A CHARACTERIZATION OF NON-LINEAR FUNCTIONALS ON $W^p_1$ POSSESSING AUTONOMOUS KERNELS. I

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Let $\Omega$ be a domain in $\mathbb{R}^n$ and let a nonlinear functional $N$ be given on the first order Sobolev space $W^1_+(\Omega)$, $1 \leq p \leq \infty$. We are concerned with obtaining a characterization of those functionals $N$ of the form

\begin{equation}
N(u) = \int g(u, D_1u, \ldots, D_n u) dm, \quad u \in W^1_+(\Omega),
\end{equation}

where $g: \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function, $D_i u (i = 1, \ldots, n)$ denotes the distribution derivative of $u$ relative to its $i$th coordinate variable and $m$ denotes Lebesgue measure. In the present paper we confine ourselves to the case $n = 1$. The general case will be considered in the second part of this work.

In recent years, characterizations have been obtained for nonlinear functionals defined on Banach lattices of functions such as the $L^p$ spaces and Orlicz spaces [1], [3], [4], [6], but the methods utilized in those works all depend crucially on the normality of these lattices and are unavailable in the present context. On the other hand, very recently a characterization was obtained for nonlinear functionals of the form

\begin{equation}
N(u) = \int g(t, D^k u(t)) dt, \quad u \in \dot{W}^k (J), \quad k \geq 1,
\end{equation}

where $J$ is an interval on the line, $g$ satisfies Caratheodory conditions and $g(t, 0) \equiv 0$ [2]. Such a functional possesses the property of $D^k$-disjoint additivity,

\[ N(u + v) = N(u) + N(v) \quad \text{provided} \quad D^k u \cdot D^k v = 0, \]

which permits a reduction of the problem to that for a disjointly additive functional on a closed subspace of $L^p(J)$. However even when $n = 1$ functionals of the form (1.1) are generally not $D$-disjointly additive and hence the methods of [2] do not apply in this case.

If $g$ in (1.1) satisfies a suitable growth condition then the integrand $G_u$, where

\begin{equation}
(G_u)(x) = g(u(x), D_1 u(x), \ldots, D_n u(x)),
\end{equation}

belongs to $L^1(\Omega)$ for all $u \in W^1_+(\Omega)$, so that $N$ is real-valued. In such
cases it can be seen that $N$ possesses the following properties:

(A) $N$ is additively invariant under swapping: if $H$ is a hyperplane in $\mathbb{R}^n$ which partitions $\Omega$ into sets $\{\Omega_1, \Omega_2\}$ and $u, v$ are swappable across $H$ in the sense that
\[ u\mathcal{L}_{\Omega_1} + v\mathcal{L}_{\Omega_2}, v\mathcal{L}_{\Omega_1} + u\mathcal{L}_{\Omega_2} \text{ are in } W_f(\Omega) \]
(either of these conditions implies the other), then
\[ N(u) + N(v) = N(u\mathcal{L}_{\Omega_1} + v\mathcal{L}_{\Omega_2}) + N(v\mathcal{L}_{\Omega_1} + u\mathcal{L}_{\Omega_2}) ; \]

(B) $N$ is invariant under 1-equimeasurability:
\[ N(u) = N(v) \]
whenever $u, v \in W^p(\Omega)$ are such that the $(n + 1)$-tuples
\[ u = (u, D_1u, \cdots, D_nu), \quad v = (v, D_1v, \cdots, D_nv) \]
are stochastically equivalent as mappings from $\Omega$ to $\mathbb{R}^{n+1}$:
\[ m(u^{-1}(B)) = m(v^{-1}(B)) \]
for every Borel set $B \subset \mathbb{R}^{n+1}$.

(C) $N$ is continuous:
\[ N(u_k) \longrightarrow N(u_0) \quad \text{whenever} \quad \| u_k - u_0 \|_{W^p(\Omega)} \longrightarrow 0 . \]

We are concerned with the extent to which properties (A), (B), (C) characterize functionals of the form (1.1). It will be shown that in conjunction with an additional hypothesis of "locally uniform continuity in variation," the conditions (A)-(C) do characterize functionals of the form (1.1).

A similar characterization is given for nonlinear operators $G$: $W^p(\Omega) \longrightarrow L^q(\Omega)$ having the form (1.2).

The present paper is devoted to an exposition of these results when $\mathbb{R}^n$ is the real line and $\Omega$ is a bounded open interval. The situation for $n > 1$ is as follows. The case $p > n$ can be treated by essentially the same method that is used in this paper, but the details are more subtle and intricate. On the other hand, in the case $p \leq n$ the characterization involves several new problems. This work will appear in the second part.

2. Representation for functionals. We develop here the desired representation result for a class of nonlinear functionals and operators on the spaces $W^p(J), 1 \leq p \leq \infty$, where $J$ is a bounded interval on the line.

Recall that $W^p(J)$ denotes the Sobolev space
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$W_1^p(J) = \{ u \in L^p(J): D u \in L^p(J) \}$

where $D u$ denotes the distribution derivative of $u$. (All functions are taken to be real.) It is known that a function $u$ belongs to $W_1^p(J)$ if and only if it is equivalent to a function in $C(J)$ which is absolutely continuous and whose first derivative belongs to $L^p(J)$. Hereafter any function in $W_1^p(J)$ will be assumed to be the continuous representative of its equivalence class. The space $W_1^p(J)$ is a Banach space under the norm

$$\| u \|_{W_1^p(J)} = \| u \|_{L^p(J)} + \| D u \|_{L^p(J)}.$$ 

Its structure under the metric

$$(2.1) \quad \rho(u, v) = \sigma(u - v) + \sigma(D u - D v)$$

where

$$\sigma(f) = \inf_{\varepsilon > 0} \{ \varepsilon + \mu(\{ t: | f(t) | > \varepsilon \}) \}$$

will also play a role in what follows.

The class of functionals to be characterized consists of those functionals representable in the form

$$N(u) = \int_J g(u(t), u'(t)) dt \quad u \in W_1^p(J),$$

for an appropriate function $g: \mathbb{R}^2 \to \mathbb{R}$. It will be necessary for our purposes to analyze the behavior of such functionals in some detail.

We adopt the following notations and conventions. Given a function $f$ we set $K(f) = \{ t: f(t) \neq 0 \}$. The functions $u, v \in W_1^p(J)$ are 1-equimeasurable, denoted $u \sim_1 v$, provided that the pairs

$$u = (u, D u), \quad v = (v, D v)$$

are stochastically equivalent: for each Borel set $B \subset \mathbb{R}^2$

$$(2.2) \quad m(u^{-1}(B)) = m(v^{-1}(B)).$$

The functions $u, v \in W_1^p(J)$ are 1-disjoint provided that $K(D u) \cap K(D v)$ is a null set. A 1-disjoint pair $u, v$ is said to be envelope compatible provided there exists a partition of $J$ into two subintervals $J', J''$ such that

$$(2.3) \quad (\begin{array}{c}
(1) \quad K(D u) \subset J', \ K(D v) \subset J'' \\
(2) \quad u \chi_{J'} + v \chi_{J''} =: u \oplus v \text{ is in } W_1^p(J)
\end{array})$$

The function $z = u \oplus v$, which is independent of the choice of partition $\{ J', J'' \}$, is called the 1-envelope of $u$ and $v$. Note that by

\[ a =: b \] means "is, by definition, ".
(2.3), for any finite partition of \( J \) into subintervals \( \{ J_i \}_{i=1}^l \), numbered from left to right, say, it is possible to decompose each \( u \in W^p(J) \) into functions \( u^{J_i} \in W^p(J_i) \), \( 1 \leq i \leq l \), such that

\[
\begin{align*}
(2.4) & \quad (a) \quad K(D(u^{J_i})) \subset J_i, \quad i \geq 1, \ldots, l \\
& \quad (b) \quad u = (u^{J_1} \oplus u^{J_2} \oplus \cdots) \oplus u^{J_l}.
\end{align*}
\]

Indeed, (2.4) holds if, for any subinterval \( J' \subset J \), \( u^{J'} \) denotes the element of \( C(J) \) which coincides with \( u \) in \( J' \) and is constant on the left and right of \( J' \).

Now let \( g: \mathbb{R}^2 \to \mathbb{R} \) be a continuous function. For each \( u \in W^p(J) \) the function \( G_u \) defined by

\[
(2.5) \quad G_u(t) = g(u(t), u'(t)) \quad t \in J,
\]

is measurable. Hence if \( G_u \) is in \( L^p(J) \) for all \( u \in W^p(J) \), in particular if \( g \) satisfies a growth condition of the form

\[
(2.6) \quad |g(x_0, x_1)| \leq K_2(1 + |x_1|^p) \quad \text{whenever} \quad |x_0| \leq M,
\]

then one can form the nonlinear functional

\[
(2.7) \quad N(u) = \int_J G_u \quad u \in W^p(J).
\]

As mentioned in §1 the functional \( N \) has the following properties:

(A) \( N \) is additively invariant under swapping.

Note that in the case \( n = 1 \) two functions \( u, v \) are swappable across a point \( \alpha \) in \( J \) if \( u(\alpha) = v(\alpha) \).

(B) \( N \) is invariant under 1-equimeasurability:

\[
N(u) = N(v) \quad \text{whenever} \quad u \approx v.
\]

(C) \( N \) is continuous:

\[
N(u_n) \to N(u_0) \quad \text{whenever} \quad \|u_n - u_0\|_{W^1_p(J)} \to 0.
\]

For the continuity of \( g \) implies that the sequence \( \{G_{u_n}\} \) converges to \( G_{u_0} \) in measure. Moreover, the \( L^p \)-convergence of \( D_{u_n} \) to \( D_{u_0} \) implies that every subsequence \( \{u_{n_i}\} \) possesses a subsequence \( \{u_{n_{i'}}\} \) for which the sequence \( \{D_{u_{n_{i'}}}\} \) is dominated by an \( L^p(J) \) function. Hence it follows from (2.6) that \( \{G_{u_{n_{i'}}}\} \) converges to \( G_{u_0} \) in measure, dominatedly in \( L^p(J) \). Thus \( \{N(u_{n_{i'}})\} \) converges to \( N(u_0) \) and the continuity of \( N \) follows.

Now suppose in addition that \( g: \mathbb{R}^2 \to \mathbb{R} \) satisfies

\[
(2.8) \quad g(x_0, 0) = 0 \quad \text{for all} \quad x_0 \in \mathbb{R}.
\]
In this case condition (A) is readily seen to imply

\[ (A') \quad \text{\textit{N is 1-envelope additive:}} \]
\[ N(u \oplus v) = N(u) + N(v) \quad \text{whenever} \quad u, v \in W_1^f(J) \]

are envelope compatible,

and condition (B) is seen to imply

\[ (B') \quad \text{\textit{N is invariant under generalized 1-equimeasurability:}} \]
\[ N(u) = N(v) \quad \text{whenever} \quad u \sim v , \]

where \( u \sim v \) means that \( u = (u, Du), v = (v, Dv) \) are stochastically equivalent on \( K(Du) \) and \( K(Dv) \), respectively: for each Borel set \( B \subset R^2 \)

\[ m(u^{-1}(B) \cap K(Du)) = m(v^{-1}(B) \cap K(Dv)) \cdot \]

Note that it follows from (2.4) and (A') that \( N \) determines for each \( u \in W_1^f(J) \) an additive set function \( \nu_u \) defined on the subintervals of \( J \) by

\[ \nu_u(J') = \int_{J'} Gv = N(w') \quad u \in W_1^f(J) . \]

This enables us to deduce the following additional property of \( N \).

\[ (D) \quad \text{\textit{N is locally uniformly continuous in (interval) variation:}} \]
\[ \lim_{\delta \to 0} V_M(\delta; N) = 0 \quad \text{for each} \quad M > 0 , \]

where the quantity \( V_M(\delta; N) \) is given by

\[ V_M(\delta; N) = \sup \sum_{i=1}^t |N(u_i^r) - N(v_i^t)| , \]

with the supremum being taken over all finite partitions of \( J \) into intervals \( \{J_i\}_{i=1} \) and all sets of pairs \( u, v \in W_1^f(J) \) satisfying

\[ \|u\|_{\infty} \leq M, \quad \rho(U, V) \leq \delta , \]

where \( U = \Sigma u_i \mathcal{X}_i, DU = \Sigma Du_i \mathcal{X}_i \) and similarly for \( V \).

This follows from the uniform continuity and boundedness of \( g \) on sets of the form

\[ \{x = (x_0, x_1); |x_i| \leq M \} \subset R^2 . \]

In fact, putting \(|x| = \max \{|x_0|, |x_1|\} \) we have
We are concerned here with the extent to which these properties characterize functionals of the form (2.7). The principal part of our result is the following theorem.

**THEOREM 2.1.** Let $J$ be a bounded interval and let $N$ be a real functional on $W^1(J)$, $1 \leq p \leq \infty$, which possesses the properties:

(A') $N(u \oplus v) = N(u) + N(v)$ whenever $u, v$ are envelope compatible,

(B') $N(u) = N(v)$ whenever $u \sim v$,

(C) $N(u_m) \to N(u_0)$ whenever $\|u_m - u_0\|_{W^1(J)} \to 0$, $1 \leq p < \infty$,

(D) $\lim_{\delta \to 0} V_N(\delta; N) = 0$ for each $M > 0$.

Then there exists a unique continuous function $g: \mathbb{R}^2 \to \mathbb{R}$ satisfying:

(2.9) $g(x_0, 0) = 0$ for all $x_0 \in \mathbb{R}$,

such that

(2.10) $N(u) = \int_J g(u(t), u'(t))dt$ for all $u \in W^1(J)$.

Moreover, if $1 \leq p < \infty$, then $g$ satisfies a growth condition of the form

(2.11) $|g(x_0, x_i)| \leq K_M(1 + |x_i|)^p$ whenever $|x_0| \leq M$.

The proof of this result utilizes the Lebesgue differentiability of the interval function $v_u$, for each $u \in W^1(J)$. The Lebesgue derivative $f_u$ of $v_u$ is shown to belong to $L^1(J)$, and $v_u$ is shown to be the indefinite integral of $f_u$. Then it is shown that there exists a unique continuous function $g: \mathbb{R}^2 \to \mathbb{R}$ such that

$$f_u(t) = g(u(t), u'(t)) \text{ a.e., for each } u \in W^1(J).$$

The representation (2.10) follows immediately for piecewise linear functions in $W^1(J)$. It is then extended to arbitrary $u \in W^1(J)$ and eventually to all of $W^1(J)$ by a limiting process.

We continue with the detailed proof of the theorem.

**LEMMA 2.1.** The interval function $v_u$ is finitely additive on the semi-algebra $S_f$ of subintervals of $J$, for every function $u \in W^1(J)$. 

\[\sum_{i=1}^t |N(u_i) - N(v_i)| = \sum_{i=1}^t |\nu_{u_i}(J_i) - \nu_{v_i}(J_i)| \leq \sum_{i=1}^t \int_{J_i} |g((u_i(t), u'_i(t)) - g(v_i(t), v'_i(t))|dt \leq \left( \sup_{|x|, |y| \leq M} |g(x) - g(y)| \right)m(J) + 2 \left( \max_{|x| \leq M} |g(x)| \right)\delta.\]
If \( u \) is in \( W^+_\alpha(J) \) then \( v \) is absolutely continuous and countably additive.

Proof. By (2.4), whenever \( \{J_i\}_{i=1} \) are disjoint intervals, indexed from left to right, whose union is an interval \( J_o \), then the relation

\[
u^{i_0} = (\cdots (u^{i_1} \oplus u^{i_2}) \oplus \cdots ) \oplus u_i
\]

holds. Thus the finite additivity of \( v \) on \( S_j \) follows from (A').

The absolute continuity of \( v \) for \( u \in W^+_\alpha(J) \) is proved as follows. Let \( \{J'_i\}_{i=1} \) denote any family of disjoint intervals in \( J \) and let \( \{J_i\}_{i=1} \) denote the minimal partition of \( J \) (into subintervals) generated by the \( \{J'_i\} \), indexed so that \( J_i = J'_i, 1 \leq i \leq l \). Then we may write:

\[
\sum_{i=1}^l |v(J'_i)| = \sum_{i=1}^l |v(J_i) - v(J'_i)| \leq V_u(\delta; N),
\]

where

\[
u_i = \begin{cases} u & i = 1, \ldots, l \\ 0 & \text{otherwise} \end{cases}, \quad v_i = 0, i = 1 \cdots, L,
\]

and \( M = \|u\|_{\Pi^m(J)}, \delta = \sum_{i=1}^l m(J'_i) \). The countable additivity and absolute continuity of \( v \) is, by (D), an immediate consequence of (2.12).

By a result of Lebesgue's on differentiation of interval functions [5, pp. 115, 119], Lemma 1 implies that whenever \( u \) is in \( W^+_\alpha(J) \) then the Lebesgue derivative \( f_u \) of \( v \) is defined almost everywhere, belongs to \( L'(J) \) and satisfies:

\[
v_u(I) = \int_I f_u, \quad \text{for all } I \in S_j.
\]

We have need of a somewhat more precise result.

**Lemma 2.2.** Given a point \( x = (x_0, x_1) \in R^2 \), let \( \mathcal{F} \) denote the family of all affine functions \( u \) on \( J \) with the property that for some \( t_o = t_o(u) \) interior to \( J \),

\[
(u(t_o), u'(t_o)) = x.
\]

Then for each \( u \in \mathcal{F} \), the Lebesgue derivative of \( v \) at \( t_o \) exists and has a value which is independent of the choice of \( u \in \mathcal{F} \):

\[
(Dv_u)(t_o) = g(x) \quad \text{for all } u \in \mathcal{F}.
\]

Moreover the function \( g: R^2 \to R \) defined in this way is continuous and satisfies (2.9).
Proof. For simplicity of notation assume that $J$ is an open
interval. Given a function $u \in \mathcal{F}_x$ and a closed subinterval $I_0 \subseteq J$
containing the point $t_0(u)$, put $\eta = m(I_0)$ and denote
\begin{equation}
(2.14) \quad I_j = I_0 + j\eta, \quad u_j(t) = u(t - j\eta), \quad j = \pm 1, \pm 2, \cdots
\end{equation}
Let $\mathcal{M}(I_0) = \{I_j\}_{j=-k}^k$ denote the maximal family of
these intervals containing points of $J$. Let $u_{x(I_0)}$ denote the (single-valued)
piecewise continuous function on $J$ given by
\begin{equation}
(2.15) \quad u_{x(I_0)} = \sum_{-k}^l u_j I_j,
\end{equation}
where $I'_0 = I_0$, and $I_j, j \neq 0$, is $I_j \cap J$ with either its right endpoint
excised ($j < 0$) or its left endpoint excised ($j > 0$).

We examine the quantity $\hat{N}(u_{x(I_0)})$ which is defined as follows:
\begin{equation}
(2.16) \quad \hat{N}(u_{x(I_0)}) = \sum_{-k}^l \nu_{u_j}(I'_j) = \sum_{-k}^l N(u_j^j).
\end{equation}
Since
\begin{equation}
(2.17) \quad u_j^j \sim u^0, \quad -k + 1 \leq j \leq l - 1,
\end{equation}
we deduce from (B') the relation
\begin{equation}
(2.18) \quad \hat{N}(u_{x(I_0)}) = (k + l - 1)N(u^0) + N(u_{x}^{l-k}) + N(u_{x}^{l}) .
\end{equation}
Moreover, by our construction
\begin{equation}
(2.19) \quad (k + l - 1)\eta < m(J) \leq (k + l + 1)\eta.
\end{equation}
Thus (2.15) implies:
\begin{equation}
(2.20) \quad \frac{1}{\eta} \nu_{u}(I_0) = \frac{1}{(k + l - 1)\eta} \hat{N}(u_{x(I_0)}) - \frac{1}{(k + l - 1)\eta} [N(u_{x}^{l-k}) + N(u_{x}^{l})]
\end{equation}
\begin{equation}
= \frac{1}{m(J) - \theta \cdot 2\eta} \hat{N}(u_{x(I_0)}) - \frac{1}{m(J) - \theta \cdot 2\eta} [N(u_{x}^{l-k}) + N(u_{x}^{l})],
\end{equation}
for some $\theta \in (0, 1)$.

In view of properties (D) and (A'), (2.20) implies that
\begin{equation}
(2.21) \quad \frac{1}{m(I_0)} \nu_{u}(I_0) - \frac{1}{m(J)} \hat{N}(u_{x(I_0)}) \longrightarrow 0 \quad \text{when} \quad m(I_0) \longrightarrow 0.
\end{equation}
Moreover, given a positive $\varepsilon$, property (D) implies that
\begin{equation}
(2.22) \quad |\hat{N}(u_{x(I_0)}) - \hat{N}(u_{x(I_0)^*})| < \varepsilon
\end{equation}
whenever $I_0, I_0^*$ are closed intervals of sufficiently small measure
containing the point $t_0$. Hence $\nu_u$ possesses a Lebesgue derivative
at $t_0 = t_0(u)$. 
Next we observe that the Lebesgue derivative 
\[(Dv_u)(t_0(u)) = \lim_{m(I) \to 0} \frac{\nu_u(I)}{m(I)}\]
is the same for all \(u \in \mathcal{F}_x\). For if \(u, \tilde{u}\) are in \(\mathcal{F}_x\) then whenever the interval \(I_0 \ni t_0(u)\) is sufficiently small, there is a corresponding interval \(\tilde{I}_0 \ni t_0(\tilde{u})\), of the same length, situated so that 
\[\tilde{u}^\approx_{t_0} \sim u^\approx_{t_0}.
\]
Property (B') then implies that 
\[\frac{\nu_u(I)}{m(I)} = \frac{\nu_{\tilde{u}}(\tilde{I})}{m(\tilde{I})},\]
from which we deduce that 
\[(Dv_u)(t_0(u)) = (Dv_{\tilde{u}})(t_0(\tilde{u})).\]

We denote the common value of all these Lebesgue derivatives by 
\[g(x): = (Dv_u)(t_0(u)) \quad \text{for all } u \in \mathcal{F}_x.
\]
It is evident from property (A') that this definition implies, when \(x = (x_0, 0)\), that \(\nu_u = 0\) for all \(u \in \mathcal{F}_x\), so that (2.9) holds.

Finally, let \(x\) and \(x^*\) be two points in \(R^2\) and let \(u\) be a function in \(\mathcal{F}_x\), with \(I_0 \subset J\) a closed interval containing \(t_0(u)\). Select \(v \in \mathcal{F}_x\), such that \(t_0(v) = t_0(u)\). Then given a positive \(\varepsilon\), property (D) implies that if \(x, x^*\) are sufficiently close and if \(m(I_0)\) is sufficiently small then 
\[(2.19) \quad |\tilde{N}(u_{x \cup I_0}) - \tilde{N}(v_{x \cup I_0})| < \varepsilon .\]
Thus by (2.17), (2.18) and (2.19), \(g\) is a continuous function on \(R^2\). This completes the proof.

**Corollary 2.1.** *The representation* 
\[N(u) = \int_J g(u(t), u'(t))dt\]
*is valid whenever* \(u \in W^1_1(J)\) *is piecewise linear.*

**Proof.** By (A') 
\[N(u) = \sum_{j=1}^{n} N(u^j)\]
where \([I_j]_{j=1}^{n}\) is a partition of \(J\) into subintervals on each of which
$u$ is linear. Hence by (2.9) it suffices to prove that for any fixed $u$ which is linear on $J$ the Lebesgue derivative (equivalently by (2.13), the Radon-Nikodym derivative) of $v_u$ satisfies:

$$f_u(t) = (Dv_u)(t) = g(u(t), u'(t)) \quad \text{a.e. } t \in J .$$

However the linear function $u$ satisfies

$$\forall \tau \in J' \quad u \in \mathcal{F}_{x(\tau)} ,$$

where

$$x(\tau) = (u(\tau), u'(\tau)), \quad \tau \in J'.$$

Consequently (2.20) follows from Lemma 2.2.

We now extend the representation to all $u \in W^1_r(J)$.

**Lemma 2.3.** The representation

$$N(u) = \int_J g(u(t), u'(t)) dt$$

is valid whenever $u$ is in $W^1_r(J)$.

**Proof.** Given $u \in W^1_r(J)$ we proceed to construct a sequence $\{u_n\} \in W^1_r(J)$ of piecewise linear functions satisfying

$$\|u_n\|_{W^1_r(J)} \leq 2 \|u\|_{W^1_r(J)} , \quad u_n \rightarrow u \quad \text{and} \quad u'_n \rightarrow u' \quad \text{a.e.}$$

(It then follows by (D) that $N(u_n) \rightarrow N(u)$.)

Consider the function $Du \in L^\infty(J)$. Now there exists a sequence $\{z_n\}$ of step functions based on subintervals of $J$ such that

$$\|z_n\|_{L^\infty(J)} \leq \|Du\|_{L^\infty(J)} , \quad z_n \rightarrow Du \quad \text{a.e.}$$

Select a point $t_0 \in J$ and define the sequence $\{u_n\}$ as follows:

$$u_n(t) = u(t_0) + \int_{t_0}^t z_n(\tau)d\tau , \quad n \geq 1 .$$

Clearly $u_n$ is piecewise linear and by (2.22) satisfies the relation

$$\|u_n\|_{W^1_r(J)} \leq 2 \|u\|_{W^1_r(J)} =: M , \quad \text{for sufficiently large } n; \quad u_n \rightarrow u \quad \text{and} \quad u'_n \rightarrow u' \quad \text{a.e.}$$

Hence by property (D)

$$N(u) = \lim_{n \to \infty} N(u_n) = \lim_{n \to \infty} \int_J g(u_n(t), u'_n(t)) dt ,$$

while by the continuity of $g$
\( \left| g(u_n(t), u'_n(t)) \right| \leq L_M = \sup_{|x| \leq M} |g(x)| , \)

\( g(u_n(t), u'_n(t)) \rightarrow g(u(t), u'(t)) \) a.e. The use of Lebesgue’s dominated convergence theorem now yields the desired result.

In order to complete the proof of the theorem in the case \( 1 \leq p < \infty \) it will be necessary to establish the growth condition (2.11).

**Lemma 2.4.** The function \( g \) defined by (*) in Lemma 2.2 satisfies, if \( 1 \leq p < \infty \), (**) a growth condition of the form:

\[
(*) \quad |g(x_0, x_i)| \leq K_p(1 + |x_i|)^p \quad \text{whenever} \quad |x_0| \leq M ,
\]

for all \( M > 0 \).

**Proof.** The given growth condition is equivalent to the assertion that for each \( M > 0 \)

\[
(2.23) \quad \sup_{|x_0| \leq M} k_{x_0} < \infty
\]

where

\[
k_{x_0} = \sup_{x_1 \in \mathbb{R}} \frac{|g(x_0, x_1)|}{(1 + |x_1|)^p} .
\]

Suppose that for some \( M \) the assertion in (2.23) is false. Select a sequence \( \{\theta_j\} \) in \( (0, 1) \) and an increasing sequence \( \{D_j\} \) in \( \mathbb{R}^+ \) such that

\[
(2.24) \quad \sum \theta_j = \frac{1}{2} , \quad \sum D_j \theta_j = \infty .
\]

We construct a sequence \( \{(c_n, d_n)\} \in [-M, M] \times \mathbb{R} \) as follows. Let \( c_n \in [-M, M] \) satisfy \( k_{c_n} > 2D_n \); then select \( d_n \in \mathbb{R} \) so that

\[
(2.25) \quad |g(c_n, d_n)| > 2D_n(1 + |d_n|)^p , \quad n \geq 1 .
\]

Clearly we can suppose without loss of generality that all \( \{d_n\} \) are of the same sign and that all \( \{g(c_n, d_n)\} \) are of the same sign, say positive in both cases. Moreover, since any infinite family \( \{c_n\} \) satisfying \( k_{c_n} > D_n, |c_n| \leq M \), has a cluster point we may suppose, by going to a subsequence if necessary, that

\[
(2.26) \quad \sum |c_{n+1} - c_n| \leq \frac{1}{2} m(J) .
\]

Now, starting from the left endpoint of \( J \) construct a family of subintervals \( \{J'_i\} \) of \( J \) satisfying
The existence of such a family follows from (2.24) and (2.26). Denote by \( \{J''_j\} \) the sequence of intervals forming the gaps between the \( \{J'_j\} \):

\[ J''_j \text{ is contiguous to } J'_j \text{ and } J'_{j+1}, \ j \geq 1 \]  

(Note that \( J''_j \) is empty whenever \( c_{j+1} = c_j \).)

We construct on each interval \( J'_j \) a continuous piecewise linear function \( v_j \) as follows:

(i) each linear piece of \( v_j \) has slope \( d_j \) or \( -1 \),

(ii) \( v_j \) attains the value \( c_j \) on each (maximal) subinterval of \( J'_j \) where \( v_j \) is linear, as well as at the endpoints of \( J'_j \),

(iii) \[ |g(v_j(t), v'_j(t)) - g(c_j, d_j)| < \frac{1}{2} |g(c_j, d_j)| \]  

wherever \( v_j(t) = d_j \).

It can be seen from (i), (ii) that the sets

\[ A_j = \{ t \in J'_j; v'_j(t) = d_j \}, B_j = \{ t \in J'_j; v'_j(t) = -1 \} \]

satisfy:

\[ m(A_j) = \frac{1}{1 + d_j} m(J'_j) = \frac{\theta_j}{(1 + d_j)^p} m(J) , \]

\[ m(B_j) = \frac{d_j}{1 + d_j} m(J'_j) = \frac{d_j \theta_j}{(1 + d_j)^p} m(J), \ j \geq 1 . \]  

By (2.27), we can construct on each nonempty interval \( J''_j \) a linear function \( w_j \), of slope either 1 or \( -1 \), taking the value \( c_j \) at the left endpoint of \( J''_j \) and the value \( c_{j+1} \) at the right endpoint.

Note that by (2.29)(i), (ii) and (2.30), \( |v_j(t) - c_j| \leq m(J'_j) \) for \( j = 1, 2, \ldots \). Thus,

\[ \sup_{t \in J'_j} |v_j(t)| \leq M + m(J) =: M_1 , \ j = 1, 2, \ldots . \]

By the definition of \( w_j \) we have also \( \sup_{t \in J''_j} |w_j(t)| \leq M, j = 1, 2, \ldots \).

Now put \( c_0 = \lim_{j \to \infty} c_j \) (see (2.26)) and examine the continuous function defined by

\[ u = \sum_{j=1}^\infty v_j \chi_{J'_j} + \sum_{j=1}^\infty w_j \chi_{J''_j} + c_0 \chi_{\bar{J}} \]
where \( \tilde{J} = J' \cup \bigcup_{j=1}^{r} (J_j' \cup J_j'') \). Obviously \( u \) is locally absolutely continuous on the interior of the intervals \( \tilde{J} \) and \( J \setminus \tilde{J} \). We now note that \( u \) is in \( W^p(\tilde{J}) \) since by (2.29), (2.30), (2.24):

\[
\|Du\|_{L^p(\tilde{J})} = \left[ \sum_{j=1}^{\infty} \int_{J_j} |v_j'(t)|^p \, dt + \sum_{j=1}^{\infty} \int_{J_j'} |w_j'(t)|^p \, dt \right]^{1/p} \\
\leq \left[ \sum_{j=1}^{\infty} (d_j^p m(A_j) + m(B_j)) + m(\tilde{J}) \right]^{1/p} \\
\leq \left( \frac{5}{2} \right)^{1/p} m(\tilde{J})^{1/p} < \infty.
\]

Moreover, the sequence \( \{u^n\} \subset W^p(\tilde{J}) \) which is defined by

\[
u^n = \sum_{j=1}^{n} v_j \chi_{J_j} w_{j'} \chi_{J_j'} + \sum_{j=1}^{n} w_{j'} \chi_{J_j} + c_n i \chi_{J_n}
\]

where \( \tilde{J}_n = J' \cup \bigcup_{j=1}^{n} (J_j' \cup J_j'') \), is easily seen to converge to \( u \):

\[
\|u - u^n\|_{W^p(\tilde{J})} \longrightarrow 0.
\]

Consider the sequence \( \{N(u^n)\} \). By use of Lemma 2.3, 2.29(iii), (2.24), (2.25) and (2.30) we have:

\[
N(u^n) = \sum_{j=1}^{n} \int_{J_j} g(v_j(t), v_j'(t)) \, dt + \int_{J_j'} g(w_{j'}(t), w_{j'}'(t)) \, dt \\
\leq \sum_{j=1}^{n} \left[ \frac{1}{2} g(c_j, d_j) m(A_j) - \left( \sup_{[x_0, -1]} |g(x_0, -1)| \right) m(B_j) \right] \\
- \sum_{j=1}^{n} \left( \sup_{[x_0, \pm 1]} |g(x_0, \pm 1)| \right) m(J_j') \\
\leq \left( \sum_{j=1}^{n} D_j \theta_j - \sup_{[x_0, \pm 1]} |g(x_0, \pm 1)| \right) m(\tilde{J}) \longrightarrow \infty.
\]

However, by (2.32) this contradicts property (C). The lemma is proved.

We are now in a position to complete the proof of the theorem in the case \( 1 \leq p < \infty \).

\textbf{Proof of Theorem 2.1.} Given a function \( u \in W^p(\tilde{J}) \), let \( \{u_n\} \) denote a sequence in \( W^p(\tilde{J}) \) converging to \( u \):

\[
\|u_n - u\|_{W^p(\tilde{J})} \longrightarrow 0.
\]

Then it follows that the (continuous) functions \( \{u_n\} \) converge uniformly to \( u \); in particular,

\[
|u_n(t)| \leq M < \infty, \quad t \in J, \quad n \geq 1.
\]

Moreover, (2.33) implies that, by selecting a subsequence \( \{u_{n_k}\} \), we
can require:

\begin{equation}
(2.35) \quad D\nu_n \rightarrow D\nu \text{ a.e. and } \sum_{i=1}^\infty ||D\nu_n - D\nu||_{L^p(J)} < \infty.
\end{equation}

It follows from (2.35) that the functions \{D\nu_n\} satisfy:

\begin{equation}
(2.36) \quad |D\nu_n(t)| \leq z(t) \text{ a.e., for some } z \in L^p(J).
\end{equation}

Hence, by (2.34) and Lemma 2.4 we deduce that the functions \{G\nu_n\} are dominated by an integrable function:

\begin{equation}
(2.37) \quad |G\nu_n(t)| = |g(\nu_n(t), u'_n(t))| \leq K_{\nu}(1 + |u'_n(t)|)^p \\
\leq K_{\nu}(1 + |z(t)|)^p \text{ a.e., } i \geq 1.
\end{equation}

Now by Lemma 2.3

\[ N(\nu_n) = \int G\nu_n, \quad n \geq 1. \]

Combining (2.36) and (2.37) we deduce by means of property (C) the relation:

\[ N(u) = \lim_{n \rightarrow \infty} N(\nu_n) = \lim_{n \rightarrow \infty} \int g(\nu_n(t), u'_n(t))dt = \int g(u(t), u'(t))dt, \]

where the last equality utilizes the Lebesgue dominated convergence theorem. This completes the argument.

**Remark 2.1.** We point out that the proof of Theorem 2.1 actually utilized only the following weaker form of condition (B):

(B'') $N$ is invariant under *inessential translation*:

\[ N(T_h u) = N(u) \text{ whenever } T_h u \sim u, \]

where $T_h u, |h| < \text{mes } J$, is defined by:

\[ (T_h u)(t) = \begin{cases} 
 u(\inf J), & t < h + \inf J \\
 u(t - h), & t - h \in J \\
 u(\sup J), & t > h + \sup J.
\end{cases} \]

Given any functional $N: W_1^p(J) \rightarrow R$ which satisfies conditions (A), (B), (C) there is a canonical decomposition of $N$

\begin{equation}
(2.38) \quad N = \tilde{N} + M_f,
\end{equation}

where $\tilde{N}$ is envelope additive and thus satisfies (A'), (B), (C) while $M_f$ is a “lower order” functional of the form:
\[ M_f(u) = \int_J f(u(t))dt. \]

To see this, define the function \( f: \mathbb{R} \to \mathbb{R} \) in terms of the values that \( N \) assumes on the one-dimensional subspace of constant functions:

\[ f(c) = \frac{N(c)}{m(J)}, \quad c \in \mathbb{R}. \tag{2.39} \]

Then \( f \) is continuous by property (C) of \( N \), from which it readily follows that \( M_f \) satisfies (A), (B), (C).

Consequently, the function \( \tilde{N} \) satisfies (A), (B), (C) and in addition annihilates the constant functions:

\[ \tilde{N}(c) = 0, \quad c \in \mathbb{R}. \tag{2.40} \]

These facts imply that \( \tilde{N} \) also satisfies the envelope additivity condition (A'); for whenever \( u, v \in W^1_f(J) \) are envelope compatible we have from (A)

\[ N(u \oplus v) = N(u) + N(v) - N(v\chi_{J''} + u\chi_{J''}), \]

while (2.40) ensures the vanishing of the last term, since the function \( v\chi_{J''} + u\chi_{J''} \) is necessarily constant. The fact that \( \tilde{N} \) also satisfies \((B')\) lies deeper; a proof is given in the appendix.

Utilizing the above remark we deduce from Theorem 2.1 the following result:

**Theorem 2.2.** Let \( J \) be a bounded interval and let \( N \) be a real functional on \( W^1_f(J) \), \( 1 \leq p \leq \infty \), which possesses the properties:

- (A) \( N(u) + N(v) = N(u\chi_{J'} + v\chi_{J''}) + N(v\chi_{J'} + u\chi_{J''}) \) whenever \( u, v \) are swappable on \( \{J', J''\} \),
- (B) \( N(u) = N(v) \) whenever \( u \sim v \),
- (C) \( N(u_m) \to N(u) \) whenever \( \|u_m - u\|_{W^1_f(J)} \to 0 \), \( 1 \leq p < \infty \),
- (D) \( \lim_{\delta \to 0^+} V_{\delta}(\delta; \tilde{N}) = 0 \) for each \( M > 0 \), where \( \tilde{N} \) is the envelope additive part of \( N \), as in (2.38).

Then there exists a unique continuous function \( g: \mathbb{R}^2 \to \mathbb{R} \) such that

\[ N(u) = \int_J g(u) \text{ for all } u \in W^1_f(J). \tag{2.41} \]

Moreover, when \( 1 \leq p < \infty \) the function \( g \) satisfies a growth condition of the form:

\[ |g(x_m, x)| \leq K_p(1 + |x|)^p \text{ for } |x_0| \leq M, \text{ for all } M > 0. \]

The argument proceeds as follows. Let \( \tilde{N} \) and \( f \) be defined as in (2.38) and (2.39). Then \( f \) is a continuous function and as is shown
in the appendix, \( \hat{N} \) satisfies the assumptions of Theorem 2.1. Let \( \hat{g}: \mathbb{R}^2 \to \mathbb{R} \) be a kernel for \( \hat{N} \) as in Theorem 2.1. Then the function \( g: \mathbb{R}^2 \to \mathbb{R} \) given by
\[
g(x_0, x_1) = \hat{g}(x_0, x_1) + f(x_0)
\]
possesses all the properties stated in Theorem 2.2. The uniqueness of \( g \) is clear from the proof.

Note that the converse assertion that every functional of the form (2.41) that fulfills the stated conditions, satisfies (A), (B), (C), (D) is immediate from what has gone before.

3. Representation for operators. Here we obtain a characterization for those non-linear operators \( G: W^p_1(J) \to L^q(J), 1 \leq p \leq \infty, 1 < q < \infty \), possessing the form (2.5).

It should be noted that each of the conditions (A), (A'), (B), (B'), (C), (D) possesses an analogue which is applicable to such operators. One interprets inequalities in the almost everywhere sense and replaces absolute values by norms. Thus one can formulate the following properties:

(Ao) \( G \) is additively invariant under swapping:
\[
Gu + Gv = G(u\chi_{J'} + v\chi_{J''}) + G(v\chi_{J'} + u\chi_{J''})
\]
whenever \( u, v \in W^p_1(J) \) are swappable across the partition \( \{J', J''\} \).

(\(\Lambda'\)) \( G \) is 1-envelope additive:
\[
G(u \oplus v) = Gu + Gv \quad \text{whenever} \quad u, v \in W^p_1(J)
\]
are envelope compatible.

(Bo) \( G \) is invariant, up to equimeasurability, under 1-equimeasurability:
\[
Gu \approx Gv \quad \text{whenever} \quad u \approx v, \ u, v \in W^p_1(J).
\]

(B'0) \( G \) is invariant, up to equimeasurability, under generalized 1-equimeasurability:
\[
Gu \approx Gv \quad \text{whenever} \quad u \approx v, \ u, v \in W^p_1(J).
\]

(Co) \( G \) is continuous:
\[
\|G(u_m) - G(u_0)\|_{L^q(J)} \to 0 \quad \text{whenever} \quad \|u_m - u_0\|_{W^p_1(J)} \to 0,
\]
\[
1 \leq p < \infty.
\]

(Do) \( G \) is locally uniformly continuous in (interval) variation:
\[
\lim_{\delta \to 0^+} V_M(\delta; G) = 0 \quad \text{for each} \quad M > 0
\]
where $V_M(\beta; G)$ is defined by

$$V_M(S; G) = \sup \sum_{i=1}^l ||G(u_i') - G(v_i')||_{L^1(J_i)}$$

with the supremum being taken over all finite partitions $\{J_i\}_{i=1}^l$ of $J$ into subintervals and all sets of pairs $u_i, v_i \in W_1^1(J_i)$ satisfying:

$$||u_i||_{W_1^1(J_i)}, ||v_i||_{W_1^1(J_i)} \leq M, \quad \rho(U, V) \leq \delta,$$

where $U = \sum u_i \chi_{J_i}$, $Du = \sum Du_i \chi_{J_i}$ and similarly for $V$.

It is easily verified, by arguments similar to those used for functionals $N$ of the form (2.7), that any $G$ of the form (2.5), (2.6) satisfies (A$G$), (B$G$), (C$G$). Similarly, it is seen that when $g$ satisfies (2.8) then (A$G$), (B$G$) and (D$G$) also hold. However these conditions do not suffice to characterize operators of the form (2.5); indeed the linear transformation $G: W_1^1(J) \rightarrow L^q(J)$ (with one-dimensional range) which is given by

$$(Gu)(t) = \int J Du = \text{const},$$

also satisfies (A$G$), (A$G$), (B$G$)-(D$G$). An additional localization condition is required.

(E$G$) $G$ is local:

$$K(Gu - Gv) \subset K(u - v) \quad \text{for all} \quad u, v \in W_1^1(J).$$

The inclusion should be interpreted as inclusion modulo a null set.

We can now give our characterization results for operators of the form (2.5). Again our main efforts will be directed to obtaining the result under the assumption of envelope additivity.

**Theorem 3.1.** Let $J$ be a bounded interval and let $G$ be a transformation from $W_1^1(J)$ to $L^q(J)$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, which satisfies the conditions (A$G$), (B$G$)-(E$G$).

Then there exists a unique continuous real function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$, satisfying

$$(*) \quad g(x_0, 0) = 0 \quad \text{for all} \quad x_0 \in \mathbb{R},$$

such that

$$g(x_0, u'(t)) = (Gu)(t) = g(u(t), u'(t)) \quad \text{a.e.} \quad \text{for all} \quad u \in W_1^1(J).$$

Moreover, for $1 \leq p < \infty$, $g$ satisfies a growth condition of the form:
\( |g(x_0, x_1)| \leq K_m(1 + |x_1|)^{p/q} \) whenever \( |x_0| \leq M \),
for all \( M > 0 \).

**Proof.** Note that, in view of \( (A'_G) \), the mapping \( G \) takes all constants to the zero function. Hence \( (E_0) \) ensures the validity of:

\( (3.2) \quad K(Gu) \subseteq \mathcal{J} \setminus A_u \) where \( A_u \) denotes the union of all intervals in which \( u \) is constant.

First, suppose that the Theorem holds for \( q = 1 \). In order to establish the result for \( q > 1 \) we proceed as follows. Consider \( G \) as a mapping into \( L^1 \). Then \( G \) satisfies all the assumptions of the Theorem for the case \( q = 1 \). Hence there exists a continuous function \( g: R \to R \) such that \((3.1)\) and \((^*)\) hold. The fact that \( g \) satisfies the growth condition \((3.2)\) now follows by the same argument used in the proof of Lemma 2.4. We do not repeat this argument since only minor modifications are needed. Thus in the definition of \( k_{a_0} \) and in \((2.24)\) \( p \) will be replaced by \( p/q \). In \((2.23)(iii)\) we add the condition

\[ |g(v_j(t), v_j'(t)) - g(c_j, d_j)| \leq \frac{1}{2} g(c_j, d_j), \]

whenever \( v_j'(t) = d_j \).

Recall that, by assumption \( g(c_j, d_j) > 0 \) \((j = 1, 2, \ldots)\) and condition \((2.29)(iii)\), as stated in Section 2, ensures that \( g(v_j(t), v_j'(t)) > 0 \) whenever \( v_j'(t) = d_j \((j = 1, 2, \ldots)\). In the final part of the argument we obtain a contradiction by considering the inequalities

\[
\int_{J} |Gw|\,^q dt = \sum_{j=1}^{n} \left( \int_{J_j} |g(v_j(t), v_j'(t))\,^q dt + \int_{J_j} |g(w_j(t), w_j'(t)|\,^q dt \right) \\
\geq \sum_{j=1}^{n} \frac{1}{2} g(c_j, d_j)\,^q m(A_j) - \sup_{|x_0| \leq M} |g(x_0, \pm 1)|\,^q m(J) \\
\geq \left( \sum_{j=1}^{n} D_j \theta_j - \sup_{|x_0| \leq M} |g(x_0, \pm 1)| \right) m(J) \to \infty .
\]

We turn now to the proof of the theorem in the case \( q = 1 \). For any interval \( I \subset J \) we can utilize the operator \( G \) to define a functional \( N': W^1(J) \to R \) as follows.

\[(3.3) \quad N'(u) = \int_{J} G'(u) = \int_{I} G(u') \quad \text{for all} \quad u \in W^1(J), \]

where \( u' \in W^1(J) \) is obtained by extending \( u \) to all of \( J \) by the use of constants to the right and left of \( I \) so that the resulting function is continuous on \( J \), and where the second equality follows from \( (E_0') \). Now the fact that \( G \) satisfies \( (A'_G) - (E_0) \) implies that \( N' \) satisfies \( (A') - (D) \) of Theorem 2.1. Hence there exists a unique continuous function
$g': \mathbb{R}^2 \to \mathbb{R}$ such that (*) holds and (if $p < \infty$) (3.2) holds and, in addition,

$$N'(u) = \int_I g'(u(t), u'(t)) dt \text{ for all } u \in W^1(I).$$

Next it will be shown that the functions $\{g'\}_{I \subset J}$ are identical. By (3.3) and (3.4):

$$N'(u) = N'(u') = \int_I g'(u(t), u'(t)) dt \text{ for all } u \in W^1(I),$$

where the second equality follows from (*). On comparing (3.3) and (3.5) and applying the uniqueness statement of Theorem 2.1 we deduce that

$$g' = g' \text{ whenever } I \subset J.$$

Finally, given $u \in W^1(J)$ and any $I \subset J$ let $\{I', I, I''\}$ denote the partition of $J$ (into intervals) which is induced by $I$. Then by $(A'\sigma)$$G u = Gu'' + Gu' + Gu''$

and hence by the use of $(E'\sigma)$ and (3.6)

$$\int_I Gu = \int_I Gu' = N'(u) = \int_I g'(u(t), u'(t)) dt, \ u \in W^1(J).$$

The validity of (3.7) for all $I \subset J$ clearly implies

$$(Gu)(t) = g'(u(t), u'(t)) \text{ a.e. for all } u \in W^1(J).$$

This completes the argument.

REMARK 3.1. In order to extend our results to more general operators $G$ we decompose $G$ as follows. Supposing that $G$ takes constant functions to constant functions, we let $f: \mathbb{R} \to \mathbb{R}$ be given by:

$$\sigma = f(c) = \text{const. a.e., for all } c \in \mathbb{R}.$$  

Then condition $(C\sigma)$ implies that $f: \mathbb{R} \to \mathbb{R}$ is continuous.

Now if we decompose

$$(Gu)(t) = [(Gu)(t) - f(u(t))] + f(u(t)) = (\tilde{G}u)(t) + (Fu)(t),$$

$u \in W^1(J),$

then $F$ satisfies $(A\sigma)-(E\sigma)$. Hence $\tilde{G}$ is easily seen to satisfy $(A\sigma)-(E\sigma)$ and, by construction, $\tilde{G}$ takes all constants to the zero function. It follows from this that $\tilde{G}$ also satisfies $(A'\sigma)$ and $(B'\sigma)$ and hence Theorem 3.1 is applicable. In this way we could obtain as a corollary
of Theorem 3.1 a result pertaining to transformations \(G: W^p(J) \rightarrow L^q(J)\) which are not envelope additive but instead satisfy \((A_0)\) and map constants to constants.

4. Appendix. Here it will be shown that, under the hypotheses of Theorem 2.2, the functional \(\tilde{N}\) defined in (2.38) satisfies condition \((B'')\) in addition to \((A)'\), \((B), (C), (D)\). By Theorem 2.1 (see Remark 2.1) \(\tilde{N}\) then possesses the representation (2.10), (2.11), which yields Theorem 2.2.

The first stage of the argument involves showing that in any event there exists a continuous function \(\gamma: R \rightarrow R\) such that \(N^* = \tilde{N} - P_\gamma\) satisfies \((A'), (B'')\) and \((C)\), where

\[
P_\gamma(u) = \int_J t \delta(\gamma_\circ u)(t) = t \gamma(u(t)) \left\{ \begin{array}{l}
\sup_{\inf J} K(Du) - \int_J (\gamma_\circ u)(t) dt \\
\inf_{\sup J} K(Du) - \int_J (\gamma_\circ u)(t) dt,
\end{array} \right.
\]

where \(I = [\inf K(Du), \sup K(Du)]\).

Let \(u \in W^p(J), h \in R\) be such that \(T_hu \sim u\) and \(K(Du) \subset J^\circ\). Put

\[
\alpha_u(h) = \tilde{N}(T_hu) - \tilde{N}(u).
\]

Now there exists a function \(v \in W^p(J), K(Dv) \subset J^\circ\), satisfying \(v(\inf J) = u(\sup J), v(\sup J) = u(\inf J), u \ominus v\) exists,

\[
T_hv \sim v.
\]

It follows that \(T_hu \oplus T_hv \sim u \oplus v\), so that properties \((A')\) and \((B)\) of \(\tilde{N}\) imply

\[
\alpha_u(h) = -(\tilde{N}(T_hv) - \tilde{N}(v)).
\]

Hence, denoting \(u(\inf J) = a, u(\sup J) = b\) we see that the dependence of \(\alpha_u\) on \(u\) is described by

\[
\alpha_u(h) = \alpha(a, b; h).
\]

Condition \((C)\) implies that \(\alpha: R^1 \rightarrow R\) is continuous. Moreover, (4.4) and the definition of \(v\) yield the identity

\[
\alpha(b, a; h) = -\alpha(a, b; h).
\]

Next we observe that the value of \(\alpha\) is proportional to \(h\). This follows from the relations
\[
\widehat{N}(T_h u) - \widetilde{N}(u) = \sum_{j=1}^{n} [\widehat{N}(T_{i(j/n)h} u) - \widetilde{N}(T_{i(j-1/n)h} u)], \\
T_{r,h} u \sim u \text{ for } r \in [0, 1];
\]

these relations imply
\[
\alpha(a, b; \frac{h}{n}) = \frac{h}{n} \alpha(a, b; h), \quad 1 \leq j \leq n,
\]

\[n = 1, 2, \ldots.\]

Hence there exists a continuous function \( \beta: \mathbb{R}^a \to \mathbb{R} \) such that
\[(4.7) \quad \alpha(a, b; h) = \beta(a, b)h.\]

Moreover, since for any \( c \) between \( a \) and \( b \) we can decompose
\[u = w \oplus z, \text{ where } w(\inf J) = a, w(\sup J) = z(\inf J) = c, z(\sup J) = b,\]
it follows that
\[(4.8) \quad \beta(a, c)h + \beta(c, b)h = \beta(a, b)h.\]

Together (4.6)-(4.8) imply the existence of a unique continuous function \( \gamma: \mathbb{R} \to \mathbb{R} \) satisfying
\[(4.9) \quad \widehat{N}(T_h u) - \widetilde{N}(u) = [\gamma(b) - \gamma(a)]h,\]

which justifies the earlier assertion that \( N^* = \widehat{N} - P_\gamma \) satisfies (B'): \n\[N^*(u) = \widehat{N}(u) - P_\gamma(u) = \widehat{N}(T_h u) - P_\gamma(T_h u) = N^*(T_h u).\]

Since (4.1) clearly implies that \( P_\gamma \) satisfies (A'), (C), the claim that \( N^* \) satisfies (A'), (B''), (C) is proved.

We now proceed by a series of propositions. For convenience we hereafter put \( J = [0, 1] \).

**Proposition 4.1.** *The function \( \gamma \) is of class \( C' \).*

**Proof.** Given \( I_\varepsilon = [-M, M] \), there exists, by condition (D), for each \( \varepsilon > 0, a, \delta > 0 \) satisfying \( V_{\varepsilon}(\delta; \widehat{N}) < \varepsilon. \) Select \( U = \sum \alpha_i \chi_{J_i}, V = \sum \nu_i \chi_{J_i} \) subject only to the following restrictions:

\[(J_i = [t_i, \bar{t}_i]) \text{ are non-overlapping subintervals of } [0, 1/2],\]
\[(4.10) \quad \|u_i\|_{w_{1,\infty}}(J_i), \|v_i\|_{w_{1,\infty}}(J_i) \leq M, \quad \rho(U, V) < \delta/2.\]
Denote

\[ u_i(\inf J_i) =: a_i, \quad v_i(\inf J_i) =: a_i', \quad u_i(\sup J_i) =: b_i, \quad v_i(\sup J_i) =: b_i' \]

and define \( \bar{U}, \bar{V} \) by

\[ \bar{U} = U + T_{1/2} U, \quad \bar{V} = V + T_{1/2} V. \]

Then \( \rho(\bar{U}, \bar{V}) < \delta \) and it follows that

\[
\varepsilon > \sum |\bar{N}(T_{1/2} u_i') - \bar{N}(T_{1/2} v_i') + \sum |\bar{N}(u_i') - \bar{N}(v_i')| \geq \sum |\bar{N}(T_{1/2} u_i') - \bar{N}(v_i')| - \frac{1}{2} |\gamma(b_i) - \gamma(a_i)| - \frac{1}{2} |\gamma(b_i') - \gamma(a_i')|,
\]

where the last equation follows from (4.9).

Consider first the following case:

\[ v_i = 0 \forall i, \quad \sum \text{mes} J_i < \delta/2, \quad u_i|_{J_i} \text{ has constant slope of magnitude } \pm M. \]

Equation (4.11) then implies that for every family of (possibly overlapping) subintervals \([a_i, b_i] \subset [-M, M] = I_M,\)

\[
\sum |b_i - a_i| < M\delta/2 \implies \sum |\gamma(b_i) - \gamma(a_i)| < 2\varepsilon.
\]

This condition ensures that \( \gamma \) is absolutely continuous (in fact, Lipschitz continuous) on \( I_M. \) Hence the derivative \( \gamma' \) is defined on a subset \( E \) of total measure in \( I_M = (-M, M). \) We proceed to show that \( \gamma' \) is uniformly continuous on \( E. \) Thus \( \gamma' \) is equivalent to a \textit{continuous} function on \( I_M, \) from which it follows that the absolutely continuous function \( \gamma \) is actually \textit{C}^1.

Given \( a^*, b^* \in E \) we show

\[
|\tilde{a}^* - a^*| < \delta \implies |\gamma'(a^*) - \gamma'(\tilde{a}^*)| \leq 4\varepsilon/M.
\]

Let us define \( U, V \) by means of \( U = \sum u_i x_i, \quad V = \sum v_i x_i, \) where

\[ J_i = [\bar{t}_i, \bar{t}_i] = \left[ \frac{i - 1}{2n}, \frac{i}{2n} \right], 1 \leq i \leq n, \quad a_i = a^*, \quad b_i = a^* + M/2n \]

\[ a_i' = \tilde{a}^*, \quad b_i' = \tilde{a}^* + M/2n. \]

For \( n \) sufficiently large the functions \( u_i, v_i \) will satisfy \( \|u_i\|_{w_1^2(J_i)}, \|V_i\|_{w_1^2(x_i)} \leq M. \) The inequality (4.11) now reads

\[
\varepsilon > \frac{n/2}{M} |\gamma(a^* + M/2n) - \gamma(a^*) - (\gamma(\tilde{a}^* + M/2n) - \gamma(\tilde{a}^*))| \geq M/4 \left| \frac{\gamma(a^* + M/2n) - \gamma(a^*)}{M/2n} - \frac{\gamma(\tilde{a}^* + M/2n) - \gamma(\tilde{a}^*)}{M/2n} \right|. \]
Proceeding to the limit as \( n \to \infty \), we obtain the final inequality in (4.13), which completes the argument.

**Proposition 4.2.** \( P_r \) satisfies condition (D).

**Proof.** Given \( M \) and \( \varepsilon > 0 \), select \( \delta > 0 \) and let \( U = \Sigma u_i \chi_{J_i} \), \( V = \Sigma v_i \chi_{J_i} \) be chosen arbitrarily subject to:

\[
J_i \subset [0, 1]; \quad \| u_i \|_{W_{1, \infty}(J)}, \quad \| v_i \|_{W_{1, \infty}(J_i)} \leq M; \quad \rho(U, V) < \delta.
\]

We can assume, by partitioning the intervals \( J_i \) if necessary, that \( \text{mes} J_i = \bar{t}_i - t_i \leq \delta/M, \forall i \). It then follows that

\[
(4.14) \quad | a_i' - a_i | > 2\delta \implies | u_i(t) - v_i(t) | > \delta, \quad \forall t \in J_i
\]

\[
| a_i' - a_i | \leq 2\delta \implies \| u_i - v_i \|_{L^\infty(J_i)} < 3\delta.
\]

Now applying a well-known chain rule we obtain:

\[
\sum | P_r(u_i^t) - P_r(v_i^t) | = \sum \left[ \int_{J_i} t | \gamma'(u_i(t)) u_i(t) - \gamma'(v_i(t)) v_i(t) | dt \right]
\]

\[
\leq \sum_{|a_i - a_i| \leq \delta} \int_{J_i} t | \gamma'(u_i(t)) - \gamma'(v_i(t)) | | u_i(t) | + | \gamma'(v_i(t)) | | u_i(t) | dt
\]

\[
+ \sum_{|a_i - a_i| > \delta} M \left[ \sup_{|s| \leq M} | \gamma'(s) | + \sup_{|s| \leq M} | \gamma'(s) | \right] | \bar{t}_i - t_i | + 2M\delta \sup_{|s| \leq M} | \gamma'(s) | + \sum_{|a_i - a_i| > \delta} 2M \sup_{|s| \leq M} | \gamma'(s) | | \bar{t}_i - t_i |.
\]

Clearly for \( \delta \) sufficiently small the right side will be less than \( \varepsilon \), which yields the proof.

By Proposition 4.2 it follows that \( N^* = \bar{N} - P_r \) satisfies (D) as well as \( (A'), (B''), (C) \). Hence by Theorem 2.1 (see Remark 2.1) there exists a unique continuous function \( g^*: \mathbb{R}^2 \to \mathbb{R} \) such that

\[
(4.15) \quad N^*(u) = \int g^*(u(t), \dot{u}(t)) dt.
\]

**Proposition 4.3.** The function \( \gamma \) is identically zero, so that \( P_r \) is the zero functional and \( \bar{N} = N^* \).

**Proof.** Given \( a \neq b \), select \( u \in W_{1,p}(J) \) satisfying
\[ u(\inf J) = u(\sup J) = a, \quad u(t_0) = b \quad \text{for some} \quad t_0 \in J. \]

Let the function \( \bar{u} \in W_{1,\infty}(J) \) be defined by
\[ \bar{u}(t) = u(\bar{t}) \quad \text{where} \quad \bar{t} = t - t_0(\mod 1). \]

Clearly \( \bar{u} \sim u \) so that condition (B) implies \( \bar{N}(\bar{u}) = \bar{N}(u) \) while (4.15) implies \( N^*(\bar{u}) = N^*(u) \), and hence \( P_\gamma(\bar{u}) = P_\gamma(u) \). On the other hand, we deduce by (4.1) that
\[ P_\gamma(\bar{u}) - P_\gamma(u) = \gamma(b) - \gamma(a). \]

This completes the argument.

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Received October 17, 1975. Research partially supported by the U. S. National Science Foundation under Grant MPS 71-02776-A03.

Carnegie-Mellon University,
Carnegie-Mellon University and Technion
Pacific Journal of Mathematics
Vol. 65, No. 1 September, 1976

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