

Pacific Journal of Mathematics

**SUBORDINATING FACTOR SEQUENCES AND CONVEX
FUNCTIONS OF SEVERAL VARIABLES**

JAMES MILLER

SUBORDINATING FACTOR SEQUENCES AND CONVEX FUNCTIONS OF SEVERAL VARIABLES

JAMES MILLER

In this paper we consider univalent holomorphic maps of E^n , the unit disk in C^n . We generalize Wilf's subordinating factor sequences to functions on E^n and use this characterization to obtain a covering theorem and bounds for convex mappings in C^n .

1. Introduction. Let K^n denote the class of functions F which are holomorphic and univalent in $E^n = \{z = (z_1, \dots, z_n) : \text{Max}_{1 \leq i \leq n} |z_i| < 1\}$, maps E^n onto a convex region in C^n , and satisfy $F(0) = 0$ and the Jacobian J of the mapping F is nonsingular. Let G and H be holomorphic in E^n . If $G(E^n) \subset H(E^n)$, then G is subordinate to H ($G < H$). If $F = (F_1, \dots, F_n) \in K^n$ then each F_i has an expansion of the form

$$F_i(Z) = \sum_{k=1}^{\infty} \sum_{\nu_1 + \dots + \nu_n = k} a_{\nu_1 \dots \nu_n}(i) z_1^{\nu_1} \dots z_n^{\nu_n}.$$

In this paper we characterize the sequences $\{c_{\nu_1 \dots \nu_n}(i)\}$ ($i = 1, \dots, n$) such that the mapping

$$H = (H_1, \dots, H_n)$$

where

$$H_i(Z) = \sum_{k=1}^{\infty} \sum_{\nu_1 + \dots + \nu_n = k} c_{\nu_1 \dots \nu_n}(i) a_{\nu_1 \dots \nu_n}(i) z_1^{\nu_1} \dots z_n^{\nu_n}$$

is subordinate to F , for all $F \in K^n$. Then we obtain a covering theorem and bounds for convex mappings.

For $n = 1$, the class K^1 is the classical family of univalent functions $F(z) = \sum_{k=1}^{\infty} a_k z^k$ which maps the unit disk onto a convex domain. Wilf [4] has characterized the sequences $\{c_k\}$ (subordinating factor sequences) such that $h(z) = \sum c_k a_k z^k$ is subordinate to $f(z) = \sum_{k=1}^{\infty} a_k z^k$ whenever $f \in K^1$. For $n > 1$, Suffridge [3] has given the following characterization of the class K^n .

THEOREM A. *Suppose $F: E^n \rightarrow C^n$ is holomorphic, $F(0) = 0$, and that J is nonsingular for all $Z \in E^n$. Then F is a univalent map of E^n onto a convex domain if and only if there exists univalent mappings $f_j \in K^1$ ($1 \leq j \leq n$) such that $F(Z) = T(f_1(z_1), \dots, f_n(z_n))$ where T is a nonsingular linear transformation.*

From Theorem A we see that if $F = (F_1, \dots, F_n) \in K^n$ then

$$F_i(z_1, \dots, z_n) = \sum_{k=1}^{\infty} (\alpha_{i1}^k z_1^k + \dots + \alpha_{in}^k z_n^k).$$

Thus we could represent $F \in K^n$ by the column vector

$$F(Z) = \sum_{K=1}^{\infty} A_k Z^k$$

where

$$A_k = \begin{bmatrix} \alpha_{i1}^k & \dots & \alpha_{in}^k \\ \vdots & & \\ \alpha_{n1}^k & & \alpha_{nn}^k \end{bmatrix} \quad Z^k = \begin{bmatrix} z_1^k \\ \vdots \\ z_n^k \end{bmatrix}.$$

2. Subordinating factor sequences. An infinite sequence $\{C_k\}$ of $n \times n$ matrices of complex numbers will be called a subordinating factor sequence if for each $F(Z) = \sum A_k Z^k \in K^n$ we have $\sum C_k \odot A_k Z^k < F(Z)$, where $C_k \odot A_k$ is the Hadamard product. If $C = (c_{ij})$ and $A = (a_{ij})$ then $C \odot A = (c_{ij} a_{ij})$. Let \mathcal{F}^n denote the collection of subordinating factor sequences.

THEOREM 1. *If $\{C_k\} \in \mathcal{F}^n$, then for each k the rows of $C_k = (c_{ij}^k)$ are identical, that is, for each k ($k = 1, 2, \dots$) and each j ($j = 1, \dots, n$) we have $c_{1j}^k = c_{2j}^k = \dots = c_{nj}^k$.*

Proof. Let $\{C_k\} \in \mathcal{F}^n$. First consider $k = 1$. Pick $\zeta = (\zeta_1, \dots, \zeta_n) \in E^n$ where $\zeta_i \neq 0$ and if $c_{jj}^1 \neq 0$ then $\zeta_j = 1/2e^{-i\alpha}$ with $\alpha = \arg c_{jj}^1$ if $c_{jj}^1 = 0$ then $\zeta_j = 0$. Let $\delta = (c_{ji}^1 - c_{ii}^1)\zeta_i$. If $\delta = 0$, then $c_{ji}^1 = c_{ii}^1$. If $\delta \neq 0$, let $M = 1/\delta$. Then define the mapping $F = (F_1, \dots, F_n)$ where $F_i(Z) = Mz_i$, $F_j(Z) = Mz_i + z_j$, and $F_k(Z) = z_k$ when neither $k \neq i$ or $k \neq j$. The mapping F is a convex univalent map by Theorem A. Thus since $\{C_k\} \in \mathcal{F}^n$ the mapping $H = (H_1, \dots, H_n)$, where $H_i(Z) = Mc_{ii}^1 z_i$, $H_j(Z) = Mc_{ji}^1 z_i + c_{jj}^1 z_j$ and $H_k(Z) = c_{kk}^1 z_k$ for $k \neq i$ or $k \neq j$, is subordinate to F . In particular, there is a $Z \in E^n$ such that $H(\zeta) = F(Z)$, which says

$$Mz_i = Mc_{ii}^1 \zeta_i$$

and

$$Mz_i + z_j = Mc_{ji}^1 \zeta_i + c_{jj}^1 \zeta_j.$$

Solving for z_j we obtain

$$z_j = M(c_{ji}^1 - c_{ii}^1)\zeta_i + c_{jj}^1 \zeta_j = 1 + \frac{1}{2}|c_{ij}^1| \geq 1.$$

This contradicts the fact that $|Z| < 1$. Thus we have $\delta = 0$ or $c_{1j}^1 = c_{2j}^1 = \dots = c_{nj}^1$ for $j = 1, \dots, n$.

For $k > 1$ we define the mapping $F = (F_1, \dots, F_n)$ where

$$F_i(Z) = Mz_i + \frac{Mz_j^k}{k^2}, \quad F_j(Z) = Mz_i + \frac{Mz_i^k}{k^2} + z_j, \quad \text{and} \quad F_k(Z) = z_k$$

for neither $k \neq i$ or $k \neq j$. Then the proof that $c_{1j}^k = c_{2j}^k = \dots = c_{nj}^k$ is similar to the proof for $k = 1$.

From Theorem 1 we have that if $\{C_k\} \in \mathcal{F}^n$, then for each k the rows of C_k are identical. For the $n \times n$ matrices C_k we will use the notation

$$C_k = \begin{bmatrix} c_1^k & \dots & c_n^k \\ \vdots & & \vdots \\ c_1^k & \dots & c_n^k \end{bmatrix} = (c_1^k, \dots, c_n^k).$$

Using Theorem 1 we are now able to characterize class \mathcal{F}^n .

THEOREM 2. *The following are equivalent:*

- (i) $\{C_k\} \in \mathcal{F}^n$ where $C_k = (c_1^k, \dots, c_n^k)$.
- (ii) For each $j = 1, \dots, n$ we have

$$\operatorname{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} c_j^k z_j^k \right\} > 0 \quad \text{for} \quad |z_j| < 1.$$

- (iii) For each $j = 1, \dots, n$ there is a nondecreasing function Ψ_j on $[0, 2\pi]$ such that

$$c_j^k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} d\Psi_j(\theta) \quad \text{and} \quad c_j^0 = 1.$$

Proof. The Herglotz's integral representation for positive harmonic functions proves that (ii) and (iii) are equivalent. Let $\{C_k\} \in \mathcal{F}^n$, where $C_k = (c_1^k, \dots, c_n^k)$. Let $f_i(z_i) = z_i/(1 - z_i)$. Then by Theorem A the mapping F is in K^n . We may write

$$F(Z) = \sum_{k=1}^{\infty} A_k Z^k$$

where $A_k = (a_{ij}^k)$ and $a_{ij}^k = 0$ if $i \neq j$ and $a_{ii}^k = 1$ then the mapping

$$H(Z) = \sum_{k=1}^{\infty} C_k \odot A_k Z^k$$

is subordinate to F . The mapping H has components $H_i(Z) = \sum_{k=1}^{\infty} c_i^k z_i^k$. Since $H < F$ we have that $H_i(F_i) \subset f_i(E_i)$ or $\operatorname{Re} \{H_i(E_i)\} \geq -1/2$ where $E_i = \{z_i: |z_i| < 1\}$. Thus $\operatorname{Re} \{ \sum_{k=1}^{\infty} c_i^k z_i^k \} > -1/2$ for $i =$

$1, \dots, n$, Now suppose (iii) holds. Let $F \in K^n$. Then by Theorem A there exists a nonsingular matrix T and functions $f_1, \dots, f_n \in K^1$, where $f_i(z_i) = \sum_{k=1}^{\infty} a_k(i)z_i^k$, such that

$$F(Z) = T \begin{bmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{bmatrix}$$

where F is a column vector. Then

$$\begin{aligned} H(Z) &= \sum C_k \odot A_k z^k = T \begin{bmatrix} \sum_{k=1}^{\infty} c_1^k a_k(1) z_1^k \\ \vdots \\ \sum_{k=1}^{\infty} c_n^k a_k(n) z_n^k \end{bmatrix} \\ &= T \begin{bmatrix} \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{ik\phi} d\Psi_1(\phi) a_k(1) z_1^k \\ \vdots \\ \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{ik\phi} d\Psi_n(\phi) a_k(n) z_n^k \end{bmatrix} \\ &= T \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{\infty} a_k(1) r_1^k e^{i(j(\theta_1+\phi))} d\Psi_1(\phi) \\ \vdots \\ \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{\infty} a_k(n) r_n^k e^{i(k(\theta_n+\phi))} d\Psi_n(\phi) \end{bmatrix} \\ &= T \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} f_1(r_1 e^{i(\theta_1+\psi)}) d\Psi_1(\phi) \\ \vdots \\ \frac{1}{2\pi} \int_0^{2\pi} f_n(r_n e^{i(\theta_n+\psi)}) d\Psi_n(\phi) \end{bmatrix} \end{aligned}$$

where $z_j = r_j e^{i\theta_j}$. Since each integral in the left hand side is the centroid of a nonnegative mass distribution of total mass one on a convex curve, the value of each integral must lie inside its convex curve. Further since T is a nonsingular linear transformation $H(Z)$ lies inside the image of the polydisk of radius (r_1, \dots, r_n) . (A polydisk or radius (r_1, \dots, r_n) is the set $\{(z_1, \dots, z_n) : |z_i| \leq r_i \text{ for } i = 1, \dots, n\}$.) Thus $H \prec F$.

3. Convex mappings in C^n . We now apply Theorem 2 to obtain some results for mapping in K^n .

COROLLARY 1. For $n > 1$ let $G \in K^n$, where $G(Z) = \sum B_k Z^k$.

Then the mapping

$$G_F^*(Z) = \sum B_k \odot A_k Z^k ,$$

where $F(Z) = \sum A_k Z^k \in K^n$, is not subordinate to F for all $F \in K^n$.

Proof. If $G_F^* < F$ for all $F \in K^n$, then the sequence $\{B_k\}$ belongs to \mathcal{S}^n . This says that the rows of each B_k are identical by Theorem 1. Hence the Jacobian of G will be identically zero. Thus G_F^* is not subordinate to F for all $F \in K^n$.

Let $T = (t_{ij})$ be a $n \times n$ nonsingular matrix. Let K be the functions $f \in K^1$ where $f'(0) = 1$. Let KT denote the subclass of K^n which is defined by $F \in KT$ if and only if there exist functions $f_i \in K (i = 1, 2, \dots, n)$ such that

$$F(Z) = T \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}$$

where F is represented as a column vector.

COROLLARY 2. *The image of E^n under a mapping $F \in KT$ contains the polydisk $|w| < 1/2(\sum_{j=1}^n |t_{ij}|, \dots, \sum_{j=1}^n |t_{nj}|)$. The radius is sharp.*

Proof. Since the sequence $\{C_k\}$ where $C_1 = (1/2, 1/2, \dots, 1/2)$ and $C_k = (0, \dots, 0)$ for $k \geq 2$, belongs to \mathcal{S}^n , we see that the image of E^n under a mapping $F \in KT$ contains $|W| < 1/2(\sum_{j=1}^n |t_{1j}|, \dots, \sum_{j=1}^n |t_{nj}|)$. The sharpness follows by using the function

$$F(Z) = T \begin{bmatrix} \frac{z_1}{1 - z_1} \\ \vdots \\ \frac{z_n}{1 - z_n} \end{bmatrix} .$$

Ruscheweyh and Sheil-Small [2] have proven Pólya and Schoenberg's [1] conjecture that if $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum b_k z^k$ are elements of K^1 then so is the function $h(z) = \sum a_k b_k z^k$. In general for K^n this is not true as shown by the example $F(Z) = \begin{pmatrix} z_1 - z_2 \\ z_1 + z_2 \end{pmatrix} = G(Z)$. However, we do have the following Pólya and Schoenberg type of theorem.

THEOREM 3. *Let $T_1 = (p_{ij})$ and $T_2 = (q_{ij})$ be $n \times n$ nonsingular matrices such that $T = T_1 \odot T_2 = (p_{ij}q_{ij})$ is nonsingular. If $F(Z) =$*

$\sum_{k=1}^{\infty} A_k Z^k \in KT_1$ and $G(Z) = \sum_{k=1}^{\infty} B_k Z^k \in KT_2$, then $H(Z) = \sum_{k=1}^{\infty} A_k \odot B_k Z^k$ belongs to KT .

Proof. Let $F \in KT_1$ and $G \in KT_2$. Then there exists functions $f_i, g_i \in K (i = 1, \dots, n)$ such that

$$F(Z) = T_1 \begin{bmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{bmatrix}$$

and

$$G(Z) = T_2 \begin{bmatrix} g_1(z_1) \\ \vdots \\ g_n(z_n) \end{bmatrix}$$

The mapping $H(Z) = \sum_{k=1}^{\infty} A_k \odot B_k z^k$ may be written as

$$H(Z) = T \begin{pmatrix} z_1 + \sum_{k=1}^{\infty} a_k(1)b_k(1)z_1^k \\ \vdots \\ z_n + \sum_{k=2}^{\infty} a_k(n)b_k(n)z_n^k \end{pmatrix}.$$

Thus $H \in KT$ since $z_i + \sum a_k(i)b_k(i)z_i^k$ belongs to K for each i [2].

4. Bounds on Mapping in K_n . Let $F \in K^n$. Then by Suffridge's representation of mappings in K^n (Theorem A), there exist an $n \times n$ nonsingular matrix $T = (t_{ij})$ and functions $f_i(z_i) = \sum_{k=1}^{\infty} a_k(i)z_i^k (i = 1, \dots, n)$ in K^1 with $f_i'(0) = 1$ such that

$$F(Z) = T \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}.$$

Then

$$A_k = (a_{ij}) = T \begin{pmatrix} a_k(1) \\ \vdots \\ a_k(n) \end{pmatrix}$$

where $F(z) = \sum_{k=1}^{\infty} A_k Z^k$. Since

$$|a_k(i)| < 1 \quad \text{and} \quad \frac{|z_i|}{1 + |z_i|} < |f_i(z_i)| < \frac{|z_i|}{1 - |z_i|},$$

we have the following theorem.

THEOREM 4. Let $F(z) = \sum_{k=1}^{\infty} A_k Z^k$ belongs to K^n . Let T be an $n \times n$ nonsingular matrix and let $f_1, \dots, f_n \in K^1$ such that

$$F(Z) = T \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}.$$

Then

$$|a_{ij}^k| < |t_{ij}|$$

for each k, i , and j , where $A_k = (a_{ij}^k)$. Let $F = (F_1, \dots, F_n)$. Then

$$\sum_{j=1}^n |t_{ij}| \frac{|z_j|}{1 + |z_j|} \leq |F_i(Z)| < \sum_{j=1}^n |t_{ij}| \frac{|z_j|}{1 - |z_j|}.$$

Both inequality are sharp.

REFERENCES

1. G. Pólya and I. J. Schoenberg, *Remarks on de la Vallie Poussin means and convex conformal maps of the unit circle*, Pacific J. Math., **8** (1958), 295-334.
2. St. Ruscheweyh and T. Sheil-Small, *Hadamard products of schlicht functions and the Polya-Schoenberg conjecture*, Comment. Math. Helv., **48** (1973), 119-135.
3. T. J. Suffridge, *The principle of subordination applied to functions of several variables*, Pacific J. Math., **33** (1970), 241-248.
4. H. S. Wilf, *Subordinating factor sequences for convex maps of the unit circle*, Proc. Amer. Math. Soc., **12** (1961), 689-693.

Received April 30, 1975.

WEST VIRGINIA UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Pacific Journal of Mathematics

Vol. 65, No. 1

September, 1976

David Lee Armacost, <i>Compactly cogenerated LCA groups</i>	1
Sun Man Chang, <i>On continuous image averaging of probability measures</i>	13
J. Chidambaraswamy, <i>Generalized Dedekind ψ-functions with respect to a polynomial. II</i>	19
Freddy Delbaen, <i>The Dunford-Pettis property for certain uniform algebras</i>	29
Robert Benjamin Feinberg, <i>Faithful distributive modules over incidence algebras</i>	35
Paul Froeschl, <i>Chained rings</i>	47
John Brady Garnett and Anthony G. O'Farrell, <i>Sobolev approximation by a sum of subalgebras on the circle</i>	55
Hugh M. Hilden, José M. Montesinos and Thomas Lusk Thickstun, <i>Closed oriented 3-manifolds as 3-fold branched coverings of S^3 of special type</i>	65
Atsushi Inoue, <i>On a class of unbounded operator algebras</i>	77
Peter Kleinschmidt, <i>On facets with non-arbitrary shapes</i>	97
Narendrakumar Ramanlal Ladhawala, <i>Absolute summability of Walsh-Fourier series</i>	103
Howard Wilson Lambert, <i>Links which are unknottable by maps</i>	109
Kyung Bai Lee, <i>On certain g-first countable spaces</i>	113
Richard Ira Loebel, <i>A Hahn decomposition for linear maps</i>	119
Moshe Marcus and Victor Julius Mizel, <i>A characterization of non-linear functionals on W_1^p possessing autonomous kernels. I</i>	135
James Miller, <i>Subordinating factor sequences and convex functions of several variables</i>	159
Keith Pierce, <i>Amalgamated sums of abelian l-groups</i>	167
Jonathan Rosenberg, <i>The C^*-algebras of some real and p-adic solvable groups</i>	175
Hugo Rossi and Michele Vergne, <i>Group representations on Hilbert spaces defined in terms of ∂_b-cohomology on the Silov boundary of a Siegel domain</i>	193
Mary Elizabeth Schaps, <i>Nonsingular deformations of a determinantal scheme</i>	209
S. R. Singh, <i>Some convergence properties of the Bubnov-Galerkin method</i>	217
Peggy Strait, <i>Level crossing probabilities for a multi-parameter Brownian process</i>	223
Robert M. Tardiff, <i>Topologies for probabilistic metric spaces</i>	233
Benjamin Baxter Wells, Jr., <i>Rearrangements of functions on the ring of integers of a p-series field</i>	253
Robert Francis Wheeler, <i>Well-behaved and totally bounded approximate identities for $C_0(X)$</i>	261
Delores Arletta Williams, <i>Gauss sums and integral quadratic forms over local fields of characteristic 2</i>	271
John Yuan, <i>On the construction of one-parameter semigroups in topological semigroups</i>	285