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ON CERTAIN g -FIRST COUNTABLE SPACES

KYUNG BAI LEE

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In this paper strongly* o -metrizable spaces are introduced and it is shown that a space is strongly* o -metrizable if and only if it is semistratifiable and o -metrizable (or symmetrizable); g -metrizable spaces are strongly* o -metrizable and hence quotient π -images of metric spaces. As what F. Siwiec did for (second countable, metrizable and first countable) spaces, we introduce g -developable spaces, and it is proved that a Hausdorff space is g -developable if and only if it is symmetrizable by a symmetric under which all convergent sequences are Cauchy.

1. o -metrizable spaces. Let X be a topological space and d be a nonnegative real-valued function defined on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$. Such a function d is called an o -metric [16] for X provided that a subset U of X is open if and only if $d(x, X - U) > 0$ for each $x \in U$. An o -metric d is called a strong o -metric [17] if each sphere $S(x; r) = \{y \in X: d(x, y) < r\}$ is a neighborhood of x ; a symmetric if $d(x, y) = d(y, x)$ for each x and y ; a semimetric if d is a symmetric such that $x \in \bar{M}$ if and only if $d(x, M) = 0$.

For a space X , let g be a map defined on $N \times X$ to the power-set of X such that $x \in g(n, x)$ and $g(n + 1, x) \subset g(n, x)$ for each n and x ; a subset U of X is open if for each $x \in U$ there is an n such that $g(n, x) \subset U$. We call such a map a CWC -map (=countable weakly-open covering map). Consider the following conditions on g :

- (1) if $x_n \in g(n, x)$ for each n , the sequence $\{x_n\}$ converges to x ,
- (2) if $x \in g(n, x_n)$ for each n , the sequence $\{x_n\}$ converges to x , and
- (3) each $g(n, x)$ is open.

Note that (1) is equivalent to: $\{g(n, x): n \in N\}$ is a local net at x , and (2) is equivalent to: $\{g^*(n, x): n \in N\}$ is a local net at x , where $g^*(n, x)$ is defined by $x \in g^*(n, y)$ if and only if $y \in g(n, x)$.

X is said to be g -first countable [1, 20] if X has a CWC -map satisfying (1); first countable if X has a CWC -map satisfying (1) and (3). Semistratifiable spaces [8] are characterized by spaces having CWC -maps satisfying (2) and (3); symmetrizable spaces [4] by spaces having CWC -maps satisfying (1) and (2); semimetrizable spaces [11] by spaces having CWC -maps satisfying (1), (2) and (3).

The following proposition may be found in [18], but we will

give its proof for later use.

PROPOSITION 1.1. *A space is o -metrizable if and only if it is a g -first countable T_1 -space.*

Proof. Let g be a g -first countable CWC-map for a space X . Set $d(x, y) = 1/\inf \{j: y \in g(j, x)\}$. A subset U of X is open if and only if for each $x \in U$, there exists an $n = n(x)$ such that $g(n, x) \subset U$, and hence $g(n, x) \cap (X - U) = \emptyset$, which is equivalent to $d(x, X - U) \geq 1/n$. Conversely, let d be an o -metric on X . Set $g(n, x) = S(x; 1/n)$. Then g is a g -first countable CWC-map.

Part of the following theorem appears in [18]. The remaining part is easily verified using a similar technique to 1.1.

THEOREM 1.2. *The following are equivalent:*

- (1) X is a first countable T_1 -space,
- (2) X is o -metrizable by an o -metric under which all spheres are open,
- (3) X is o -metrizable by an o -metric d such that $x \in \bar{M}$ if and only if $d(x, M) = 0$, and
- (4) X is strongly o -metrizable.

The following is a kind of dual character of strongly o -metrizable spaces.

DEFINITION 1.3. A space X is said to be *strongly** o -metrizable if it has an o -metric d such that $S^*(x; r) = \{y \in X: d(y, x) < r\}$ is a neighborhood of x for each $x \in X$ and $r > 0$.

Ja. A. Kofner [13] proved that semistratifiable o -metrizable spaces are symmetrizable. But symmetrizability is not a sufficient condition for semistratifiability. In fact,

THEOREM 1.4. *For an o -metrizable space X , the following are equivalent:*

- (1) X is semistratifiable,
- (2) X is symmetrizable and semistratifiable,
- (3) X has an o -metric d such that each $S^*(x; r)$ is open,
- (4) X has an o -metric d such that $d(\bar{M}, x) = 0$ if $x \in \bar{M}$, and
- (5) X is strongly* o -metrizable.

Proof. (1 \Rightarrow 2). See [13, Theorem 11].

(2 \Rightarrow 3). Let f, g be a symmetrizable, a semistratifiable CWC-map

for X , respectively. Set $h^*(n, x) = \text{Int}(f(n, x) \cup g(n, x))$. Note that $h(n, x) \subset f^*(n, x) \cup g^*(n, x)$. This implies that h is an o -metrizable *CWC*-map (cf. Proposition 1.1) with an additional condition: each $h^*(h, x)$ is open.

Now set $d(x, y) = 1/\inf\{j \in N: y \notin h(j, x)\}$. By the proof of 1.1, d is an o -metric for X . Furthermore, $S^*(x; 1/n) = h^*(n, x)$, which is open.

(3 \Rightarrow 4). Let d be an o -metric for X such that each $S^*(x; r)$ is open. If $d(M, x) = r > 0$, $M \cap S^*(x; r) = \emptyset$. This implies $x \notin \bar{M}$.

(4 \Rightarrow 5). Assume $x \notin \text{Int } S^*(x; r)$ for some $r > 0$. This implies that $x \in \text{cl}(X - S^*(x; r))$. Therefore, $d(X - S^*(x; r), x) = 0$, which is a contradiction.

(5 \Rightarrow 1). Let d be a strong* o -metric for X . Set $g(n, x) = \text{Int } S^*(x; 1/n)$ for each n and x . Now it is easily shown that g is a semistratifiable *CWC*-map for X .

Note that strong o -metrizability and strong* o -metrizability are independent. In fact, a space is semi-metrizable if and only if it is strongly and strongly* o -metrizable.

COROLLARY 1.5. *A g -metrizable space [19] is strongly* o -metrizable.*

Proof. A g -metrizable space has a σ -cushioned pair-net, and hence is semi-stratifiable [13 or 15]. Now apply 1.4.

A mapping f from a metric space X to a space Y is called a π -mapping [19] if for each $y \in Y$ and each neighborhood U of y ,

$$d(f^{-1}y, X - f^{-1}U) > 0.$$

F. Siwiec posed a question ([20], 1.19): Is every g -metrizable space a quotient π -image of a metric space? Ja A. Kofner answers the question.

COROLLARY 1.6. *Every g -metrizable space is a quotient π -image of a metric space.*

Proof. Kofner has shown that a strongly* o -metrizable space has a symmetric satisfying the weak condition of Cauchy ([14], Theorem 1), and hence is a quotient π -image of a metric space ([13], Theorem 19). Now 1.5 completes the proof.

EXAMPLE 1.7. (1) A countable M_1 -space which is not o -metrizable. Example 9.4 of [6].

(2) A strongly* o -metrizable space which is neither semimetrizable nor g -metrizable. Let X be the space of Example 5.1 in [9], Y a

semimetrizable nonmetrizable space. The topological sum of X and Y .

(3) Example 1 in [14] is an example of a space possessing a symmetric with the weak condition of Cauchy but which is not strongly* o -metrizable.

2. g -developable spaces. Considering definitions of g -first countable spaces, g -metrizable spaces and g -second countable spaces, symmetrizable spaces might be called g -semimetrizable spaces. (See the characterization of symmetrizable spaces by means of CWC -maps in §1.) *Developable* spaces are characterized by means of COC -maps (=countable open covering maps) by Heath [11]: If $x, x_n \in g(n, y_n)$ for each n , then the sequence $\{x_n\}$ converges to x . The g -setting of developable spaces is the following.

DEFINITION 2.1. A space is g -developable if it has a CWC -map g with the following property: If $x, x_n \in g(n, y_n)$ for each n , the sequence $\{x_n\}$ converge to x .

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots)$ be a sequence of covers of a space X such that γ_{n+1} refines γ_n for each n . Such a sequence is said to be *semi-refined* [7] if $\{st(x, \gamma_n): x \in X, n \in N\}$ is a *weak base* [1] for X . Burke and Stoltenberg [4] shows that a T_1 -space has a semi-refined sequence if and only if it is symmetrizable.

If X has a g -first countable CWC -map g such that $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots)$, where $\gamma_n = \{g(n, x): x \in X\}$, is a semi-refined sequence for X , then X is g -developable. Conversely, let g be a g -developable CWC -map for a space X . If we set $\gamma_n = \{g(n, x): x \in X\}$ for each n , $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots)$ is a semirefined sequence for X . Thus, a g -developable space is symmetrizable. F. Siwiec [20] proved symmetrizable spaces are semimetrizable if they are Fréchet. The same proof says the following.

PROPOSITION 2.2. A Hausdorff space is developable if and only if it is g -developable and Fréchet.

As D. K. Burke [5] showed, every semimetric space can be semimetrizable by a semimetric under which every convergent sequence has a Cauchy subsequence. Unfortunately, this is not true for symmetric spaces. On the other hand, Morton Brown [3] noted that a T_1 -space is developable if and only if it is semimetrizable by a semimetric under which all convergent sequences are Cauchy. Analogously we are able to characterize symmetrizable spaces with a symmetric under which all convergent sequences are Cauchy.

THEOREM 2.3. *A Hausdorff space X is g -developable if and only if X is symmetrizable by a symmetric under which all convergent sequences are Cauchy.*

Proof. Let g be a g -developable CWC-map for X , and $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots)$ the semirefined sequence mentioned above, that is, $\gamma_n = \{g(n, x) : x \in X\}$. Now define a symmetric d by $d(x, y) = 1/\inf \{j \in N : y \notin st(x, \gamma_j)\}$. Let $\{x_n\}$ be a sequence converging to x . Since X is Hausdorff and g a g -first countable CWC-map, $\{x_n\}$ is eventually in $g(k, x)$ for each $k \in N$. For any $\varepsilon > 0$, choose $k, h \in N$ such that $1/k < \varepsilon$ and $x_n \in g(k, x)$ for all $n \geq h$. Then $g(k, x) \supset \{x_h, x_{h+1}, \dots\}$. This implies that $d(x_m, x_n) < \varepsilon$ for any $m, n \geq h$.

Conversely, let d be a symmetric for X under which all convergent sequences are Cauchy. It is easily verified that d satisfies the Aleksandrov-Nemytskii condition

(AN) For any $x \in X$ and any $\varepsilon > 0$, there exists a $\delta = \delta(x, \varepsilon)$ such that $d(x, y) < \delta$ and $d(x, z) < \delta$ imply $d(y, z) < \varepsilon$.

For each x and n , choose $\delta = \delta(x, n)$ such that $d(x, y) < \delta$ and $d(x, z) < \delta$ imply $d(y, z) < 1/n$, let $g(n, x) = S(x; \delta(x, n))$. Now it is not difficult to show that g is a desired g -developable CWC-map.

COROLLARY 2.4. *A Hausdorff g -developable space is a quotient π -image of a metric space.*

EXAMPLE 2.5. (1) In symmetric spaces, g -developability and the weak condition of Cauchy are not equivalent. In fact, there exist strongly* o -metrizable spaces which are not g -developable. Non-developable semimetric spaces are such examples.

(2) Non-metrizable Moore spaces are g -developable but not g -metrizable.

Question 2.6. The author could not determine the following

- (1) Is a g -metrizable space g -developable?
- (2) Is a g -developable space semistratifiable?

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