ON CERTAIN $g$-FIRST COUNTABLE SPACES

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In this paper strongly* $o$-metrizable spaces are introduced and it is shown that a space is strongly* $o$-metrizable if and only if it is semistratifiable and $o$-metrizable (or symmetrizable); $g$-metrizable spaces are strongly* $o$-metrizable and hence quotient $\pi$-images of metric spaces. As what F. Siwiec did for (second countable, metrizable and first countable) spaces, we introduce $g$-developable spaces, and it is proved that a Hausdorff space is $g$-developable if and only if it is symmetrizable by a symmetric under which all convergent sequences are Cauchy.

1. $o$-metrizable spaces. Let $X$ be a topological space and $d$ be a nonnegative real-valued function defined on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$. Such a function $d$ is called an $o$-metric [16] for $X$ provided that a subset $U$ of $X$ is open if and only if $d(x, X - U) > 0$ for each $x \in U$. An $o$-metric $d$ is called a strong $o$-metric [17] if each sphere $S(x; r) = \{y \in X: d(x, y) < r\}$ is a neighborhood of $x$; a symmetric if $d(x, y) = d(y, x)$ for each $x$ and $y$; a semimetric if $d$ is a symmetric such that $x \in M$ if and only if $d(x, M) = 0$.

For a space $X$, let $g$ be a map defined on $N \times X$ to the power-set of $X$ such that $x \in g(n, x)$ and $g(n + 1, x) \subseteq g(n, x)$ for each $n$ and $x$; a subset $U$ of $X$ is open if for each $x \in U$ there is an $n$ such that $g(n, x) \subseteq U$. We call such a map a CWC-map (=countable weakly-open covering map). Consider the following conditions on $g$:

(1) if $x_n \in g(n, x)$ for each $n$, the sequence $\{x_n\}$ converges to $x$,

(2) if $x \in g(n, x_n)$ for each $n$, the sequence $\{x_n\}$ converges to $x$, and

(3) each $g(n, x)$ is open.

Note that (1) is equivalent to: $\{g(n, x): n \in N\}$ is a local net at $x$, and (2) is equivalent to: $\{g^*(n, x): n \in N\}$ is a local net at $x$, where $g^*(n, x)$ is defined by $x \in g^*(n, y)$ if and only if $y \in g(n, x)$.

$X$ is said to be $g$-first countable [1, 20] if $X$ has a CWC-map satisfying (1); first countable if $X$ has a CWC-map satisfying (1) and (3). Semistratifiable spaces [8] are characterized by spaces having CWC-maps satisfying (2) and (3); symmetrizable spaces [4] by spaces having CWC-maps satisfying (1) and (2); semimetrizable spaces [11] by spaces having CWC-maps satisfying (1), (2) and (3).

The following proposition may be found in [18], but we will
Proposition 1.1. A space is o-metrizable if and only if it is a 1-first countable $T_1$-space.

Proof. Let $g$ be a 1-first countable CWC-map for a space $X$. Set $d(x, y) = 1/\inf\{j: y \in g(j, x)\}$. A subset $U$ of $X$ is open if and only if for each $x \in U$, there exists an $n = n(x)$ such that $g(n, x) \subseteq U$, and hence $g(n, x) \cap (X - U) = \emptyset$, which is equivalent to $d(x, X - U) \geq 1/n$. Conversely, let $d$ be an o-metric on $X$. Set $g(n, x) = S(x; 1/n)$. Then $g$ is a 1-first countable CWC-map.

Part of the following theorem appears in [18]. The remaining part is easily verified using a similar technique to 1.1.

Theorem 1.2. The following are equivalent:
1. $X$ is a first countable $T_1$-space,
2. $X$ is o-metrizable by an o-metric under which all spheres are open,
3. $X$ is o-metrizable by an o-metric $d$ such that $x \in M$ if and only if $d(x, M) = 0$, and
4. $X$ is strongly o-metrizable.

The following is a kind of dual character of strongly o-metrizable spaces.

Definition 1.3. A space $X$ is said to be strongly* o-metrizable if it has an o-metric $d$ such that $S^*(x; r) = \{y \in X: d(y, x) < r\}$ is a neighborhood of $x$ for each $x \in X$ and $r > 0$.

Ja. A. Kofner [13] proved that semistratifiable o-metrizable spaces are symmetrizable. But symmetrizability is not a sufficient condition for semistratifiability. In fact,

Theorem 1.4. For an o-metrizable space $X$, the following are equivalent:
1. $X$ is semistratifiable,
2. $X$ is symmetrizable and semistratifiable,
3. $X$ has an o-metric $d$ such that each $S^*(x; r)$ is open,
4. $X$ has an o-metric $d$ such that $d(M, x) = 0$ if $x \in M$, and
5. $X$ is strongly* o-metrizable.

Proof. (1 $\Rightarrow$ 2). See [13, Theorem 11].
(2 $\Rightarrow$ 3). Let $f, g$ be a symmetrizable, a semistratifiable CWC-map
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for $X$, respectively. Set $h^*(n, x) = \text{Int} (f(n, x) \cup g(n, x))$. Note that $h(n, x) \subset f^*(n, x) \cup g^*(n, x)$. This implies that $h$ is an $o$-metrizable CWC-map (cf. Proposition 1.1) with an additional condition: each $h^*(h, x)$ is open.

Now set $d(x, y) = 1/\inf \{j \in N: y \not\in h(j, x)\}$. By the proof of 1.1, $d$ is an $o$-metric for $X$. Futhermore, $S^*(x; 1/n) = h^*(n, x)$, which is open.

$(3 \Rightarrow 4)$. Let $d$ be an $o$-metric for $X$ such that each $S^*(x; r)$ is open. If $d(M, x) = r > 0$, $M \cap S^*(x; r) = \emptyset$. This implies $x \in M$.

$(4 \Rightarrow 5)$. Assume $x \in \text{Int} S^*(x; r)$ for some $r > 0$. This implies that $x \in \text{cl} (X - S^*(x; r))$. Therefore, $d(X - S^*(x; r), x) = 0$, which is a contradiction.

$(5 \Rightarrow 1)$. Let $d$ be a strong* $o$-metric for $X$. Set $g(n, x) = \text{Int} S^*(x; 1/n)$ for each $n$ and $x$. Now it is easily shown that $g$ is a semistratifiable CWC-map for $X$.

Note that strong $o$-metrizability and strong* $o$-metrizability are independent. In fact, a space is semi-metrizable if and only if it is strongly and strongly* $o$-metrizable.

**Corollary 1.5.** A $g$-metrizable space [19] is strongly* $o$-metrizable.

**Proof.** A $g$-metrizable space has a $\sigma$-cushioned pair-net, and hence is semi-stratifiable [13 or 15]. Now apply 1.4.

A mapping $f$ from a metric space $X$ to a space $Y$ is called a $\pi$-mapping [19] if for each $y \in Y$ and each neighborhood $U$ of $y$,

$$d(f^{-1}y, X - f^{-1}U) > 0.$$ 

F. Siwiec posed a question ([20], 1.19): Is every $g$-metrizable space a quotient $\pi$-image of a metric space? Ja A. Kofner answers the question.

**Corollary 1.6.** Every $g$-metrizable space is a quotient $\pi$-image of a metric space.

**Proof.** Kofner has shown that a strongly* $o$-metrizable space has a symmetric satisfying the weak condition of Cauchy ([14], Theorem 1), and hence is a quotient $\pi$-image of a metric space ([13], Theorem 19). Now 1.5 completes the proof.

**Example 1.7.** (1) A countable $M_\tau$-space which is not $o$-metrizable. Example 9.4 of [6].

(2) A strongly* $o$-metrizable space which is neither semimetrizable nor $g$-metrizable. Let $X$ be the space of Example 5.1 in [9], $Y$ a
semimetrizable nonmetrizable space. The topological sum of $X$ and $Y$.

(3) Example 1 in [14] is an example of a space possessing a symmetric with the weak condition of Cauchy but which is not strongly* $o$-metrizable.

2. g-developable spaces. Considering definitions of $g$-first countable spaces, $g$-metrizable spaces and $g$-second countable spaces, symmetrizable spaces might be called $g$-semimetrizable spaces. (See the characterization of symmetrizable spaces by means of $CWC$-maps in §1.) *Developable* spaces are characterized by means of COC-maps (=countable open covering maps) by Heath [11]: If $x, x_n \in g(n, y_n)$ for each $n$, then the sequence $\{x_n\}$ converges to $x$. The $g$-setting of developable spaces is the following.

**Definition 2.1.** A space is *$g$-developable* if it has a $CWC$-map $g$ with the following property: If $x, x_n \in g(n, y_n)$ for each $n$, the sequence $\{x_n\}$ converge to $x$.

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ be a sequence of covers of a space $X$ such that $\gamma_{n+1}$ refines $\gamma_n$ for each $n$. Such a sequence is said to be *semi-refined* [7] if $\{st(x, \gamma_n): x \in X, n \in N\}$ is a weak base [1] for $X$. Burke and Stoltenberg [4] shows that a $T_\gamma$-space has a semi-refined sequence if and only if it is symmetrizable.

If $X$ has a $g$-first countable $CWC$-map $g$ such that $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$, where $\gamma_n = \{g(n, x): x \in X\}$, is a semi-refined sequence for $X$, then $X$ is $g$-developable. Conversely, let $g$ be a $g$-developable $CWC$-map for a space $X$. If we set $\gamma_n = \{g(n, x): x \in X\}$ for each $n$, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ is a semi-refined sequence for $X$. Thus, a $g$-developable space is symmetrizable. F. Siwiec [20] proved symmetrizable spaces are semimetrizable if they are Fréchet. The same proof says the following.

**Proposition 2.2.** A Hausdorff space is developable if and only if it is $g$-developable and Fréchet.

As D. K. Burke [5] showed, every semimetric space can be semimetrizable by a semimetric under which every convergent sequence has a Cauchy subsequence. Unfortunately, this is not true for symmetric spaces. On the other hand, Morton Brown [3] noted that a $T_\gamma$-space is developable if and only if it is semimetrizable by a semimetric under which all convergent sequences are Cauchy. Analogously we are able to characterize symmetrizable spaces with a symmetric under which all convergent sequences are Cauchy.
THEOREM 2.3. A Hausdorff space $X$ is $g$-developable if and only if $X$ is symmetrizable by a symmetric under which all convergent sequences are Cauchy.

Proof. Let $g$ be a $g$-developable CWC-map for $X$, and $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ the semirefined sequence mentioned above, that is, $\gamma_n = \{g(n, x) : x \in X\}$. Now define a symmetric $d$ by $d(x, y) = 1/\inf \{j \in \mathbb{N} : y \in st(x, \gamma_j)\}$. Let $\{x_n\}$ be a sequence converging to $x$. Since $X$ is Hausdorff and $g$ a $g$-first countable CWC-map, $\{x_n\}$ is eventually in $g(k, x)$ for each $k \in \mathbb{N}$. For any $\varepsilon > 0$, choose $k, h \in \mathbb{N}$ such that $1/k < \varepsilon$ and $x_n \in g(k, x)$ for all $n \geq h$. Then $g(k, x) \supseteq \{x_n, x_{h+1}, \cdots\}$. This implies that $d(x_m, x_n) < \varepsilon$ for any $m, n \geq h$.

Conversely, let $d$ be a symmetric for $X$ under which all convergent sequences are Cauchy. It is easily verified that $d$ satisfies the Aleksandrov-Nemytskii condition

(AN) For any $x \in X$ and any $\varepsilon > 0$, there exists a $\delta = \delta(x, \varepsilon)$ such that $d(x, y) < \delta$ and $d(x, z) < \delta$ imply $d(y, z) < \varepsilon$.

For each $x$ and $n$, choose $\delta = \delta(x, n)$ such that $d(x, y) < \delta$ and $d(x, z) < \delta$ imply $d(y, z) < 1/n$, let $g(n, x) = S(x; \delta(x, n))$. Now it is not difficult to show that $g$ is a desired $g$-developable CWC-map.

COROLLARY 2.4. A Hausdorff $g$-developable space is a quotient $\pi$-image of a metric space.

EXAMPLE 2.5. (1) In symmetric spaces, $g$-developability and the weak condition of Cauchy are not equivalent. In fact, there exist strongly* $\omega$-metrizable spaces which are not $g$-developable. Non-developable semimetric spaces are such examples.

(2) Non-metrizable Moore spaces are $g$-developable but not $g$-metrizable.

Question 2.6. The author could not determine the following

(1) Is a $g$-metrizable space $g$-developable?

(2) Is a $g$-developable space semistratifiable?

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