

# Pacific Journal of Mathematics

**LINKS WHICH ARE UNKNOTTABLE BY MAPS**

HOWARD WILSON LAMBERT

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**Let  $L$  be a piecewise linear (PL) link of two components in the Euclidean 3-sphere  $S^3$  (i.e.,  $L = L_1 \cup L_2$  where  $L_1, L_2$  are disjoint polygonal simple closed curves in  $S^3$ ). In Theorem 1 of this paper we give a geometric condition on  $L$  which implies it is unknottable. In Theorem 2, we show that there is an infinite class of links of two components which are unknottable.**

We call a continuous (PL) map  $f: S^3 \rightarrow S^3$  strongly 1-1 on  $L$  if  $f|L$  is a homeomorphism onto  $f(L)$ ,  $f(S^3 - L) \cap f(L) = \emptyset$  and  $f$  is locally 1-1 at each point of  $L$ . In Theorem 1 of [3], the link  $L_0 = L_{01} \cup L_{02}$  where  $L_{01}$  is unknotted and  $L_{02}$  is the square knot is shown to have the property that there is no strongly 1-1 map  $f$  on  $L_0$  such that  $f(L_{01})$  and  $f(L_{02})$  are unknotted. Call  $L$  "unknottable" if there does not exist a strongly 1-1 map  $f$  on  $L$  such that  $f(L_1)$  and  $f(L_2)$  are unknotted. This paper and [3] resulted from an attempt to generalize Hempel's result [2] that given any knot  $K$  in  $S^3$  there exists a strongly 1-1 map  $f$  on  $K$  such that  $f(K)$  is unknotted.

Let  $S_1$  be a (PL) orientable surface such that  $\text{Bd } S_1 = L_1$  and  $L_2$  intersects and pierces  $S_1$  in a finite number of points. Let  $N(L) = N(L_1) \cup N(L_2)$  be a regular neighborhood of  $L$  such that  $S_1 \cap N(L_1)$  is an annulus and  $S_1 \cap N(L_2)$  consists of transverse disks. Call  $S_1$  essential if  $S_1 - \text{Int } N(L)$  is incompressible [7] and boundary incompressible [7] in  $S^3 - \text{Int } N(L)$ .

**DEFINITION 1.**  $L$  is boundary incompressibly unlinked with respect to  $L_1$  (B.I.U.) if, whenever  $S_1$  is essential, we have  $S_1 \cap L_2 = \emptyset$ .  $L$  is said to be 1-linked [5] if  $L_1, L_2$  do not bound disjoint orientable surfaces in  $S^3$ .

**THEOREM 1.** *If  $L$  is 1-linked, B.I.U. and  $L_1$  is knotted, then  $L$  is unknottable.*

*Proof.* Suppose there exists a  $f: S^3 \rightarrow S^3$  which is strongly 1-1 on  $L$  and  $f(L_1), f(L_2)$  are unknotted. Let  $D_1$  be a disk in  $S^3$  such that  $\text{Bd } D_1 = f(L_1)$  and  $f(L_2)$  intersects and pierces  $D_1$  in a finite number  $t$  of points. Suppose also that  $t$  is chosen to be smallest possible. Now, following the techniques used in [7], we adjust  $f$  so that it is transverse to  $D_1$ , in particular  $D'_1 = f^{-1}(D_1) \cap (S^3 - \text{Int } N(L))$  is an

orientable surface with one boundary component in  $\text{Bd } N(L_1)$  which is a longitude of  $N(L_1)$  and  $t$  boundary components in  $\text{Bd } N(L_2)$ , each of which is a meridian of  $N(L_2)$ . Now suppose  $D'_1$  is compressible in  $S^3 - \text{Int } N(L)$ , i.e. there exists a disk  $Q$  in  $S^3 - \text{Int } N(L)$  such that  $Q \cap D'_1 = \text{Bd } Q \cap D'_1 = \text{Bd } Q$  and  $\text{Bd } Q$  does not bound a disk in  $D'_1$ . Now if the loop  $f(\text{Bd } Q)$  separates a point of  $D_1 \cap f(L_2)$  from  $\text{Bd } D_1$ , we may apply Dehn's lemma [4] to conclude that  $t$  was not minimal. If  $f(\text{Bd } Q)$  separates no point of  $D_1 \cap f(L_2)$  from  $\text{Bd } D_1$ , then we may cut out a small regular neighborhood of  $\text{Bd } Q$  in  $D'_1$  and add two parallel copies of  $Q$  to form a new orientable surface  $D'_1$  with less genus than  $D'_1$ . We may then redefine the map  $f$  so that  $D'_1 = f^{-1}(D_1) \cap (S^3 - \text{Int } N(L))$ . If  $D'_1$  is boundary compressible, then there exists a disk  $Q$  such that  $\text{Int } Q \cap D'_1 = \emptyset$  and  $\text{Bd } Q$  consists of two arcs, one in  $\text{Bd } N(L_2)$ , the other in  $D'_1$  and the arc in  $D'_1$  together with any arc in  $\text{Bd } D'_1$  do not bound a disk in  $D'_1$ . In this case we may use a modified version of the loop theorem (see [6]) on the loop  $f(\text{Bd } Q)$  in  $S^3 - \text{Int } f(N(L))$  to conclude that  $t$  was not minimal. Hence we may assume that  $D'_1$  is incompressible and boundary incompressible. Since  $L$  is B.I.U. we have  $t = 0$ . Then  $f(L_2)$  bounds a disk  $D_2$  which is disjoint from  $D_1$ . We may adjust  $f$  so that  $f^{-1}(D_1)$ ,  $f^{-1}(D_2)$  are disjoint orientable surfaces, contradicting the assumption that  $L$  is 1-linked, and the proof is complete.

We now define the class of links  $L_{1j} \cup L_{2j}$  illustrated in Figures 1 and 2. Each  $L_{1j}$  is a curve with  $j$  full twists ( $j$  is any positive or negative integer and one of the full twists is shown in the figure). If  $j \neq 0$ , then in [1] it is shown that  $L_{1j}$  is knotted.

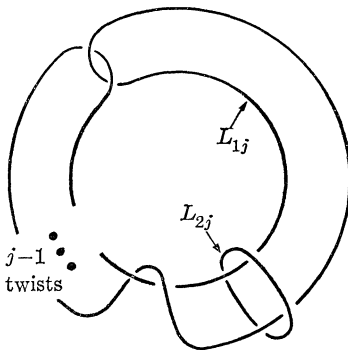


FIGURE 1

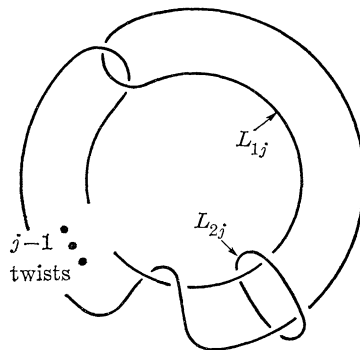


FIGURE 2

LEMMA 1.  $L_{1j} \cup L_{2j}$  is 1-linked for all  $j$ .

*Proof.* Suppose  $L_{1j}$ ,  $L_{2j}$  bound disjoint orientable surfaces  $S_{1j}$ ,  $S_{2j}$ , respectively. Let  $D'$  be a disk bounded by  $L_{2j}$  such that  $L_{1j}$  intersects and pierces  $D'$  in two points and the two components of

$L_{1j} - D'$  self link each other. By cut and paste techniques (see [7] or we used some of these methods in Theorem 1) we may assume that  $(\text{Int } D') \cap S_{2j} = \emptyset$  and  $D' \cap S_{1j}$  consists of one arc connecting the two points of  $D' \cap L_{1j}$ . Let  $D''$  be a disk whose boundary consists of the arc  $D' \cap S_{1j}$  and one of the two arcs of  $L_{1j} - D'$ . Assume  $D'' \cap D' = D' \cap S_{1j}$  and the other arc of  $L_{1j} - D'$  intersects and pierces  $D''$  in one point. But it now follows that there is a curve in  $S_{2j} \cap D''$  which is not homologous to zero in  $S^3 - L_{1j}$ , contradicting that  $S_{1j} \cap S_{2j} = \emptyset$ .

In Figure 3 we view  $L_{1j}$  as being contained in a cube with two handles  $C$  where  $N(L_{1j}) \subset \text{Int } C \subset S^3 - L_{2j}$ . Let  $H_1, H_2$  be the two annuli illustrated in Figure 3, where  $H_1 \cap H_2$  is an arc.

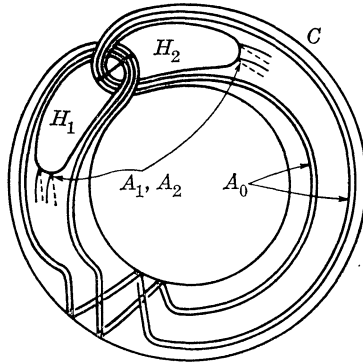


FIGURE 3

LEMMA 2. *Each link  $L_{1j} \cup L_{2j}$ ,  $j \neq 0$ , is boundary incompressibly unlinked (B.I.U.).*

*Proof.* Suppose  $S'$  is an orientable surface in the solid torus  $T = S^3 - \text{Int } N(L_{2j})$  with one boundary component  $L_{1j}$  and each of the remaining  $t$  boundary components is a meridian of  $N(L_{2j})$  in  $\text{Bd } N(L_{2j})$ . Suppose also that  $S = S' - \text{Int } N(L_{1j})$  is incompressible and boundary incompressible in  $T - \text{Int } N(L_{1j})$ . We may choose the cube with two handles  $C$  so that  $S' \cap C$  consists of an annulus  $A_0$  and  $s$  disks  $A_1, \dots, A_s$  (see Figure 3). Now, by following the techniques used in Lemma 1 of [3], we may adjust  $S'$  so that  $S' \cap H_1$  is one arc parallel to  $H_1 \cap H_2$  in  $H_1$ . (To see this, put  $S'$  in general position relative to  $H_1$  and push arcs of  $S' \cap H_1$  with both endpoints in the same component of  $\text{Bd } H_1$  off  $H_1$  and then off  $C$ , i.e. we reduce  $s$  by 1 or 2 and hence we may suppose  $s = 0$ .) By the same reasoning we may suppose further that  $S' \cap H_2$  consists of one arc parallel to  $H_1 \cap H_2$  in  $H_2$ . Let  $N(H_1), N(H_2)$  be regular neighborhoods of  $H_1, H_2$ , resp., taken in  $T - \text{Int } C$ . Let  $T'$  be the solid torus

$C \cup N(H_1) \cup N(H_2)$ . Then  $T - \text{Int } T'$  is homeomorphic to the product space  $(S^1 \times S^1) \times I$ . None of the three simple closed curves of  $S' \cap \text{Bd } T'$  is homotopic to the  $t$  curves of  $S' \cap \text{Bd } T$ . (Note that one component of  $S' \cap \text{Bd } T'$  bounds a disk in  $\text{Bd } T'$  and the other two go once around the longitude of  $T'$  and  $j$  times,  $j \neq 0$ , around the meridian of  $T'$ .) Since  $S$  is incompressible and boundary incompressible, it follows that  $\pi_1(S \cap (T - \text{Int } T'))$  injects into the abelian group  $\pi_1(T - \text{Int } T')$ . Hence  $S \cap (T - \text{Int } T')$  consists of one disk and one annulus, so  $t = 0$  and the proof of Lemma 2 is finished.

Theorem 1, Lemma 1 and Lemma 2 now imply the following:

**THEOREM 2.** *Each of the links  $L_{1j} \cup L_{2j}$ ,  $j \neq 0$ , is unknottable, i.e. there does not exist a strongly 1 - 1 map  $f$  on  $L_{1j} \cup L_{2j}$  such that  $f(L_{1j})$  and  $f(L_{2j})$  are unknotted.*

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