{A}^\sharp$. Let $H_1 = \bar{A}^*\bar{A}$ and $H_2 = ((A^\sharp)^*)^*(A^\sharp)^*$. Clearly we have $H_1 \supset \overline{A^\sharp A}$ and $H_2 \supset \overline{A^\sharp A}$. Since $(A^\sharp A)^\sharp = A^\sharp A$, $\overline{A^\sharp A}$ is self-adjoint. Since self-adjoint operators are maximal, it follows that $H_1 = H_2 = \overline{A^\sharp A}$. Hence we have $\mathfrak{D}(\bar{A}) = \mathfrak{D}(H_1^{1/2}) = \mathfrak{D}(H_2^{1/2}) = \mathfrak{D}((A^\sharp)^*)$. Therefore we get $\bar{A} = (A^\sharp)^*$, and so $\bar{A}^* = \bar{A}^\sharp$.

We shall that $\bar{A} + \bar{B} = \overline{A + B}$ for all A and B in $\mathfrak{A}$. Since $\bar{A} + \bar{B}$ $\overline{\bar{A} + \bar{B}}$, clearly $\overline{A + B} \subset \bar{A} + \bar{B}$. Since $\bar{A} = (A^\sharp)^*$, we have

$$\overline{\bar{A} + \bar{B}} = \overline{(A^\sharp)^* + (B^\sharp)^*} \subset (A^\sharp + B^\sharp)^* = ((A + B)^\sharp)^* = \overline{A + B} .$$

Similarly we can show that $\bar{A}\cdot\bar{B} = \overline{AB}$ and $\lambda\cdot\bar{A} = \overline{\lambda A}$. For all $A \in \mathfrak{A}$, since $(\bar{I} + \bar{A}^*\bar{A})^{-1} = \overline{(I + A^\sharp A)^{-1}}$ and $(I + A^\sharp A)^{-1} \in \mathfrak{A}_b$, $(\bar{I} + \bar{A}^*\bar{A})^{-1}$ lies in $\bar{\mathfrak{A}}_b$.

Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. Then there is a natural induced topology τ_0 on $\mathfrak{D}$. This topology is defined as follows. Suppose that $\mathfrak{S}$ is a finite subset of elements of $\mathfrak{A}$. We define the seminorm $\|\ \|_{\mathfrak{S}}$ on $\mathfrak{D}$ as

$$\|\xi\|_{\mathfrak{S}} = \sum_{A \in \mathfrak{S}} \|A\xi\| ,$$

where $\|\xi\|$ is the Hilbert space norm of ξ. We define the induced topology τ_0 on $\mathfrak{D}$ as the topology generated by the family $\{\|\ \|_{\mathfrak{S}}; \mathfrak{S}\}$ of the seminorms.

DEFINITION 2.4. Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. If $\mathfrak{D}$ is complete under the topology τ_0, then $\mathfrak{A}$ is said to be closed.

PROPOSITION 2.5. *Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. Let*

$$\tilde{D} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\bar{A}),\ \tilde{A}x = \bar{A}x,\ (x \in \tilde{\mathfrak{D}}) .$$

Then $\tilde{\mathfrak{A}} = \{\tilde{A}; A \in \mathfrak{A}\}$ is a closed symmetric $\sharp$-algebra on $\tilde{\mathfrak{D}}$ and a minimal closed extension of $\mathfrak{A}$. Hereafter we call $\tilde{\mathfrak{A}}$ the closure of $\mathfrak{A}$.

Proof. By a slight modification of ([11] Lemma 2.6). Proposition 2.5 is easily shown.

PROPOSITION 2.6. *If $\mathfrak{A}$ is a closed symmtric $\sharp$-algebra on $\mathfrak{D}$, then we have*

$$\mathfrak{D} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\bar{A}) = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(A^*) .$$

Proof. By Proposition 2.5 we get $\mathfrak{D} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\bar{A})$. Since $A^* = \bar{A}^\sharp$ for all $A \in \mathfrak{A}$, we have

$$\bigcap_{A \in \mathfrak{A}} \mathfrak{D}(A^*) = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\bar{A}^\sharp) = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\bar{A}) = \mathfrak{D} .$$

We define several locally convex topologies in a symmetric $\sharp$-algebra $\mathfrak{A}$ on $\mathfrak{D}$.

(1) The weak topology. The locally convex topology, induced by the seminorms;

$$T \in \mathfrak{A} \longrightarrow P_{\xi,\eta}(T) = |(T\xi|\eta)|$$

for each $\xi, \eta \in \mathfrak{D}$, is called the weak topology. Under the weak topology $\mathfrak{A}$ is a locally convex $\sharp$-algebra. Since

$$\begin{aligned} 4(T\xi|\eta) = &(T(\xi+\eta)|\xi+\eta) - (T(\xi-\eta)|\xi-\eta) \\ &+ i(T(\xi+i\eta)|\xi+i\eta) - i(T(\xi-i\eta)|\xi-i\eta) , \end{aligned}$$

the weak topology is in accord with the topology induced by the seminorms $\{P_{\xi,\xi}(\ );\ \xi\in\mathfrak{D}\}$.

If $\mathfrak{A}$ is an $EC^{\#}$-algebra on $\mathfrak{D}$, then $\overline{\mathfrak{A}}$ is a GB^{*}-algebra defined by P. G. Dixon [4] under the weak topology.

(2) The strong topology. The strong topology is the locally convex topology induced by the seminorms;

$$T\in\mathfrak{A}\longrightarrow P_{\xi}(T)=\|T\xi\|,\ \xi\in\mathfrak{D}\ .$$

(3) The σ-weak topology. Let

$$\mathfrak{D}_{\infty}(\mathfrak{A})=\{\xi_{\infty}=(\xi_1,\xi_2,\cdots,\xi_n,\cdots);\ \xi_n\in\mathfrak{D},\ n=1,2,\cdots \text{ and } \sum_{n=1}^{\infty}\|T\xi_n\|^2<\infty \quad \text{for all} \quad T\in\mathfrak{A}\}\ .$$

For each $\xi_{\infty}=(\xi_1,\xi_2,\cdots,\xi_n,\cdots)$ and $\eta_{\infty}=(\eta_1,\eta_2,\cdots,\eta_n,\cdots)$ in $\mathfrak{D}_{\infty}(\mathfrak{A})$, putting

$$P_{\xi\infty,\eta\infty}(T)=\left|\sum_{n=1}^{\infty}(T\xi_n|\eta_n)\right|,\ \ T\in\mathfrak{A}\ ,$$

$P_{\xi\infty,\eta\infty}(\ )$ is a seminorm on $\mathfrak{A}$. We call the σ-weak topology in $\mathfrak{A}$ the locally convex topology in $\mathfrak{A}$ induced by the family $\{P_{\xi\infty,\eta\infty}(\ );\ \xi_{\infty},\eta_{\infty}\in\mathfrak{D}_{\infty}(\mathfrak{A})\}$ of seminorms. Under the σ-weak topology $\mathfrak{A}$ is a locally convex $\#$-algebra. The σ-weak topology is in accord with the topology induced by the seminorms $\{P_{\xi\infty,\xi\infty}(\ );\ \xi_{\infty}\in\mathfrak{D}_{\infty}(\mathfrak{A})\}$.

(4) The σ-strong topology. For each $\xi_{\infty}=(\xi_1,\xi_2,\cdots,\xi_n,\cdots)\in\mathfrak{D}_{\infty}(\mathfrak{A})$, putting

$$P_{\xi\infty}(T)=\left(\sum_{n=1}^{\infty}\|T\xi_n\|^2\right)^{1/2},\quad T\in\mathfrak{A}\ ,$$

$P_{\xi\infty}(\ )$ is a seminorm on $\mathfrak{A}$. The locally convex topology induced by the family $\{P_{\xi\infty}(\ );\ \xi_{\infty}\in\mathfrak{D}_{\infty}(\mathfrak{A})\}$ of seminorms is called the σ-strong topology in $\mathfrak{A}$.

DEFINITION 2.7. Let $\mathfrak{A}$ be a symmetric $\#$-algebra on $\mathfrak{D}$. We define the commutant $\mathfrak{A}'$ of $\mathfrak{A}$ by

$$\mathfrak{A}'=\{C\in\mathscr{B}(\mathfrak{H});\ (CA\xi|\eta)=(C\xi|A^{\#}\eta) \text{ for all } A\in\mathfrak{A} \text{ and } \xi,\eta\in\mathfrak{D}\}\ ,$$

where let $\mathscr{B}(\mathfrak{H})$ be the set of all bounded linear operators on $\mathfrak{H}$.

PROPOSITION 2.8. *Let $\mathfrak{A}$ be a* (resp. closed) *symmetric $\#$-algebra on $\mathfrak{D}$. Then $\mathfrak{A}'$ is a von Neumann algebra and furthermore for each $C\in\mathfrak{A}'$ we have $C\mathfrak{D}\subset\tilde{\mathfrak{D}}$* (resp. $C\mathfrak{D}\subset\mathfrak{D}$) *and $CA\xi=\tilde{A}C\xi$* (resp. $CA\xi=AC\xi$) *for all $A\in\mathfrak{A}$ and $\xi\in\mathfrak{D}$.*

Proof. This follows from ([11] Lemma 4.6).

Let $\mathfrak{A}$ be an $EW^{\#}$-algebra. Then we investigate the relations between the von Neumann algebra $\overline{\mathfrak{A}}_b$ and the von Neumann algebra $\mathfrak{A}''$.

PROPOSITION 2.9. *Let $\mathfrak{A}$ be an $EC^{\#}$-algebra on $\mathfrak{D}$. Then we have $\mathfrak{A}' = (\overline{\mathfrak{A}}_b)'$ and $\mathfrak{A}'' = (\overline{\mathfrak{A}}_b)''$. In particular, if $\mathfrak{A}$ is an $EW^{\#}$-algebra on $\mathfrak{D}$, then we have $\mathfrak{A}'' = \overline{\mathfrak{A}}_b$.*

Proof. Let $C \in \mathfrak{A}'$. By Proposition 2.8 we have $CA\xi = \tilde{A}C\xi$ for all $A \in \mathfrak{A}$ and $\xi \in \mathfrak{D}$. In particular, we have $CA\xi = \tilde{A}C\xi$ for all $A \in \mathfrak{A}_b$ and hence $C\bar{A} = \bar{A}C$, i.e., $C \in (\overline{\mathfrak{A}}_b)'$.

Conversely suppose that $C \in (\overline{\mathfrak{A}}_b)'$. By ([5] Prop. 2.4) $\bar{A}$ is affiliated with $(\mathfrak{A}_b)''(\bar{A}\eta(\overline{\mathfrak{A}}_b)'')$ for every $A \in \mathfrak{A}$ and it follows that for each $\xi, \eta \in \mathfrak{D}$

$$(CA\xi | \eta) = (\bar{A}C\xi | \eta) = (C\xi | A^*\eta) = (C\xi | A^{\#}\eta) .$$

Therefore we get $C \in \mathfrak{A}'$.

DEFINITION 2.10. Let $\mathfrak{A}$ be a symmetric $\#$-algebra on $\mathfrak{D}$. An element T of $\mathfrak{A}$ is called hermitian, if $T^{\#} = T$ and we denote by $\mathfrak{A}_h$ the set of all hermitian elements of $\mathfrak{A}$. Let $T \in \mathfrak{A}_h$. If $(T\xi | \xi) \geqq 0$ for all $\xi \in \mathfrak{D}$, then T is called positive and write $T \geqq 0$. The set of all positive hermitian elements of $\mathfrak{A}$ is denoted $\mathfrak{A}_h^+$.

PROPOSITION 2.11. *Let $\mathfrak{A}$ be an $EC^{\#}$-algebra on $\mathfrak{D}$ and let $T \in \mathfrak{A}_h$. Then the following conditions are equivalent;*

(1) $T \geqq 0$,

(2) $T = A^2$ *for some* $A \in \mathfrak{A}_h^+$,

(3) $T = S^{\#}S$ *for some* $S \in \mathfrak{A}$,

(4) $\bar{T} \geqq 0$ (i.e., $(\bar{T}x | x) \geqq 0$ for every $x \in \mathfrak{D}(\bar{T})$).

Proof. If $\mathfrak{A}$ is an $EC^{\#}$-algebra, $\overline{\mathfrak{A}}$ is a GB^*-algebra under the weak topology. Therefore, by ([4] Prop. 5.1) and Theorem 2.3 we can easily prove the above proposition.

PROPOSITION 2.12. *Let $\mathfrak{A}$ be an $EW^{\#}$-algebra on $\mathfrak{D}$ and $T \in \mathfrak{A}$. Then there exist $U \in \mathfrak{A}_b$ and $|T| \in \mathfrak{A}_h^+$ such that $T = U|T|$, where $\bar{U}$ is a partial isometry whose initial domain is $\overline{\mathfrak{R}(T^*)}$* (we denote the range of T by $\mathfrak{R}(T)$) *and $\overline{|T|}$ is a positive self-adjoint operator such that $\mathfrak{R}(\overline{|T|}) = \overline{\mathfrak{R}(T^*)}$. Furthermore such decomposition is unique.*

Proof. By the polar decomposition of a closed operator $\bar{T}$, Theorem 2.3 and $\bar{T}\eta\bar{\mathfrak{A}}_b$ (Prop. 2.9) we can easily prove the above propositition.

DEFINITION 2.13. The decomposition $T = U|T|$ of Proposition 2.12 is called the polar decomposition of T.

3. Elementary operations on $EW^\sharp$-algebras. We define reduced and induced $EW^\sharp$-algebras. Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. Define $\mathfrak{A}_p = \{E \in \mathfrak{A};\ E^2 = E^\sharp = E\}$ and let $E \in \mathfrak{A}_p$. For each $T \in \mathfrak{A}$ we define $T_E = ET/E\mathfrak{D}$ (the restriction of ET onto $E\mathfrak{D}$) and $\mathfrak{A}_E = \{T_E;\ T \in \mathfrak{A}\}$. Then T_E is a linear operator on $E\mathfrak{D}$. We put $\mathfrak{B} = \{T \in \mathfrak{A};\ TE = ET = T\}$. Then $\mathfrak{B}$ is a $\sharp$-subalgebra of $\mathfrak{A}$ and we have $\mathfrak{B} = E\mathfrak{A}E$. The mapping $T \rightarrow T_E$ is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}_E$.

THEOREM 3.1. *Let $\mathfrak{A}$ be a symmetric $\sharp$-algebra on $\mathfrak{D}$. Suppose $E \in \mathfrak{A}_p$. Then $\mathfrak{A}_E$ is a smmetric $\sharp$-algebra on $E\mathfrak{D}$. In particular, if $\mathfrak{A}$ is an $EW^\sharp$-algebra on $\mathfrak{D}$, then $\mathfrak{A}_E$ is an $EW^\sharp$-algebra on $E\mathfrak{D}$ and we have*

$$(\mathfrak{A}_E)' = (\mathfrak{A}')_{\bar{E}} = ((\bar{\mathfrak{A}}_b)')_{\bar{E}} = (\overline{(\mathfrak{A}_E)_b})' .$$

Proof. We can easily show that $\mathfrak{A}_E$ is a symmetric $\sharp$-algebra on $E\mathfrak{D}$. Suppose that $\mathfrak{A}$ is an $EW^\sharp$-algebra. Then we have only to show that $\overline{(\mathfrak{A}_E)_b}$ is a von Neumann algebra. Clearly we have $(\bar{\mathfrak{A}}_b)_{\bar{E}} \subset \overline{(\mathfrak{A}_E)_b}$ and it follows that $(\overline{(\mathfrak{A}_E)_b})' \subset ((\bar{\mathfrak{A}}_b)_{\bar{E}})'$. Since $\bar{\mathfrak{A}}_b$ is a von Neumann algebra and $(\bar{\mathfrak{A}}_b)' = \mathfrak{A}'$, we have $((\bar{\mathfrak{A}}_b)_{\bar{E}})' = ((\bar{\mathfrak{A}}_b)')_{\bar{E}} = (\mathfrak{A}')_{\bar{E}}$. Next we shall show that $(\mathfrak{A}')_{\bar{E}} \subset (\mathfrak{A}_E)'$. Let $C \in \mathfrak{A}'$. For each $\xi, \eta \in \mathfrak{D}$ and $T \in \mathfrak{A}$ we have

$$\begin{aligned}(C_{\bar{E}}T_E E\xi \,|\, E\eta) &= (CETE\xi \,|\, E\eta) = (CE\xi \,|\, (ETE)^\sharp E\eta) \\ &= (CE\xi \,|\, ET^\sharp E\eta) = (C_{\bar{E}}E\xi \,|\, T_E^\sharp E\eta)\end{aligned}$$

and hence $C_{\bar{E}} \in (\mathfrak{A}_E)'$, and so $(\mathfrak{A}')_{\bar{E}} \subset (\mathfrak{A}_E)'$. On the other hand we have $(\overline{(\mathfrak{A}_E)_b})' \supset (\mathfrak{A}_E)'$. Therefore we have

$$(\overline{(\mathfrak{A}_E)_b})' \subset ((\bar{\mathfrak{A}}_b)_{\bar{E}})' = (\bar{\mathfrak{A}}_b')_{\bar{E}} = (\mathfrak{A}')_{\bar{E}} \subset (\mathfrak{A}_E)' \subset (\overline{(\mathfrak{A}_E)_b})'$$

and it follows that

$$(\overline{(\mathfrak{A}_E)_b})' = ((\bar{\mathfrak{A}}_b)_{\bar{E}})' = (\mathfrak{A}')_{\bar{E}} = (\mathfrak{A}_E)' .$$

Therefore we have

$$(\overline{(\mathfrak{A}_E)_b})'' = ((\bar{\mathfrak{A}}_b)_{\bar{E}})'' = (\bar{\mathfrak{A}}_b)_{\bar{E}} \subset \overline{(\mathfrak{A}_E)_b} .$$

Consequently $\mathfrak{A}_E$ is an $EW^\sharp$-algebra on $E\mathfrak{D}$.

DEFINITION 3.2. Let $\mathfrak{A}$ be an $EW^\sharp$-algebra on $\mathfrak{D}$ and let $E \in \mathfrak{A}_p$.

We call $\mathfrak{A}_E$ the reduced $EW^\#$-algebra of $\mathfrak{A}$.

PROPOSITION 3.3. *Let $\mathfrak{A}$ be a closed symmetric $\sharp$-algebra on $\mathfrak{D}$ and let $\mathfrak{M}$ be an $\mathfrak{A}$-invariant τ_0-closed subspace of $\mathfrak{D}$. Let $A_{\mathfrak{M}}$ be the restriction of A onto $\mathfrak{M}$ and let $\mathfrak{A}_{\mathfrak{M}} = \{A_{\mathfrak{M}}; A \in \mathfrak{A}\}$. Then the following conditions are satisfied.*

(1) *$\mathfrak{A}_{\mathfrak{M}}$ is a closed symmetric $\sharp$-algebra on $\mathfrak{M}$ and we have*

$$\mathfrak{M} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\overline{A_{\mathfrak{M}}}) = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(A_{\mathfrak{M}}^*) .$$

(2) *Let $E_{\mathfrak{M}}$ be the projection onto $\bar{\mathfrak{M}}$. Then we have $E_{\mathfrak{M}}\mathfrak{D} = \mathfrak{M}$ and $E_{\mathfrak{M}} \in \mathfrak{A}'$.*

(3) *If $E \in (\mathfrak{A}')_p$, then $E\mathfrak{D}$ is an $\mathfrak{A}$-invariant τ_0-closed subspace of $\mathfrak{D}$.*

Proof. (1) Under $(A_{\mathfrak{M}})^\# = (A^\#)_{\mathfrak{M}}$, clearly $\mathfrak{A}_{\mathfrak{M}}$ is a symmetric $\sharp$-algebra on $\mathfrak{M}$. By Theorem 2.3 we have $\overline{(A_{\mathfrak{M}})^\#} = \overline{(A^\#)_{\mathfrak{M}}} = (A_{\mathfrak{M}})^*$ for all $A \in \mathfrak{A}$. Furthermore, since $\mathfrak{M}$ is τ_0-closed, $\mathfrak{A}_{\mathfrak{M}}$ is closed. Therefore we have

$$\mathfrak{M} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\overline{A_{\mathfrak{M}}}) = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(A_{\mathfrak{M}}^*) .$$

(2) We shall show $E_{\mathfrak{M}}\mathfrak{D} = \mathfrak{M}$. Clearly we have $\mathfrak{M} \subset E_{\mathfrak{M}}\mathfrak{D}$. Let $\xi \in \mathfrak{D}$. For each $\eta \in \mathfrak{M}$ and $A \in \mathfrak{A}$ we have

$$(A_{\mathfrak{M}}\eta \,|\, E_{\mathfrak{M}}\xi) = (E_{\mathfrak{M}}A_{\mathfrak{M}}\eta \,|\, \xi) = (A\eta \,|\, \xi) = (\eta \,|\, A^*\xi)$$

and it follows that $E_{\mathfrak{M}}\xi \in \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(A_{\mathfrak{M}}^*) = \mathfrak{M}$. Consequently we have $\mathfrak{M} = E_{\mathfrak{M}}\mathfrak{D}$. We shall show $E_{\mathfrak{M}} \in \mathfrak{A}'$. For each $A \in \mathfrak{M}$ and $\xi, \eta \in \mathfrak{D}$ we have

$$\begin{aligned}(E_{\mathfrak{M}}A\xi \,|\, \eta) &= (A\xi \,|\, E_{\mathfrak{M}}\eta) = (\xi \,|\, A^\# E_{\mathfrak{M}}\eta) = (E_{\mathfrak{M}}\xi \,|\, A^\# E_{\mathfrak{M}}\eta) \\ &= (AE_{\mathfrak{M}}\xi \,|\, E_{\mathfrak{M}}\eta) = (E_{\mathfrak{M}}AE_{\mathfrak{M}}\xi \,|\, \eta) = (AE_{\mathfrak{M}}\xi \,|\, \eta)\end{aligned}$$

and hence $E_{\mathfrak{M}} \in \mathfrak{A}'$.

(3) By Proposition 2.8 it is clear that $E\mathfrak{D}$ is an $\mathfrak{A}$-invariant subspace of $\mathfrak{D}$. We can easily show that $E\mathfrak{D}$ is τ_0-closed.

DEFINITION 3.4 Let $\mathfrak{A}$ be a closed symmetric $\sharp$-algebra on $\mathfrak{D}$ and let $E \in (\mathfrak{A}')_p$. By Proposition 3.3 (3), $\mathfrak{M} = E\mathfrak{D}$ is an $\mathfrak{A}$-invariant τ_0-closed subspace of $\mathfrak{D}$. We define

$$A_E = A_{\mathfrak{M}} \quad \text{and} \quad \mathfrak{A}_E = \{A_E; A \in \mathfrak{A}\} .$$

By Proposition 3.3 (1), $\mathfrak{A}_E$ is a closed symmetric $\sharp$-algebra on $\mathfrak{M}$. Clearly the map $A \to A_E$ of $\mathfrak{A}$ onto $\mathfrak{A}_E$ is a homomorphism. We call this homomorphism the induction of $\mathfrak{A}$ and $\mathfrak{A}_E$ is called the induced algebra of $\mathfrak{A}$.

THEOREM 3.5. *Let $\mathfrak{A}$ be a closed $EW^{\#}$-algebra on $\mathfrak{D}$ and let $E \in (\mathfrak{A}')_p$. Then $\mathfrak{A}_E$ is a closed $EW^{\#}$-algebra on $E\mathfrak{D}$ and we have $(\mathfrak{A}_E)' = (\mathfrak{A}')_E$.*

Proof. We shall show that $((\overline{\mathfrak{A}_E})_b)' = (\mathfrak{A}')_E$. Let $C \in ((\overline{\mathfrak{A}_E})_b)'$, i.e., C is a bounded linear operator on $\overline{E\mathfrak{D}}$ such that $C\overline{A_E} = A_E C$ for every $A_E \in (\mathfrak{A}_E)_b$. We shall show $CE \in \mathfrak{A}'$. For each $A \in \mathfrak{A}$ let $A = U|A|$ be the polar decomposition of A. Let $\overline{|A|} = \int_0^\infty \lambda d\overline{E_A}(\lambda)$ be the spectral decomposition of $\overline{|A|}$. Then we have $U, E_A(\lambda) \in \mathfrak{A}_b$ for all λ and hence $U_E, E_A(\lambda)_E \in (\mathfrak{A}_E)_b$. Since $(\overline{U_E}) = (\bar{U})_E$, $(\overline{E_A(\lambda)_E}) = (\overline{E_A(\lambda)})_E$ and $C \in ((\overline{\mathfrak{A}_E})_b)'$, we have $C(\bar{U})_E = (\bar{U})_E C$ and $C(\overline{E_A(\lambda)})_E = (\overline{E_A(\lambda)})_E C$ and hence CE commutes with $\bar{U}$ and $\overline{E_A(\lambda)}$. Therefore CE commutes with $\bar{A}$. Then, clearly we have $CE \in \mathfrak{A}'$ and hence $C = (CE)_E \in (\mathfrak{A}')_E$. Therefore we get $((\overline{\mathfrak{A}_E})_b)' \subset (\mathfrak{A}')_E$. Conversely we can easily show $((\overline{\mathfrak{A}_E})_b)' \supset (\mathfrak{A}')_E$. Consequently we have $((\overline{\mathfrak{A}_E})_b)' = (\mathfrak{A}')_E$. We shall show $((\overline{\mathfrak{A}_E})_b)'' = (\overline{\mathfrak{A}_E})_b$. By the above argument, ([3] Ch. I, §2, Prop. 1) and Proposition 2.9 we have

$$((\overline{\mathfrak{A}_E})_b)'' = ((\mathfrak{A}')_E)' = (\mathfrak{A}'')_E = (\bar{\mathfrak{A}}_b)_E .$$

On the other hand, clearly we have $(\bar{\mathfrak{A}}_b)_E \subset (\overline{\mathfrak{A}_E})_b$ and hence $((\overline{\mathfrak{A}_E})_b)'' = (\bar{\mathfrak{A}}_b)_E \subset (\overline{\mathfrak{A}_E})_b$ and it follows that $((\overline{\mathfrak{A}_E})_b)'' = (\overline{\mathfrak{A}_E})_b$. Consequently $\mathfrak{A}_E$ is an $EW^{\#}$-algebra on $E\mathfrak{D}$. Furthermore we have $(\mathfrak{A}_E)' = ((\overline{\mathfrak{A}_E})_b)' = (\mathfrak{A}')_E$, by Proposition 2.9.

DEFINITION 3.6. Let $\mathfrak{A}$ be a closed $EW^{\#}$-algebra on $\mathfrak{D}$ and let $E \in (\mathfrak{A}')_p$. Then $\mathfrak{A}_E$ is called the induced $EW^{\#}$-algebra of $\mathfrak{A}$.

Next we shall study the product of $EW^{\#}$-algebras. Let $\{\mathfrak{A}, \mathfrak{D}_\iota\}_{\iota \in \Lambda}$ be a family of symmetric $\#$-algebras $\mathfrak{A}_\iota$ on $\mathfrak{D}_\iota$. Let $\mathfrak{H}_\iota$ be the completion of $\mathfrak{D}_\iota$ for each $\iota \in \Lambda$ and let $\mathfrak{H}$ be the direct sum of $\{\mathfrak{H}_\iota\}_{\iota \in \Lambda}$. We denote the product of $\{\mathfrak{A}_\iota\}_{\iota \in \Lambda}$ by $\mathfrak{A} = \prod_{\iota \in \Lambda} \mathfrak{A}_\iota$ and define $\mathfrak{A}$ as follows. Let

$$\mathfrak{D}(\mathfrak{A}) = \{(\xi_\iota)_{\iota \in \Lambda} \in \mathfrak{H};\ \xi_\iota \in \mathfrak{D}_\iota \quad \text{for all} \quad \iota \in \Lambda \quad \text{and} \quad \sum_{\iota \in \Lambda} \|A_\iota \xi_\iota\|^2 < \infty \quad \text{for all} \quad A_\iota \in \mathfrak{A}_\iota\} .$$

We define

$$A\xi = (A_\iota)_{\iota \in \Lambda} (\xi_\iota)_{\iota \in \Lambda} = (A_\iota \xi_\iota)_{\iota \in \Lambda}$$

for all $\xi = (\xi_\iota)_{\iota \in \Lambda} \in \mathfrak{D}(\mathfrak{A})$ and $A = (A_\iota)_{\iota \in \Lambda} \in \mathfrak{A}$. It is clear that $\prod_{\iota \in \Lambda} \mathfrak{A}_\iota$

is a $\sharp$-algebra on $\mathfrak{D}(\mathfrak{A})$ under the following operations; $A+B=(A_\iota+B_\iota)_{\iota\in\Lambda}$, $\lambda A=(\lambda A_\iota)_{\iota\in\Lambda}$, $AB=(A_\iota B_\iota)_{\iota\in\Lambda}$, $A^\sharp=(A_\iota^\sharp)$, for each $A=(A_\iota)_{\iota\in\Lambda}$, $B=(B_\iota)_{\iota\in\Lambda}\in\prod_{\iota\in\Lambda}\mathfrak{A}_\iota$ and $\lambda\in\mathfrak{C}$.

THEOREM 3.7. *Let $\{\mathfrak{A}_\iota\}_{\iota\in\Lambda}$ be a family of* (resp. closed) *symmetric $\sharp$-algebras $\mathfrak{A}_\iota$ on $\mathfrak{D}_\iota$. Then $\mathfrak{A}=\prod_{\iota\in\Lambda}\mathfrak{A}_\iota$ is a* (resp. closed) *symmetric $\sharp$-algebra on $\mathfrak{D}(\mathfrak{A})$. In particular, if $\mathfrak{A}_\iota$ is an $EW^\sharp$-algebra on $\mathfrak{D}_\iota$ for every $\iota\in\Lambda$, then $\mathfrak{A}$ is an $EW^\sharp$-algebra on $\mathfrak{D}(\mathfrak{A})$ and we have*

$$\mathfrak{A}'=\bigoplus_{\iota\in\Lambda}\mathfrak{A}'_\iota \quad \textit{and} \quad \bar{\mathfrak{A}}_b=\bigoplus_{\iota\in\Lambda}(\overline{\mathfrak{A}_\iota})_b\ ,$$

where we denote by $\bigoplus_{\iota\in\Lambda}\mathfrak{B}_\iota$ the direct sum of a family $\{\mathfrak{B}_\iota\}_{\iota\in\Lambda}$ of von Neumann algebras.

Proof. If $\mathfrak{A}_\iota$ is a (resp. closed) symmetric $\sharp$-algebra on $\mathfrak{D}_\iota$ for all $\iota\in\Lambda$, it is easily shown that $\prod_{\iota\in\Lambda}\mathfrak{A}_\iota$ is a (resp. closed) symmetric $\sharp$-algebra on $\mathfrak{D}(\mathfrak{A})$. We shall show that $\bar{\mathfrak{A}}_b=\bigoplus_{\iota\in\Lambda}(\overline{\mathfrak{A}_\iota})_b$, if $\mathfrak{A}_\iota$ is an $EW^\sharp$-algebra on $\mathfrak{D}_\iota$ for every $\iota\in\Lambda$. Suppose that $A=(A_\iota)_{\iota\in\Lambda}\in\mathfrak{A}_b$. We can easily show that $A_\iota\in(\mathfrak{A}_\iota)_b$ for every $\iota\in\Lambda$ and $\sup_{\iota\in\Lambda}\|\bar{A}_\iota\|\leqq\|\bar{A}\|$. For each $\xi=(\xi_\iota)_{\iota\in\Lambda}\in\mathfrak{D}(\mathfrak{A})$ we have $A\xi=(\bar{A}_\iota)_{\iota\in\Lambda}\xi$ and hence $\bar{A}=(\bar{A}_\iota)_{\iota\in\Lambda}$, and so $\bar{A}\in\bigoplus_{\iota\in\Lambda}(\overline{\mathfrak{A}_\iota})_b$. Conversely suppose $X=(X_\iota)_{\iota\in\Lambda}\in\bigoplus_{\iota\in\Lambda}(\overline{\mathfrak{A}_\iota})_b$. Then there is an element A_ι in $(\mathfrak{A}_\iota)_b$ such that $X_\iota=\bar{A}_\iota$ for all $\iota\in\Lambda$. Let $A=(A_\iota)_{\iota\in\Lambda}$. We can easily show that $A\in\mathfrak{A}_b$ and $\bar{A}=(\bar{A}_\iota)_{\iota\in\Lambda}=X$. Therefore we have $X\in\bar{\mathfrak{A}}_b$. Consequently we have $\bar{\mathfrak{A}}_b=\bigoplus_{\iota\in\Lambda}(\overline{\mathfrak{A}_\iota})_b$. Since $\bigoplus_{\iota\in\Lambda}(\overline{\mathfrak{A}_\iota})_b$ is a von Neumann algebra, $\mathfrak{A}$ is an $EW^\sharp$-algebra on $\mathfrak{D}(\mathfrak{A})$. Furthermore we have $\mathfrak{A}'=(\bar{\mathfrak{A}}_b)'=(\bigoplus_{\iota\in\Lambda}(\overline{\mathfrak{A}_\iota})_b)'=\bigoplus_{\iota\in\Lambda}((\overline{\mathfrak{A}_\iota})_b)'=\bigoplus_{\iota\in\Lambda}\mathfrak{A}'_\iota$, by Proposition 2.9.

DEFINITION 3.8. Let $\mathfrak{A}$(resp. $\mathfrak{B}$) be a symmetric $\sharp$-algebra on $\mathfrak{D}$(resp. $\mathfrak{E}$). The map. Φ of $\mathfrak{A}$ into $\mathfrak{B}$ is called a homomorphism if it is linear, if $\Phi(ST)=\Phi(S)\Phi(T)$, S, $T\in\mathfrak{A}$, and if $\Phi(S^\sharp)=\Phi(S)^\sharp$, $S\in\mathfrak{A}$. If Φ is a bijective homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then it is called an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are called isomorphic. Let Φ be an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. If there is an isometric mapping U of $\mathfrak{D}$ onto $\mathfrak{E}$ such that $\Phi(S)=USU^{-1}$ for every $S\in\mathfrak{A}$, then Φ is called a spatial isomorphism and we call $\mathfrak{A}$ and $\mathfrak{B}$ are spatial isomorphic and write by $\mathfrak{A}\underset{(s)}{\cong}\mathfrak{B}$.

PROPOSITION 3.9. *Let $\mathfrak{A}$ be a closed $EW^\sharp$-algebra on $\mathfrak{D}$ and let $\{E_\iota\}_{\iota\in\Lambda}$ be a family of mutually orthogonal projections in $\mathfrak{A}'$ such that $\sum_{\iota\in\Lambda}E_\iota=I$. Then there exist a family $\{\mathfrak{A}_\iota\}_{\iota\in\Lambda}$ of $EW^\sharp$-algebras and a spatial isomorphism Φ of $\mathfrak{A}$ onto the $EW^\sharp$-subalgebra of $\prod_{\iota\in\Lambda}\mathfrak{A}_\iota$ such that $\overline{\Phi(\mathfrak{A})}_b=\bigoplus_{\iota\in\Lambda}\overline{(\mathfrak{A}_\iota)}_b$.*

Proof. Let $\mathfrak{D}_\iota = E_\iota\mathfrak{D}$ and let $\mathfrak{H}_\iota$ be the completion of $\mathfrak{D}_\iota$. Then $\mathfrak{A}_\iota = \mathfrak{A}_{E_\iota}$ is a closed $EW^\#$-algebra on $\mathfrak{D}_\iota$ by Theorem 3.5. It is easy to show that Φ; $A \longrightarrow (A_\iota)_{\iota\in A}(A_\iota = A_{E_\iota})$ is an isomorphism of $\mathfrak{A}$ into $\prod_{\iota\in A}\mathfrak{A}_\iota$. We define the mapping U of $\mathfrak{D}$ into $\bigoplus_{\iota\in A}\mathfrak{H}_\iota$ by $U\xi = (E_\iota\xi)_{\iota\in A}$. Then U is an isometric mapping of $\mathfrak{D}$ onto $\mathfrak{D}\Phi(\mathfrak{A})$. In fact, let $\xi\in\mathfrak{D}$ and then $\xi_\iota = E_\iota\xi\in\mathfrak{D}_\iota$ for all $\iota\in A$ and we have, for each $A_\iota\in\mathfrak{A}_\iota$,

$$\sum_{\iota\in A}\|A_\iota\xi_\iota\|^2 = \sum_{\iota\in A}\|AE_\iota\xi\|^2 = \sum_{\iota\in A}\|E_\iota A\xi\|^2 = \|A\xi\|^2 < \infty$$

and hence $(E_\iota\xi)_{\iota\in A}\in\mathfrak{D}\Phi(\mathfrak{A})$. Conversely suppose that $(\xi_\iota)_{\iota\in A}\in\mathfrak{D}(\prod_{\iota\in A}\mathfrak{A})$, i.e., $\xi_\iota\in\mathfrak{D}_\iota = E_\iota\mathfrak{D}$ and $\sum_{\iota\in A}\|A_\iota\xi_\iota\|^2 < \infty$ for all $A\in\mathfrak{A}$. Let $\xi = \sum_{\iota\in A}\xi_\iota$. Then we have

$$\sum_{\iota\in A}\|A\xi_\iota\|^2 = \sum_{\iota\in A}\|AE_\iota\xi\|^2 = \sum_{\iota\in A}\|A_\iota\xi_\iota\|^2 < \infty$$

for all $A\in\mathfrak{A}$ and therefore $\xi\in\mathfrak{D}(\bar{A})$ for all $A\in\mathfrak{A}$. Since $\mathfrak{A}$ is closed, we have $\xi\in\mathfrak{D}$ and $U\xi = (\xi_\iota)_{\iota\in A}$. Consequently U is onto. Furthermore we have

$$\|U\xi\|^2 = \|(E_\iota\xi)_{\iota\in A}\|^2 = \sum_{\iota\in A}\|E_\iota\xi\|^2 = \|\xi\|^2$$

and hence $\bar{U}$ is an isometric mapping of $\mathfrak{H}$ onto $\bigoplus_{\iota\in A}\mathfrak{H}_\iota$. Finally we shall show that $UAU^{-1} = (A_\iota)_{\iota\in A}$ for all $A\in\mathfrak{A}$. For each $\xi\in\mathfrak{D}$ we have

$$UAU^{-1}(E_\iota\xi)_{\iota\in A} = UA\xi = (E_\iota A\xi)_{\iota\in A} = (AE_\iota\xi)_{\iota\in A} = (A_\iota E_\iota\xi)_{\iota\in A}$$

and

$$(A_\iota)_{\iota\in A}(E_\iota\xi)_{\iota\in A} = (A_\iota E_\iota\xi)_{\iota\in A}$$

and hence $UAU^{-1} = (A_\iota)_{\iota\in A}$. By ([3] Ch. I, §2, 2) it is easy to show that $\overline{\Phi(\mathfrak{A})_b} = \bigoplus_{\iota\in A}\overline{(\mathfrak{A}_\iota)_b}$. Consequently $\Phi(\mathfrak{A})$ is an $EW^\#$-subalgebra of $\prod_{\iota\in A}\mathfrak{A}_\iota$ with $\overline{\Phi(\mathfrak{A})_b} = \bigoplus_{\iota\in A}\overline{(\mathfrak{A}_\iota)_b}$.

PROPOSITION 3.10. *Let $\mathfrak{A}_\iota$ be a closed $EW^\#$-algebra on $\mathfrak{D}_\iota$ for all $\iota\in A$ and let $\mathfrak{A} = \prod_{\iota\in A}\mathfrak{A}_\iota$. If $F_\iota\in(\mathfrak{A}'_\iota)_p$ for every $\iota\in A$, then $F = (F_\iota)_{\iota\in A}\in(\mathfrak{A}')_p$ and furthermore we have*

$$\mathfrak{A}_F = \prod_{\iota\in A}(\mathfrak{A}_\iota)_{F_\iota} \quad \textit{and} \quad (\mathfrak{A}_F)' = \bigoplus_{\iota\in A}(\mathfrak{A}'_\iota)_{F_\iota}.$$

Proof. Clearly $F = (F_\iota)_{\iota\in A}\in(\mathfrak{A}')_p$. Let $\mathfrak{B} = \prod_{\iota\in A}(\mathfrak{A}_\iota)_{F_\iota}$. Then we have

$$\mathfrak{D}(\mathfrak{A}_F) = F\mathfrak{D}(\mathfrak{A}) = \{(F_\iota\xi_\iota)_{\iota\in A};\ \xi = (\xi_\iota)_{\iota\in A}\in\mathfrak{D}(\mathfrak{A})\}$$

and

$$\mathfrak{D}(\mathfrak{B}) = \{(F_\iota \xi_\iota)_{\iota \in \Lambda};\ \xi_\iota \in \mathfrak{D}_\iota \ \text{ for all } \ \iota \in \Lambda \ \text{ and } \sum_{\iota \in \Lambda} \|(A_\iota)_{F_\iota} F_\iota \xi_\iota\|^2 < \infty \ \text{ for all } \ A_\iota \in \mathfrak{A}_\iota\} ,$$

and so it is easy to show that $\mathfrak{D}(\mathfrak{A}_F) = \mathfrak{D}(\mathfrak{B})$. Consequently we have $\mathfrak{A}_F = \prod_{\iota \in \Lambda} (\mathfrak{A}_\iota)_{F_\iota}$. By Theorem 3.5 and Theorem 3.7 we have

$$(\mathfrak{A}_F)' = \Big(\prod_{\iota \in \Lambda} (\mathfrak{A}_\iota)_{F_\iota}\Big)' = \bigoplus_{\iota \in \Lambda} ((\mathfrak{A}_\iota)_{F_\iota})' = \bigoplus_{\iota \in \Lambda} (\mathfrak{A}'_\iota)_{F_\iota} .$$

We define the amplification of an $EW^\#$-algebra $\mathfrak{A}$ onto $\mathfrak{A} \tilde{\otimes} I$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be $EW^\#$-algebras on $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively. Let $\mathfrak{D}$ be the subspace of $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ generated by $\{\xi_1 \otimes \xi_2;\ \xi_1 \in \mathfrak{D}_1, \xi_2 \in \mathfrak{D}_2\}$ and denoted by $\mathfrak{D}_1 \otimes \mathfrak{D}_2$. Clearly $\mathfrak{D}$ is a dense subspace of $\mathfrak{H}$. For each $T_1 \in \mathfrak{A}_1$ and $T_2 \in \mathfrak{A}_2$ we get an operator $T_1 \otimes T_2$ on $\mathfrak{D}$ defined by $(T_1 \otimes T_2)(\xi_1 \otimes \xi_2) = (T_1\xi_1) \otimes (T_2\xi_2)$, for each $\xi_1 \in \mathfrak{D}_1$ and $\xi_2 \in \mathfrak{D}_2$ Then we have, for each $T_1, S_1 \in \mathfrak{A}_1$ and $T_2, S_2 \in \mathfrak{A}_2$, $T_1 \otimes T_2$ is a bilinear function of T_1 and T_2; $(T_1 \otimes T_2)(S_1 \otimes S_2) = T_1S_1 \otimes T_2S_2$; $(T_1 \otimes T_2)^\# = T_1^\# \otimes T_2^\#$. Then the following proposition is easily shown.

PROPOSITION 3.11. *Let $\mathfrak{A}_1$ be an $EW^\#$-algebra on $\mathfrak{D}_1$ and let $\mathfrak{H}_2$ be a Hilbert space. Putting*

$$\mathfrak{A}_1 \otimes I_{\mathfrak{H}_2} = \{T_1 \otimes I_{H_2};\ T_1 \in \mathfrak{A}_1\} ,$$

where $I_{\mathfrak{H}_2}$ is an identity operator on $\mathfrak{H}_2$, $\mathfrak{A}_1 \otimes I_{\mathfrak{H}_2}$ is an $EW^\#$-algebra on $\mathfrak{D}_1 \otimes \mathfrak{H}_2$ and we have

$$(\overline{\mathfrak{A}_1 \otimes I_{\mathfrak{H}_2}})_b = (\overline{\mathfrak{A}_1})_b \otimes I_{\mathfrak{H}_2} .$$

Putting

$$\mathfrak{D}_1 \tilde{\otimes} \mathfrak{H}_2 = \bigcap_{T_1 \in \mathfrak{A}_1} \mathfrak{D}(\overline{T_1 \otimes I_{\mathfrak{H}_2}}),\ (T_1 \tilde{\otimes} I_{\mathfrak{H}_2})x = \overline{T_1 \otimes I_{\mathfrak{H}_2}}x,\ x \in \mathfrak{D}_1 \tilde{\otimes} \mathfrak{H}_2 ,$$

$\mathfrak{A}_1 \tilde{\otimes} I_{\mathfrak{H}_2} = \{T_1 \tilde{\otimes} I_{\mathfrak{H}_2};\ T_1 \in \mathfrak{A}_1\}$ is the closure of $\mathfrak{A}_1 \otimes I_{\mathfrak{H}_2}$, and so $\mathfrak{A}_1 \tilde{\otimes} I_{\mathfrak{H}_2}$ is a closed $EW^\#$-algebra on $\mathfrak{D}_1 \tilde{\otimes} \mathfrak{H}_2$.

DEFINITION 3.12. The isomorphism; $T_1 \to T_1 \tilde{\otimes} I_{\mathfrak{H}_2}$ is called an amplification of $\mathfrak{A}_1$ onto $\mathfrak{A}_1 \tilde{\otimes} I_{\mathfrak{H}_2}$.

4. Preduals of $EW^\#$-algebras. Let $\mathfrak{A}$ be a symmetric $\#$-algebra on $\mathfrak{D}$. Let $\mathfrak{H}_\infty = \bigoplus_{n=1}^\infty \mathfrak{H}_n$, where $\mathfrak{H}_n$ is the replica of $\mathfrak{H}$ for $n = 1, 2, \cdots$. For each $\xi = (\xi_1, \xi_2, \cdots, \xi_n, \cdots) \in \mathfrak{D}_\infty(\mathfrak{A})$ and $T \in \mathfrak{A}$, putting $T_\infty\xi_\infty = (T\xi_1, T\xi_2, \cdots, T\xi_n, \cdots)$, we get a linear operator T_∞ on $\mathfrak{D}_\infty(\mathfrak{A})$. Let $\mathfrak{A}_\infty = \{T_\infty;\ T \in \mathfrak{A}\}$. Then we have, for each S and T in $\mathfrak{A}$, $T_\infty + S_\infty = (T + S)_\infty$, $\lambda T_\infty = (\lambda T)_\infty$, $T_\infty S_\infty = (TS)_\infty$, $T_\infty^\# = (T^\#)_\infty$, and so the following lemma is easily shown.

LEMMA 4.1. *Let* $\mathfrak{A}$ *be a* (resp. closed) *symmetric* $\sharp$*-algebra on* $\mathfrak{D}$. *Then* $\mathfrak{A}_\infty$ *is a* (resp. closed) *symmetric* $\sharp$*-algebra on* $\mathfrak{D}_\infty(\mathfrak{A})$. *Furthermore, if* $\mathfrak{A}$ *is an* $EW^\sharp$*-algebra on* $\mathfrak{D}$, *then* $\mathfrak{A}_\infty$ *is an* $EW^\sharp$*-algebra on* $\mathfrak{D}_\infty(\mathfrak{A})$.

Let $\mathfrak{A}$ *be a symmetric* $\sharp$*-algebra on* $\mathfrak{D}$. *A linear functional* φ *on* $\mathfrak{A}$ *is called positive if* $\varphi(A^\sharp A) \geqq 0$ *for every* $A \in \mathfrak{A}$ *and we denote by* $\varphi \geqq 0$.

For each $\xi \in \mathfrak{D}$ *and* $y \in \mathfrak{H}$, *putting*

$$\omega_{\xi,y}(T) = (T\xi \mid y)\,, \quad T \in \mathfrak{A}\,,$$

$\omega_{\xi,y}$ *is a strongly continuous linear functional on* $\mathfrak{A}$. *In particular, we denote* $\omega_{\xi,\xi}(\xi \in \mathfrak{D})$ *by* ω_ξ.

LEMMA 4.2. *Let* $\mathfrak{A}$ *be a closed symmetric* $\sharp$*-algebra on* $\mathfrak{D}$. *Suppose that* φ *is a positive linear functional on* $\mathfrak{A}$ *and* $\xi \in \mathfrak{D}$. *If* $\varphi \leqq \omega_\xi$, *then there exists a* $C \in \mathfrak{A}'$ *such that* $0 \leqq C \leqq I$ *and* $\varphi = \omega_{C\xi}$.

Proof. For each $S, T \in \mathfrak{A}$ we have

$$|\varphi(S^\sharp T)|^2 \leqq \varphi(S^\sharp S)\varphi(T^\sharp T) \leqq \|S\xi\|^2 \|T\xi\|^2\,.$$

Putting $B(T\xi, S\xi) = \varphi(S^\sharp T)$, $B(\ ,\ )$ is an hermitian positive sesquilinear form on $\mathfrak{A}\xi$ with norm $\leqq 1$, so that there is an hermitian positive operator C_0 in $\mathscr{B}(\overline{\mathfrak{A}\xi})$ such that $\|C_0\| \leqq 1$ and for all S and T in $\mathfrak{A}$ $\varphi(S^\sharp T) = (T\xi \mid C_0 S\xi)$. Since $\mathfrak{A}\xi$ is an $\mathfrak{A}$-invariant subspace of $\mathfrak{D}$, the projection E_ξ onto $\overline{\mathfrak{A}\xi}$ belongs to $\mathfrak{A}'$ (Proposition 3.3). Putting $C' = C_0 E_\xi$, for each A, B and T in $\mathfrak{A}$ we have

$$\begin{aligned}(TC'A\xi \mid B\xi) &= (TC_0E_\xi A\xi \mid B\xi) = (TC_0A\xi \mid B\xi) = (A\xi \mid C_0T^\sharp B\xi)\\ &= \varphi((T^\sharp B)^\sharp A) = \varphi(B^\sharp TA) = (TA\xi \mid C_0B\xi)\\ &= (C_0TA\xi \mid B\xi) = (C'TA\xi \mid B\xi)\end{aligned}$$

and since $\mathfrak{A}\xi$ is dense in $E_\xi\mathfrak{D}$ under the induced topology τ_0, we get

$$(TC'E_\xi\xi_1 \mid E_\xi\eta_1) = (C'TE_\xi\xi_1 \mid E_\xi\eta_1)$$

for every $\xi_1, \eta_1 \in \mathfrak{D}$ and furthermore we have

$$(TC'(I - E_\xi)\xi_1 \mid \eta_1) = 0 = (C_0TE_\xi(I - E_\xi)\xi_1 \mid \eta_1) = (C'T(I - E_\xi)\xi_1 \mid \eta_1)\,.$$

Hence we have $(TC'\xi_1 \mid \eta_1) = (C'T\xi_1 \mid \eta_1)$ for every $T \in \mathfrak{A}$ and $\xi_1, \eta_1 \in \mathfrak{D}$. Consequently we get $C' \in \mathfrak{A}'$ and clearly C' is an hermitian positive operator and $\|C'\| \leqq 1$. Now, putting $C = C'^{1/2}$, for all $T \in \mathfrak{A}$,

$$\varphi(T) = (T\xi \mid C'\xi) = \omega_{C\xi}(T)\,.$$

PROPOSITION 4.3. *Let* $\mathfrak{A}$ *be a closed symmetric* $\sharp$*-algebra on* $\mathfrak{D}$

and let φ *be a positive linear functional on* $\mathfrak{A}$. *Then*

(I) *the following conditions are equivalent*;

(1) φ *is weakly continuous*;

(2) $\varphi = \sum_{i=1}^{n} \omega_{\xi_i}$, $\xi_i \in \mathfrak{D}$, $i = 1, 2, \cdots, n$;

(II) *the following conditions are equivalent*;

(3) φ *is* σ-*weakly continuous*;

(4) $\varphi = \sum_{n=1}^{\infty} \omega_{\xi_n}$, $\xi_\infty = (\xi_1, \xi_2, \cdots, \xi_n, \cdots) \in \mathfrak{D}_\infty(\mathfrak{A})$.

Proof. (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3); clear.

(3) $\Rightarrow$ (4); By Lemma 4.1. $\mathfrak{A}_\infty$ is a closed symmetric $\sharp$-algebra on $\mathfrak{D}_\infty(\mathfrak{A})$. Putting $\varphi_\infty(T_\infty) = \varphi(T)$, $T \in \mathfrak{A}$, φ_∞ is a positive linear functional on $\mathfrak{A}_\infty$. Furthermore, since φ is σ-weakly continuous, there is an $\eta_\infty = (\eta_1, \eta_2, \cdots, \eta_n, \cdots)$ in $\mathfrak{D}_\infty(\mathfrak{A})$ such that

$$|\varphi_\infty(T_\infty)| = |\varphi(T)| \leqq |\sum_{n=1}^{\infty} (T\eta_n | \eta_n)| = |(T_\infty \eta_\infty | \eta_\infty)| \,.$$

Hence φ_∞ is a positive linear functional on $\mathfrak{A}_\infty$ and $\varphi_\infty \leqq \omega_{\eta_\infty}$. By Lemma 4.2. there is a $\xi_\infty = (\xi_1, \xi_2, \cdots, \xi_n, \cdots)$ in $\mathfrak{D}_\infty(\mathfrak{A})$ such that $\varphi_\infty = \omega_{\xi_\infty}$. For each $T \in \mathfrak{A}$ we have

$$\varphi(T) = \varphi_\infty(T_\infty) = \omega_{\xi_\infty}(T) = \sum_{n=1}^{\infty} (T\xi_n | \xi_n) = \sum_{n=1}^{\infty} \omega_{\xi_n}(T) \,.$$

(1) $\Rightarrow$ (2); By a slight modification of the argument (3) $\Rightarrow$ (4) we can easily show (1) $\Rightarrow$ (2).

DEFINITION 4.4. We denote by $\mathfrak{A}_*$(resp. $\mathfrak{A}_*^+$) the set of all σ-weakly continuous (resp. positive) linear functionals on $\mathfrak{A}$ and $\mathfrak{A}_*$ is called the predual of $\mathfrak{A}$.

For $A \in \mathfrak{A}$ and $f \in \mathfrak{A}_*$, we define actions of $\mathfrak{A}$ on f by;

$$(fA)(T) = f(AT), (Af)(T) = f(TA)$$

for each $T \in \mathfrak{A}$. Then we have $fA, Af \in \mathfrak{A}_*$.

Let $\mathfrak{A}$ be a closed $EW^\sharp$-algebra on $\mathfrak{D}$. By Lemma 4.1. $\mathfrak{A}_\infty$ is a closed $EW^\sharp$-algebra on $\mathfrak{D}_\infty(\mathfrak{A})$. For each $T \in \mathfrak{A}$ and $\varphi \in \mathfrak{A}_*$(resp. $\mathfrak{A}_*^+$) putting $\varphi_\infty(T_\infty) = \varphi(T)$, φ_∞ is a weakly continuous (resp. positive) linear functional on $\mathfrak{A}_\infty$. Moreover, for each $T \in \mathfrak{A}_b$ and $\varphi \in \mathfrak{A}_*$(resp. $\mathfrak{A}_*^+$) putting $\bar{\varphi}(\bar{T}) = \varphi(T)$, $\bar{\varphi}$ belongs to the predual $(\overline{\mathfrak{A}}_b)_*$(resp. $(\overline{\mathfrak{A}}_b)_*^+$) of a von Neumann algebra $\overline{\mathfrak{A}}_b$.

LEMMA 4.5. *Let* $\mathfrak{A}$ *be a closed* $EW^\sharp$-*algebra on* $\mathfrak{D}$. *Let* φ *and* ψ *in* $\mathfrak{A}_*$.

(1) *If* $\bar{\varphi} = \bar{\psi}$, *then* $\varphi = \psi$.

(2) *If* $\bar{\varphi} \geqq 0$, *then* $\varphi \geqq 0$.

Proof. (1) For each $T \in \mathfrak{A}$, let $T_\infty = U_\infty |T_\infty|$ be the polar decomposition of T_∞. Then we have $U_\infty \in (\mathfrak{A}_\infty)_b$ and $|T_\infty| \in (\mathfrak{A}_\infty)_h^+$. Let $\overline{|T_\infty|} = \int_0^\infty \lambda dE(\lambda)$ be the spectral decomposition of $\overline{|T_\infty|}$ and for each n, putting $\bar{X}_n = \int_0^n \lambda dE(\lambda)$, we get $X_n \in (\mathfrak{A}_\infty)_b$. Since $\mathfrak{D}_\infty(\mathfrak{A}) \subset \mathfrak{D}(\overline{|T_\infty|})$, for each $\xi_\infty \in \mathfrak{D}_\infty(\mathfrak{A})$ we have $\lim_{n\to\infty} X_n\xi_\infty = \overline{|T_\infty|}\xi_\infty = |T_\infty|\xi_\infty$ and hence $\lim_{n\to\infty} U_\infty X_n \xi_\infty = U_\infty \overline{|T_\infty|}\xi_\infty = T_\infty \xi_\infty$. That is, $U_\infty X_n$ converges strongly to T_∞. Since φ_∞ and ψ_∞ are weakly continuous, we have

$$\lim_{n\to\infty} \varphi_\infty(U_\infty X_n) = \varphi_\infty(T_\infty) = \varphi(T)$$

and $\lim_{n\to\infty} \psi_\infty(U_\infty X_n) = \psi_\infty(T) = \psi(T)$ and furthermore $\bar{\varphi} = \bar{\psi}$ and $U_\infty X_n \in (\mathfrak{A}_\infty)_b$, and so we have $\varphi_\infty(U_\infty X_n) = \psi_\infty(U_\infty X_n)$. Therefore we get $\varphi(T) = \psi(T)$.

(2) Suppose $T \in \mathfrak{A}_h^+$. Then it is easy to show $T_\infty \in (\mathfrak{A}_\infty)_h^+$. Let $\overline{T_\infty} = \int_0^\infty \lambda dE(\lambda)$ be the spectral decomposition of $\overline{T_\infty}$ and putting, for each n, $\overline{X_n} = \int_0^n \lambda dE(\lambda)$. By (1), we have $\lim_{n\to\infty} \varphi_\infty(X_n) = \varphi_\infty(T_\infty)$. Furthermore, since $\bar{\varphi} \geqq 0$ and $X_n \in (\mathfrak{A}_\infty)_b^+$, $\varphi_\infty(X_n) \geqq 0$ for each n. Therefore we get $\varphi(T) = \varphi_\infty(T_\infty) \geqq 0$.

PROPOSITION 4.6. *Suppose that $\mathfrak{A}$ is a closed $EW^\sharp$-algebra on $\mathfrak{D}$ and $f \in \mathfrak{A}_*$. Then there exists a couple (φ, U) with the following properties;*

(a) $\varphi \in \mathfrak{A}_*^+$ *and* $\|\bar{\varphi}\| = \|\bar{f}\|$;

(b) $\bar{U}$ *is a partial isometry of* $\bar{\mathfrak{A}}_b$ *having* $S(\bar{\varphi})$ *as the final projection* $\bar{U}\bar{U}^* = \overline{UU^\sharp}$, *where* $S(\bar{\varphi})$ *denotes the support of* $\bar{\varphi}$;

(c) $f = \varphi U$, $\varphi = f U^\sharp$;

(d) *such decomposition is unique.*

Proof. Using Lemma 4.5 and the polar decomposition of a σ-weakly continuous linear functional $\bar{f}$ on a von Neumann algebra $\bar{\mathfrak{A}}_b$, we can easily show Proposition 4.6.

DEFINITION 4.7. The φ of Proposition 4.6 is called the absolute value of f and we denote φ by $|f|$. This decomposition is called the polar decomposition of f.

THEOREM 4.8. *Let $\mathfrak{A}$ be a closed $EW^\sharp$-algebra on $\mathfrak{D}$.*

(I) *The following conditions are equivalent;*

(1) f *is weakly continuous;*

(2) $f = \sum_{i=1}^n \omega_{\xi_i, \eta_i}$, $\xi_i, \eta_i \in \mathfrak{D}(i = 1, 2, \cdots, n)$.

(II) *The following conditions are equivalent;*

(3) $f \in \mathfrak{A}_*$;

(4) $f = \sum_{n=1}^{\infty} \omega_{\xi_n, \eta_n}$, $\xi_\infty = (\xi_1, \xi_2, \cdots, \xi_n, \cdots)$, $\eta_\infty = (\eta_1, \cdots, \eta_n, \cdots) \in \mathfrak{D}_\infty(\mathfrak{A})$.

Proof. (2) ⇒ (1) and (4) ⇒ (3); clear.

(3) ⇒ (4) Suppose $f \in \mathfrak{A}_*$. Let $f = |f|U$ be the polar decomposition of f. By Proposition 4.3 there is a $\xi_\infty = (\xi_1, \xi_2, \cdots, \xi_n, \cdots) \in \mathfrak{D}_\infty(\mathfrak{A})$ such that $|f| = \sum_{n=1}^{\infty} \omega_{\xi_n}$. For each $T \in \mathfrak{A}$ we have

$$f(T) = (|f|U)(T) = \sum_{n=1}^{\infty} (UT\xi_n | \xi_n) = \sum_{n=1}^{\infty} (T\xi_n | U^\# \xi_n),$$

and so putting $\eta_n = U^\# \xi_n$, $n = 1, 2, \cdots$, $\eta_\infty = (\eta_1, \eta_2, \cdots, \eta_n, \cdots) \in \mathfrak{D}_\infty(\mathfrak{A})$ and $f = \sum_{n=1}^{\infty} \omega_{\xi_n, \eta_n}$.

(1) ⇒ (2) By a slight modification of the argument (3) ⇒ (4), (1) ⇒ (2) is easily shown.

5. The structure of a σ-weakly continuous homomorphism. In this section we shall show that a σ-weakly continuous homomorphism of a closed $EW^\#$-algebra is decomposed in the following three types; a spatial isomorphism, an induction and an amplification.

DEFINITION 5.1. Let $\mathfrak{A}$(resp. $\mathfrak{B}$, $\mathfrak{B}_1$) be a symmetric $\#$-algebra on $\mathfrak{D}$(resp. $\mathfrak{E}$, $\mathfrak{E}_1$). Let Φ(resp. Φ_1) be a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$(resp. $\mathfrak{B}_1$). Then Φ and Φ_1 are called unitarily equivalent if there is an isometric isomorphism U of $\mathfrak{E}$ onto $\mathfrak{E}_1$ such that

$$U\Phi(T)\xi = \Phi_1(T)U\xi$$

for all $T \in A$ and $\xi \in \mathfrak{E}$ and we denote by $\Phi \cong \Phi_1$.

LEMMA 5.2. *Let $\mathfrak{A}$ be a closed $EW^\#$-algebra on $\mathfrak{D}$ and $\varphi = \sum_{i=1}^{n} \omega_{\xi_i}$, $\xi_i \in \mathfrak{D} (i = 1, \cdots, n)$*(resp. $\varphi = \sum_{i=1}^{\infty} \omega_{\xi_i}$, $\xi_\infty = (\xi_1, \cdots, \xi_n, \cdots) \in \mathfrak{D}_\infty(\mathfrak{A})$). *Let $\mathfrak{K}$ be a Hilbert space with dimension n*(resp. a separable Hilbert space) *and let Φ be an amplification $T \to T \tilde{\otimes} I_\mathfrak{K}$ of $\mathfrak{A}$ onto $\mathfrak{A} \tilde{\otimes} I_\mathfrak{K}$. Then there exists an element x of $\mathfrak{D} \tilde{\otimes} \mathfrak{K}$ such that $\varphi(T) = (\Phi(T)x | x)$ for all $T \in \mathfrak{A}$.*

Proof. Suppose that $\{e_i\}_{i=1,2,\dots}$ is an orthogonal basis in $\mathfrak{K}$. Let $x = \sum_{i=1}^{\infty} \xi_i \otimes e_i$. Then we have $\sum_{i=1}^{n} \xi_i \otimes e_i \to x (n \to \infty)$ and

$$\left\| (T \otimes I_\mathfrak{K}) \sum_{i=n}^{m} \xi_i \otimes e_i \right\|^2 = \left\| \sum_{i=n}^{m} T\xi_i \otimes e_i \right\|^2 = \sum_{i=n}^{m} \| T\xi_i \|^2$$
$$\longrightarrow 0 \quad (n, m \longrightarrow \infty)$$

and hence we get, for all $T \in \mathfrak{A}$, $x \in \mathfrak{D}(\overline{T \otimes I_\mathfrak{K}})$ and $\overline{T \otimes I_\mathfrak{K}} x =$

$\sum_{i=1}^{\infty} T\xi_i \otimes e_i$. That is, $x \in \mathfrak{D} \tilde{\otimes} \mathfrak{K}$ and $(T \tilde{\otimes} I_{\mathfrak{K}})x = \sum_{i=1}^{\infty} T\xi_i \otimes e_i$. Furthermore we have

$$(\Phi(T)x \mid x) = ((T \tilde{\otimes} I_{\mathfrak{K}})x \mid x)$$
$$= \left(\sum_{i=1}^{\infty} T\xi_i \otimes e_i \middle| \sum_{i=1}^{\infty} \xi_i \otimes e_i\right) = \sum_{i=1}^{\infty} (T\xi_i \mid \xi_i) = \varphi(T) .$$

Let $\mathfrak{A}$ be a closed symmetric $\sharp$-algebra on $\mathfrak{D}$ and let $\xi \in \mathfrak{D}$. We denote by $\mathfrak{X}_\xi^{\mathfrak{A}}$ the subspace of $\mathfrak{D}$ generated by $\{T\xi;\ T \in \mathfrak{A}\}$. Let $(\mathfrak{X}_\xi^{\mathfrak{A}})^-$ be the closure of $\mathfrak{X}_\xi^{\mathfrak{A}}$ under the induced topology τ_0 and let $E_\xi^{\mathfrak{A}}$ be the projection onto $\mathfrak{X}_\xi^{\mathfrak{A}}$. Then, by Proposition 3.3, $E_\xi^{\mathfrak{A}} \in \mathfrak{A}'$ and $E_\xi^{\mathfrak{A}}\mathfrak{D} = (\mathfrak{X}_\xi^{\mathfrak{A}})^-$.

DEFINITION 5.3. If $(\mathfrak{X}_\xi^{\mathfrak{A}})^- = \mathfrak{D}$, then ξ is called a strongly cyclic vector for $\mathfrak{A}$.

LEMMA 5.4. *Let* $\mathfrak{A}$(resp. $\mathfrak{B}$, $\mathfrak{B}_1$) *be a closed symmetric* $\sharp$*-algebra on* $\mathfrak{D}$(resp. $\mathfrak{E}$, $\mathfrak{E}_1$) *and let* Φ(resp. Φ_1) *be a homomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$(resp. $\mathfrak{B}_1$). *If there is a strongly cyclic vector* $\xi \in \mathfrak{E}$(resp. $\xi_1 \in \mathfrak{E}_1$) *for* $\mathfrak{B}$(resp. $\mathfrak{B}_1$) *such that*

$$(\Phi(T)\xi \mid \xi) = (\Phi_1(T)\xi_1 \mid \xi_1)$$

for all $T \in \mathfrak{A}$, *then* $\Phi \cong \Phi_1$.

Proof. Putting U_0; $\Phi(T)\xi \longrightarrow \Phi_1(T)\xi_1$, we have, for all $T \in \mathfrak{A}$,

$$\| U_0\Phi(T)\xi \|^2 = \| \Phi(T)\xi \|^2 ,$$

so that U_0 is an isometric isomorphism of $\Phi(\mathfrak{A})\xi$ onto $\Phi_1(\mathfrak{A})\xi_1$ and furthermore, since ξ(resp. ξ_1) is a cyclic vector for $\mathfrak{B}$(resp. $\mathfrak{B}_1$), U_0 is extended to an isometric isomorphism U of $\mathfrak{K} = \bar{\mathfrak{E}}$ onto $\mathfrak{K}_1 = \bar{\mathfrak{E}}_1$. For each $\eta \in \mathfrak{E}$ there is a net $\{T_\alpha\}$ in $\mathfrak{A}$ such that $\lim_\alpha \Phi(T)\Phi(T_\alpha)\xi = \Phi(T)\eta$ for all $T \in \mathfrak{A}$ and then we have $\lim_\alpha \Phi_1(T_\alpha)\xi_1 = \lim U\Phi(T_\alpha)\xi = U\eta$ and $\lim_\alpha \Phi_1(T)\Phi_1(T_\alpha)\xi_1 = \lim_\alpha U\Phi(T)\Phi(T_\alpha)\xi = U\Phi(T)\eta$, so that we get $U\eta \in \bigcap_{T \in \mathfrak{A}} \mathfrak{D}(\overline{\Phi_1(T)}) = \mathfrak{E}_1$ and $\Phi_1(T)U\eta = \overline{\Phi_1(T)}U\eta = U\Phi(T)\eta$ for all $T \in \mathfrak{A}$. Similarly we can show $U\mathfrak{E} \supset \mathfrak{E}_1$ and therefore $\Phi \cong \Phi_1$.

THEOREM 5.5. *Let* $\mathfrak{A}$(resp. $\mathfrak{B}$) *be a closed* $EW^\sharp$*-algebra on* $\mathfrak{D}$(resp. $\mathfrak{E}$) *and let* Φ *be a* σ*-weakly continuous homomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$. *Then there exist an amplification* Φ_1 *of* $\mathfrak{A}$ *onto a closed* $EW^\sharp$*-algebra* $\mathfrak{A}_1$ *on* $\mathfrak{D}_1$, *an induction* Φ_2 *of* $\mathfrak{A}_1$ *onto a closed* $EW^\sharp$*-algebra* $\mathfrak{A}_2$ *on* $\mathfrak{D}_2$ *and a spatial isomorphism* Φ_3 *of* $\mathfrak{A}_2$ *onto* $\mathfrak{B}$ *such that* $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$.

Proof. (1) Suppose that $\mathfrak{B}$ has a strongly cyclic vector $\eta \in \mathfrak{E}$. Putting

$$\varphi(T) = (\Phi(T)\eta \mid \eta)\ ,\quad T \in \mathfrak{A}\ ,$$

φ is a σ-weakly continuous positive linear functional on $\mathfrak{A}$. By Proposition 4.3 there exists a $\xi_\infty = (\xi_1, \xi_2, \cdots, \xi_n, \cdots) \in \mathfrak{D}_\infty(\mathfrak{A})$ such that $\varphi = \sum_{i=1}^{\infty} \omega_{\xi_i}$. Let $\mathfrak{D}_1 = \tilde{\mathfrak{D}} \tilde{\otimes} \mathfrak{K}_1$($\mathfrak{K}_1$; a separable Hilbert space), let $\mathfrak{A}_1 = \mathfrak{A} \tilde{\otimes} I_{\mathfrak{K}_1}$ and let Φ_1 be an amlification of $\mathfrak{A}$ onto $\mathfrak{A}_1$. By Lemma 5.2 there exists an element x of $\mathfrak{D}_1$ such that $\varphi(T) = (\Phi_1(T)x \mid x)$ for all $T \in \mathfrak{A}$. By Proposition 3.11 $\mathfrak{A}_1$ is a closed $EW^{\#}$-algebra on $\mathfrak{D}_1$. Let $\mathfrak{D}_2 = (\mathfrak{X}_x^{\mathfrak{A}_1})^-$ and let $E = E_x^{\mathfrak{A}_1}$. Let $\mathfrak{A}_2 = (\mathfrak{A}_1)_E$ and let Φ_2 be an induction of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. By Theorem 3.5 $\mathfrak{A}_2$ is a closed $EW^{\#}$-algebra on $\mathfrak{D}_2$ and

$$(\Phi(T)\eta \mid \eta) = \varphi(T) = (\Phi_1(T)x \mid x) = ((\Phi_2 \circ \Phi_1)(T)x \mid x)$$

for all $T \in \mathfrak{A}$. Furthermore, $\Phi_2 \circ \Phi_1$ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_2$, Φ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ and x(resp. η) is a strongly cyclic vector for $\mathfrak{A}_2$(resp. $\mathfrak{B}$), so that, by Lemma 5.4, we get $\Phi \cong \Phi_2 \circ \Phi_1$. Putting

$$\Phi_3;\ \Phi_2 \circ \Phi_1(T) \longrightarrow \Phi(T)\ ,\quad T \in \mathfrak{A}\ ,$$

Φ_3 is a spatial isomorphism of $\mathfrak{A}_2$ onto $\mathfrak{B}$. Clearly we have $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$.

(2) In a general case we shall prove the theorem. Suppose that $\{\eta_\iota\}_{\iota \in \Lambda}$ is a maximal family such that $\{\eta_\iota\}_{\iota \in \Lambda} \subset \mathfrak{E}$ and $\mathfrak{E}_\iota = (\mathfrak{X}_{\eta_\iota}^{\mathfrak{B}})^-$ is mutually orthogonal. Let $E_\iota = E_{\eta_\iota}^{\mathfrak{B}}$ for every $\iota \in \Lambda$ and then $E_\iota \in \mathfrak{B}'$ and furthermore we get $\sum_{\iota \in \Lambda} E_\iota = I$, by the maximality of $\{\eta_\iota\}_{\iota \in \Lambda}$. For each $\iota \in \Lambda$ putting

$$\mathfrak{B}_\iota = \mathfrak{B}_{E_\iota} \quad \text{and} \quad \Phi^\iota(T) = \Phi(T)_{E_\iota}\ ,\quad T \in \mathfrak{A}\ ,$$

Φ^ι is a σ-weakly continuous homomorphism and $\mathfrak{B}_\iota$ is a closed $EW^{\#}$-algebra on $\mathfrak{E}_\iota = E_\iota \mathfrak{E}$ with a strongly cyclic vector η_ι. By (1), for each $\iota \in \Lambda$, there exist an amplification Φ_1^ι of $\mathfrak{A}$ onto a closed $EW^{\#}$-algebra $\mathfrak{A}_1^\iota = \mathfrak{A} \tilde{\otimes} I_{\mathfrak{K}_1^\iota}$ on $\mathfrak{D} \tilde{\otimes} \mathfrak{K}_1^\iota$, an induction Φ_2^ι of $\mathfrak{A}_1^\iota$ onto a closed $EW^{\#}$-algebra $\mathfrak{A}_2^\iota = (\mathfrak{A}_1^\iota)_{F_\iota}(F_\iota \in (\mathfrak{A}_1^\iota)'_p)$ on $\mathfrak{D}_2^\iota = F_\iota \mathfrak{D}_1$ and a spatial isomorphism Φ_3^ι of $\mathfrak{A}_2^\iota$ onto $\mathfrak{B}_\iota$ such that $\Phi^\iota = \Phi_3^\iota \circ \Phi_2^\iota \circ \Phi_1^\iota$. Let $\mathfrak{K}_1 = \bigoplus_{\iota \in \Lambda} \mathfrak{K}_1^\iota$, $\mathfrak{A}_1 = \mathfrak{A} \tilde{\otimes} I_{\mathfrak{K}_1}$ and let Φ_1 be an amplification of $\mathfrak{A}$ onto $\mathfrak{A}_1$. It is easy to show that $\mathfrak{A}_1 \underset{(s)}{\cong} \{(T \tilde{\otimes} I_{\mathfrak{K}_1^\iota})_{\iota \in \Lambda} \in \prod_{\iota \in \Lambda} \mathfrak{A}_1^\iota;\ T \in \mathfrak{A}\}$. For each $\iota \in \Lambda$ we have $F_\iota \in (\mathfrak{A}_1^\iota)'_p = (\mathfrak{A} \otimes I_{\mathfrak{K}_1^\iota})'_p = (\mathfrak{A}' \otimes \mathscr{B}(\mathfrak{K}_1^\iota))_p$, so that $F = (F_\iota)_{\iota \in \Lambda} \in \bigoplus_{\iota \in \Lambda} (\mathfrak{A}' \otimes \mathscr{B}(\mathfrak{K}_1^\iota))_p = (\mathfrak{A}' \otimes \mathscr{B}(\mathfrak{K}_1))_p = (\mathfrak{A}_1')_p$. Let $\mathfrak{A}_2 = (\mathfrak{A}_1)_F$. Then $\mathfrak{A}_2$ is a closed $EW^{\#}$-algebra on $\mathfrak{D}_2 = F\mathfrak{D}_1$. Let Φ_2 be an induction of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ and let Φ_3; $\Phi_2 \circ \Phi_1(T) \to \Phi(T)$, $T \in \mathfrak{A}$. We shall show that Φ_3 is a spatial isomorphism of $\mathfrak{A}_2$ onto $\mathfrak{B}$. For each $T \in \mathfrak{A}$ we have

$$\begin{aligned}\Phi_2\circ\Phi_1(T) &= (T \tilde{\otimes} I_{\mathfrak{K}_1})_F = ((T\tilde{\otimes} I_{\mathfrak{K}_1^\iota})_{\iota\in\Lambda})_{(F)_{\iota\in\Lambda}} \\ &= ((T \tilde{\otimes} I_{\mathfrak{K}_1^\iota})_{F_\iota})_{\iota\in\Lambda} \text{ (by Proposition 3.10)} \\ &= ((\Phi_2^\iota\circ\Phi_1^\iota)(T))_{\iota\in\Lambda}\,.\end{aligned}$$

On the other hand, by Proposition 3.9, $\mathfrak{B}$ is spatially isomorphic to $\{(\Phi^\iota(T))_{\iota\in\Lambda}\in\prod_{\iota\in\Lambda}\mathfrak{B}_\iota;\ T\in\mathfrak{A}\}$. Furthermore, since Φ_3^ι is a spatial isomorphism for each $\iota\in\Lambda$, we get $((\Phi_2^\iota\circ\Phi_1^\iota)(T))_{\iota\in\Lambda}\longrightarrow(\Phi^\iota(T))_{\iota\in\Lambda}$ is a spatial isomorphism, i.e., Φ_3 is a spatial isomorphism of $\mathfrak{A}_2$ onto $\mathfrak{B}$.

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