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In this paper strongly* o-metrizable spaces are introduced and it is shown that a space is strongly* o-metrizable if and only if it is semistratifiable and o-metrizable (or symmetrizable); g-metrizable spaces are strongly* o-metrizable and hence quotient π-images of metric spaces. As what F. Siwiec did for (second countable, metrizable and first countable) spaces, we introduce g-developable spaces, and it is proved that a Hausdorff space is g-developable if and only if it is symmetrizable by a symmetric under which all convergent sequences are Cauchy.

1. o-metrizable spaces. Let X be a topological space and d be a nonnegative real-valued function defined on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$. Such a function d is called an *o-metric* [16] for X provided that a subset U of X is open if and only if $d(x, X - U) > 0$ for each $x \in U$. An o-metric d is called a *strong o-metric* [17] if each sphere $S(x; r) = \{y \in X: d(x, y) < r\}$ is a neighborhood of x; a *symmetric* if $d(x, y) = d(y, x)$ for each x and y; a *semimetric* if d is a symmetric such that $x \in \bar{M}$ if and only if $d(x, M) = 0$.

For a space X, let g be a map defined on $N \times X$ to the power-set of X such that $x \in g(n, x)$ and $g(n + 1, x) \subset g(n, x)$ for each n and x; a subset U of X is open if for each $x \in U$ there is an n such that $g(n, x) \subset U$. We call such a map a *CWC-map* (=countable weakly-open covering map). Consider the following conditions on g:

(1) if $x_n \in g(n, x)$ for each n, the sequence $\{x_n\}$ converges to x,

(2) if $x \in g(n, x_n)$ for each n, the sequence $\{x_n\}$ converges to x, and

(3) each $g(n, x)$ is open.

Note that (1) is equivalent to: $\{g(n, x): n \in N\}$ is a local net at x, and (2) is equivalent to: $\{g^*(n, x): n \in N\}$ is a local net at x, where $g^*(n, x)$ is defined by $x \in g^*(n, y)$ if and only if $y \in g(n, x)$.

X is said to be *g-first countable* [1, 20] if X has a CWC-map satisfying (1); *first countable* if X has a CWC-map satisfying (1) and (3). *Semistratifiable* spaces [8] are characterized by spaces having CWC-maps satisfying (2) and (3); *symmetrizable* spaces [4] by spaces having CWC-maps satisfying (1) and (2); *semimetrizable* spaces [11] by spaces having CWC-maps satisfying (1), (2) and (3).

The following proposition may be found in [18], but we will give its proof for later use.

PROPOSITION 1.1. *A space is o-metrizable if and only if it is a g-first countable T_1-space.*

Proof. Let g be a g-first countable CWC-map for a space X. Set $d(x, y) = 1/\inf\{j: y \notin g(j, x)\}$. A subset U of X is open if and only if for each $x \in U$, there exists an $n = n(x)$ such that $g(n, x) \subset U$, and hence $g(n, x) \cap (X - U) = \emptyset$, which is equivalent to $d(x, X - U) \geqq 1/n$. Conversely, let d be an o-metric on X. Set $g(n, x) = S(x; 1/n)$. Then g is a g-first countable CWC-map.

Part of the following theorem appears in [18]. The remaing part is easily verified using a similar technique to 1.1.

THEOREM 1.2. *The following are equivalent:*

(1) *X is a first countable T_1-space,*

(2) *X is o-metrizable by an o-metric under which all spheres are open,*

(3) *X is o-metrizable by an o-metric d such that $x \in \bar{M}$ if and only if $d(x, M) = 0$, and*

(4) *X is strongly o-metrizable.*

The following is a kind of dual character of strongly o-metrizable spaces.

DEFINITION 1.3. A space X is said to be *strongly* o-metrizable* if it has an o-metric d such that $S^*(x; r) = \{y \in X: d(y, x) < r\}$ is a neighborhood of x for each $x \in X$ and $r > 0$.

Ja. A. Kofner [13] proved that semistratifiable o-metrizable spaces are symmetrizable. But symmetrizability is not a sufficient condition for semistratifiability. In fact,

THEOREM 1.4. *For an o-metrizable space X, the following are equivalent:*

(1) *X is semistratifiable,*

(2) *X is symmetrizable and semistratifiable,*

(3) *X has an o-metric d such that each $S^*(x: r)$ is open,*

(4) *X has an o-metric d such that $d(M, x) = 0$ if $x \in \bar{M}$, and*

(5) *X is strongly* o-metrizable.*

Proof. $(1 \Rightarrow 2)$. See [13, Theorem 11].

$(2 \Rightarrow 3)$. Let f, g be a symmetrizable, a semistratifiable CWC-map for X, respectively. Set $h^*(n, x) = \operatorname{Int}(f(n, x) \cup g(n, x))$. Note that $h(n, x) \subset f^*(n, x) \cup g^*(n, x)$. This implies that h is an o-metrizable CWC-map (cf. Proposition 1.1) with an additional condition: each $h^*(h, x)$ is open.

Now set $d(x, y) = 1/\inf \{j \in N: y \notin h(j, x)\}$. By the proof of 1.1, d is an o-metric for X. Futhermore, $S^*(x; 1/n) = h^*(n, x)$, which is open.

$(3 \Rightarrow 4)$. Let d be an o-metric for X such that each $S^*(x; r)$ is open. If $d(M, x) = r > 0$, $M \cap S^*(x: r) = \varnothing$. This implies $x \notin \bar{M}$.

$(4 \Rightarrow 5)$. Assume $x \notin \operatorname{Int} S^*(x; r)$ for some $r > 0$. This implies that $x \in \operatorname{cl}(X - S^*(x; r))$. Therefore, $d(X - S^*(x; r), x) = 0$, which is a contradiction.

$(5 \Rightarrow 1)$. Let d be a strong* o-metric for X. Set $g(n, x) = \operatorname{Int} S^*(x; 1/n)$ for each n and x. Now it is easily shown that g is a semistratifiable CWC-map for X.

Note that strong o-metrizability and strong* o-metrizability are independent. In fact, a space is semi-metrizable if and only if it is strongly and strongly* o-metrizable.

COROLLARY 1.5. *A g-metizable space* [19] *is strongly* o-metrizable.*

Proof. A g-metrizable space has a σ-cushioned pair-net, and hence is semi-stratifiable [13 or 15]. Now apply 1.4.

A mapping f from a metric space X to a space Y is called a π-*mapping* [19] if for each $y \in Y$ and each neighborhood U of y,

$$d(f^{-1}y, X - f^{-1}U) > 0.$$

F. Siwiec posed a question ([20], 1.19): Is every g-metrizable space a quotient π-image of a metric space? Ja A. Kofner answers the question.

COROLLARY 1.6. *Every g-metrizable space is a quotient π-image of a metric space.*

Proof. Kofner has shown that a strongly* o-metrizable space has a symmetric satisfying the weak condition of Cauchy ([14], Theorem 1), and hence is a quotient π-image of a metric space ([13], Theorem 19). Now 1.5 completes the proof.

EXAMPLE 1.7. (1) A countable M_1-space which is not o-metrizable. Example 9.4 of [6].

(2) A strongly* o-metrizable space which is neither semimetrizable nor g-metrizable. Let X be the space of Example 5.1 in [9], Y a

semimetrizable nonmetrizable space. The topological sum of X and Y.

(3) Example 1 in [14] is an example of a space possessing a symmetric with the weak condition of Cauchy but which is not strongly* o-metrizable.

2. g-developable spaces. Considering definitions of g-first countable spaces, g-metrizable spaces and g-second countable spaces, symmetrizable spaces might be called g-semimetrizable spaces. (See the characterization of symmetrizable spaces by means of CWC-maps in §1.) *Developable* spaces are characterized by means of COC-maps (=countable open covering maps) by Heath [11]: If $x, x_n \in g(n, y_n)$ for each n, then the sequence $\{x_n\}$ converges to x. The g-setting of developable spaces is the following.

DEFINITION 2.1. A space is *g-developable* if it has a CWC-map g with the following property: If $x, x_n \in g(n, y_n)$ for each n, the sequence $\{x_n\}$ converge to x.

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ be a sequence of covers of a space X such that γ_{n+1} refines γ_n for each n. Such a sequence is said to be *semi-refined* [7] if $\{st(x, \gamma_n): x \in X, n \in N\}$ is a *weak base* [1] for X. Burke and Stoltenberg [4] shows that a T_1-space has a semi-refined sequence if and only if it is symmetrizable.

If X has a g-first countable CWC-map g such that $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$, where $\gamma_n = \{g(n, x): x \in X\}$, is a semi-refined sequence for X, then X is g-developable. Conversely, let g be a g-developable CWC-map for a space X. If we set $\gamma_n = \{g(n, x): x \in X\}$ for each n, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ is a semirefined sequence for X. Thus, a g-developable space is symmetrizable. F. Siwiec [20] proved symmetrizable spaces are semimetrizable if they are Fréchet. The same proof says the following.

PROPOSITION 2.2. *A Hausdorff space is developable if and only if it is g-developable and Fréchet.*

As D. K. Burke [5] showed, every semimetric space can be semimetrizable by a semimetric under which every convergent sequence has a Cauchy subsequence. Unfortunately, this is not true for symmetric spaces. On the other hand, Morton Brown [3] noted that a T_1-space is developable if and only if it is semimetrizable by a semimetric under which all convergent sequences are Cauchy. Analogously we are able to characterize symmetrizable spaces with a symmetric under which all convergent sequences are Cauchy.

THEOREM 2.3. *A Hausdorff space X is g-developable if and only if X is symmetrizable by a symmetric under which all convergent sequences are Cauchy.*

Proof. Let g be a g-developable CWC-map for X, and $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ the semirefined sequence mentioned above, that is, $\gamma_n = \{g(n, x): x \in X\}$. Now define a symmetric d *by* $d(x, y) = 1/\inf \{j \in N: y \notin st(x, \gamma_j)\}$. Let $\{x_n\}$ be a sequence converging to x. Since X is Hausdorff and g a g-first countable CWC-map, $\{x_n\}$ is eventually in $g(k, x)$ for each $k \in N$. For any $\varepsilon > 0$, choose $k, h \in N$ such that $1/k < \varepsilon$ and $x_n \in g(k, x)$ for all $n \geqq h$. Then $g(k, x) \supset \{x_h, x_{h+1}, \cdots\}$. This implies that $d(x_m, x_n) < \varepsilon$ for any $m, n \geqq h$.

Conversely, let d be a symmetric for X under which all convergent sequences are Cauchy. It is easily verified that d satisfies the Aleksandrov-Nemytskii condition

(AN) For any $x \in X$ and any $\varepsilon > 0$, there exists a $\delta = \delta(x, \varepsilon)$ such that $d(x, y) < \delta$ and $d(x, z) < \delta$ imply $d(y, z) < \varepsilon$.

For each x and n, choose $\delta = \delta(x, n)$ such that $d(x, y) < \delta$ and $d(x, z) < \delta$ imply $d(y, z) < 1/n$, let $g(n, x) = S(x; \delta(x, n))$. Now it is not difficult to show that g is a desired g-developable CWC-map.

COROLLARY 2.4. *A Hausdorff g-developable space is a quotient π-image of a metric space.*

EXAMPLE 2.5. (1) In symmetric spaces, g-developability and the weak condition of Cauchy are not equivalent. In fact, there exist strongly* o-metrizable spaces which are not g-developable. Non-developable semimetric spaces are such examples.

(2) Non-metrizable Moore spaces are g-developable but not g-metrizable.

Question 2.6. The auther could not determine the following

(1) Is a g-metrizable space g-developable?

(2) Is a g-developable space semistratifiable?

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