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**SUBORDINATING FACTOR SEQUENCES AND CONVEX
FUNCTIONS OF SEVERAL VARIABLES**

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In this paper we consider univalent holomorphic maps of E^n , the unit disk in C^n . We generalize Wilf's subordinating factor sequences to functions on E^n and use this characterization to obtain a covering theorem and bounds for convex mappings in C^n .

1. Introduction. Let K^n denote the class of functions F which are holomorphic and univalent in $E^n = \{z = (z_1, \dots, z_n) : \text{Max}_{1 \leq i \leq n} |z_i| < 1\}$, maps E^n onto a convex region in C^n , and satisfy $F(0) = 0$ and the Jacobian J of the mapping F is nonsingular. Let G and H be holomorphic in E^n . If $G(E^n) \subset H(E^n)$, then G is subordinate to H ($G < H$). If $F = (F_1, \dots, F_n) \in K^n$ then each F_i has an expansion of the form

$$F_i(Z) = \sum_{k=1}^{\infty} \sum_{\nu_1 + \dots + \nu_n = k} a_{\nu_1 \dots \nu_n}(i) z_1^{\nu_1} \dots z_n^{\nu_n}.$$

In this paper we characterize the sequences $\{c_{\nu_1 \dots \nu_n}(i)\}$ ($i = 1, \dots, n$) such that the mapping

$$H = (H_1, \dots, H_n)$$

where

$$H_i(Z) = \sum_{k=1}^{\infty} \sum_{\nu_1 + \dots + \nu_n = k} c_{\nu_1 \dots \nu_n}(i) a_{\nu_1 \dots \nu_n}(i) z_1^{\nu_1} \dots z_n^{\nu_n}$$

is subordinate to F , for all $F \in K^n$. Then we obtain a covering theorem and bounds for convex mappings.

For $n = 1$, the class K^1 is the classical family of univalent functions $F(z) = \sum_{k=1}^{\infty} a_k z^k$ which maps the unit disk onto a convex domain. Wilf [4] has characterized the sequences $\{c_k\}$ (subordinating factor sequences) such that $h(z) = \sum c_k a_k z^k$ is subordinate to $f(z) = \sum_{k=1}^{\infty} a_k z^k$ whenever $f \in K^1$. For $n > 1$, Suffridge [3] has given the following characterization of the class K^n .

THEOREM A. *Suppose $F: E^n \rightarrow C^n$ is holomorphic, $F(0) = 0$, and that J is nonsingular for all $Z \in E^n$. Then F is a univalent map of E^n onto a convex domain if and only if there exists univalent mappings $f_j \in K^1$ ($1 \leq j \leq n$) such that $F(Z) = T(f_1(z_1), \dots, f_n(z_n))$ where T is a nonsingular linear transformation.*

From Theorem A we see that if $F = (F_1, \dots, F_n) \in K^n$ then

$$F_i(z_1, \dots, z_n) = \sum_{k=1}^{\infty} (a_{i1}^k z_1^k + \dots + a_{in}^k z_n^k).$$

Thus we could represent $F \in K^n$ by the column vector

$$F(Z) = \sum_{k=1}^{\infty} A_k Z^k$$

where

$$A_k = \begin{bmatrix} a_{i1}^k & \dots & a_{in}^k \\ \vdots & & \\ a_{n1}^k & & a_{nn}^k \end{bmatrix} \quad Z^k = \begin{bmatrix} z_1^k \\ \vdots \\ z_n^k \end{bmatrix}.$$

2. Subordinating factor sequences. An infinite sequence $\{C_k\}$ of $n \times n$ matrices of complex numbers will be called a subordinating factor sequence if for each $F(Z) = \sum A_k Z^k \in K^n$ we have $\sum C_k \odot A_k Z^k < F(Z)$, where $C_k \odot A_k$ is the Hadamard product. If $C = (c_{ij})$ and $A = (a_{ij})$ then $C \odot A = (c_{ij} a_{ij})$. Let \mathcal{F}^n denote the collection of subordinating factor sequences.

THEOREM 1. *If $\{C_k\} \in \mathcal{F}^n$, then for each k the rows of $C_k = (c_{ij}^k)$ are identical, that is, for each k ($k = 1, 2, \dots$) and each j ($j = 1, \dots, n$) we have $c_{1j}^k = c_{2j}^k = \dots = c_{nj}^k$.*

Proof. Let $\{C_k\} \in \mathcal{F}^n$. First consider $k = 1$. Pick $\zeta = (\zeta_1, \dots, \zeta_n) \in E^n$ where $\zeta_i \neq 0$ and if $c_{jj}^1 \neq 0$ then $\zeta_j = 1/2e^{-i\alpha}$ with $\alpha = \arg c_{jj}^1$ if $c_{jj}^1 = 0$ then $\zeta_j = 0$. Let $\delta = (c_{ji}^1 - c_{ii}^1)\zeta_i$. If $\delta = 0$, then $c_{ji}^1 = c_{ii}^1$. If $\delta \neq 0$, let $M = 1/\delta$. Then define the mapping $F = (F_1, \dots, F_n)$ where $F_i(Z) = Mz_i$, $F_j(Z) = Mz_i + z_j$, and $F_k(Z) = z_k$ when neither $k \neq i$ or $k \neq j$. The mapping F is a convex univalent map by Theorem A. Thus since $\{C_k\} \in \mathcal{F}^n$ the mapping $H = (H_1, \dots, H_n)$, where $H_i(Z) = M c_{ii}^1 z_i$, $H_j(Z) = M c_{ji}^1 z_i + c_{jj}^1 z_j$ and $H_k(Z) = c_{kk}^1 z_k$ for $k \neq i$ or $k \neq j$, is subordinate to F . In particular, there is a $Z \in E^n$ such that $H(\zeta) = F(Z)$, which says

$$Mz_i = M c_{ii}^1 \zeta_i$$

and

$$Mz_i + z_j = M c_{ji}^1 \zeta_i + c_{jj}^1 \zeta_j.$$

Solving for z_j we obtain

$$z_j = M(c_{ji}^1 - c_{ii}^1)\zeta_i + c_{jj}^1 \zeta_j = 1 + \frac{1}{2}|c_{ij}^1| \geq 1.$$

This contradicts the fact that $|Z| < 1$. Thus we have $\delta = 0$ or $c_{1j}^i = c_{2j}^i = \dots = c_{nj}^i$ for $j = 1, \dots, n$.

For $k > 1$ we define the mapping $F = (F_1, \dots, F_n)$ where

$$F_i(Z) = Mz_i + \frac{Mz_j^k}{k^2}, F_j(Z) = Mz_i + \frac{Mz_i^k}{k^2} + z_j, \text{ and } F_k(Z) = z_k$$

for neither $k \neq i$ or $k \neq j$. Then the proof that $c_{1j}^k = c_{2j}^k = \dots = c_{nj}^k$ is similar to the proof for $k = 1$.

From Theorem 1 we have that if $\{C_k\} \in \mathcal{F}^n$, then for each k the rows of C_k are indential. For the $n \times n$ matrices C_k we will use the notation

$$C_k = \begin{bmatrix} c_1^k & \dots & c_n^k \\ \vdots & & \vdots \\ c_1^k & \dots & c_n^k \end{bmatrix} = (c_1^k, \dots, c_n^k).$$

Using Theorem 1 we are now able to characterize class \mathcal{F}^n .

THEOREM 2. *The following are equivalent:*

- (i) $\{C_k\} \in \mathcal{F}^n$ where $C_k = (c_1^k, \dots, c_n^k)$.
- (ii) For each $j = 1, \dots, n$ we have

$$\operatorname{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} c_j^k z_j^k \right\} > 0 \text{ for } |z_j| < 1.$$

- (iii) For each $j = 1, \dots, n$ there is a nondecreasing function Ψ_j on $[0, 2\pi]$ such that

$$c_j^k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} d\Psi_j(\theta) \text{ and } c_j^0 = 1.$$

Proof. The Herglotz's integral representation for positive harmonic functions proves that (ii) and (iii) are equivalent. Let $\{C_k\} \in \mathcal{F}^n$, where $C_k = (c_1^k, \dots, c_n^k)$. Let $f_i(z_i) = z_i/(1 - z_i)$. Then by Theorem A the mapping F is in K^n . We may write

$$F(Z) = \sum_{k=1}^{\infty} A_k Z^k$$

where $A_k = (a_{ij}^k)$ and $a_{ij}^k = 0$ if $i \neq j$ and $a_{ii}^k = 1$ then the mapping

$$H(Z) = \sum_{k=1}^{\infty} C_k \odot A_k Z^k$$

is subordinate to F . The mapping H has components $H_i(Z) = \sum_{k=1}^{\infty} c_i^k z_i^k$. Since $H < F$ we have that $H_i(F_i) \subset f_i(E_i)$ or $\operatorname{Re} \{H_i(E_i)\} \geq -1/2$ where $E_i = \{z_i: |z_i| < 1\}$. Thus $\operatorname{Re} \{ \sum_{k=1}^{\infty} c_i^k z_i^k \} > -1/2$ for $i =$

1, ..., n, Now suppose (iii) holds. Let $F \in K^n$. Then by Theorem A there exists a nonsingular matrix T and functions $f_1, \dots, f_n \in K^1$, where $f_i(z_i) = \sum_{k=1}^{\infty} a_k(i)z_i^k$, such that

$$F(Z) = T \begin{bmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{bmatrix}$$

where F is a column vector. Then

$$\begin{aligned} H(Z) &= \sum C_k \odot A_k z^k = T \begin{bmatrix} \sum_{k=1}^{\infty} c_1^k a_k(1) z_1^k \\ \vdots \\ \sum_{k=1}^{\infty} c_n^k a_k(n) z_n^k \end{bmatrix} \\ &= T \begin{bmatrix} \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{ik\phi} d\Psi_1(\phi) a_k(1) z_1^k \\ \vdots \\ \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{ik\phi} d\Psi_n(\phi) a_k(n) z_n^k \end{bmatrix} \\ &= T \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{\infty} a_k(1) r_1^k e^{ij(\theta_1 + \phi)} d\Psi_1(\phi) \\ \vdots \\ \frac{1}{2} \int_0^{2\pi} \sum_{k=1}^{\infty} a_k(n) r_n^k e^{ik(\theta_n + \phi)} d\Psi_n(\phi) \end{bmatrix} \\ &= T \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} f_1(r_1 e^{i(\theta_1 + \psi)}) d\Psi_n(\phi) \\ \vdots \\ \frac{1}{2\pi} \int_0^{2\pi} f_n(r_n e^{i(\theta_n + \phi)}) d\Psi_n(\phi) \end{bmatrix} \end{aligned}$$

where $z_j = r_j e^{i\theta_j}$. Since each integral in the left hand side is the centroid of a nonnegative mass distribution of total mass one on a convex curve, the value of each integral must lie inside its convex curve. Further since T is a nonsingular linear transformation $H(Z)$ lies inside the image of the polydisk of radius (r_1, \dots, r_n) . (A polydisk or radius (r_1, \dots, r_n) is the set $\{(z_1, \dots, z_n) : |z_i| \leq r_i \text{ for } i = 1, \dots, n\}$.) Thus $H < F$.

3. Convex mappings in C^n . We now apply Theorem 2 to obtain some results for mapping in K^n .

COROLLARY 1. For $n > 1$ let $G \in K^n$, where $G(Z) = \sum B_k Z^k$.

Then the mapping

$$G_F^*(Z) = \sum B_k \odot A_k Z^k,$$

where $F(Z) = \sum A_k Z^k \in K^n$, is not subordinate to F for all $F \in K^n$.

Proof. If $G_F^* < F$ for all $F \in K^n$, then the sequence $\{B_k\}$ belongs to \mathcal{S}^n . This says that the rows of each B_k are identical by Theorem 1. Hence the Jacobian of G will be identically zero. Thus G_F^* is not subordinate to F for all $F \in K^n$.

Let $T = (t_{ij})$ be a $n \times n$ nonsingular matrix. Let K be the functions $f \in K^1$ where $f'(0) = 1$. Let KT denote the subclass of K^n which is defined by $F \in KT$ if and only if there exist functions $f_i \in K (i = 1, 2, \dots, n)$ such that

$$F(Z) = T \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}$$

where F is represented as a column vector.

COROLLARY 2. *The image of E^n under a mapping $F \in KT$ contains the polydisk $|w| < 1/2(\sum_{j=1}^n |t_{1j}|, \dots, \sum_{j=1}^n |t_{nj}|)$. The radius is sharp.*

Proof. Since the sequence $\{C_k\}$ where $C_1 = (1/2, 1/2, \dots, 1/2)$ and $C_k = (0, \dots, 0)$ for $k \geq 2$, belongs to \mathcal{S}^n , we see that the image of E^n under a mapping $F \in KT$ contains $|W| < 1/2(\sum_{j=1}^n |t_{1j}|, \dots, \sum_{j=1}^n |t_{nj}|)$. The sharpness follows by using the function

$$F(Z) = T \begin{bmatrix} \frac{z_1}{1 - z_1} \\ \vdots \\ \frac{z_n}{1 - z_n} \end{bmatrix}.$$

Ruscheweyh and Sheil-Small [2] have proven Pólya and Schoenberg's [1] conjecture that if $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum b_k z^k$ are elements of K^1 then so is the function $h(z) = \sum a_k b_k z^k$. In general for K^n this is not true as shown by the example $F(Z) = \begin{pmatrix} z_1 - z_2 \\ z_1 + z_2 \end{pmatrix} = G(Z)$. However, we do have the following Pólya and Schoenberg type of theorem.

THEOREM 3. *Let $T_1 = (p_{ij})$ and $T_2 = (q_{ij})$ be $n \times n$ nonsingular matrices such that $T = T_1 \odot T_2 = (p_{ij}q_{ij})$ is nonsingular. If $F(Z) =$*

$\sum_{k=1}^{\infty} A_k Z^k \in KT_1$ and $G(Z) = \sum_{k=1}^{\infty} B_k Z^k \in KT_2$, then $H(Z) = \sum_{k=1}^{\infty} A_k \odot B_k Z^k$ belongs to KT .

Proof. Let $F \in KT_1$ and $G \in KT_2$. Then there exists functions $f_i, g_i \in K (i = 1, \dots, n)$ such that

$$F(Z) = T_1 \begin{bmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{bmatrix}$$

and

$$G(Z) = T_2 \begin{bmatrix} g_1(z_1) \\ \vdots \\ g_n(z_n) \end{bmatrix}$$

The mapping $H(Z) = \sum_{k=1}^{\infty} A_k \odot B_k z^k$ may be written as

$$H(Z) = T \begin{pmatrix} z_1 + \sum_{k=1}^{\infty} a_k(1)b_k(1)z_1^k \\ \vdots \\ z_n + \sum_{k=1}^{\infty} a_k(n)b_k(n)z_n^k \end{pmatrix}.$$

Thus $H \in KT$ since $z_i + \sum a_k(i)b_k(i)z_i^k$ belongs to K for each i [2].

4. Bounds on Mapping in K_n . Let $F \in K^n$. Then by Suffridge's representation of mappings in K^n (Theorem A), there exist an $n \times n$ nonsingular matrix $T = (t_{ij})$ and functions $f_i(z_i) = \sum_{k=1}^{\infty} a_k(i)z_i^k (i=1, \dots, n)$ in K^1 with $f_i'(0) = 1$ such that

$$F(Z) = T \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}.$$

Then

$$A_k = (a_{ij}) = T \begin{pmatrix} a_k(1) \\ \vdots \\ a_k(n) \end{pmatrix}$$

where $F(z) = \sum_{k=1}^{\infty} A_k Z^k$. Since

$$|a_k(i)| < 1 \quad \text{and} \quad \frac{|z_i|}{1 + |z_i|} < |f_i(z_i)| < \frac{|z_i|}{1 - |z_i|},$$

we have the following theorem.

THEOREM 4. Let $F(z) = \sum_{k=1}^{\infty} A_k Z^k$ belongs to K^n . Let T be an $n \times n$ nonsingular matrix and let $f_1, \dots, f_n \in K^1$ such that

$$F(Z) = T \begin{pmatrix} f_1(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}.$$

Then

$$|a_{ij}^k| < |t_{ij}|$$

for each k, i , and j , where $A_k = (a_{ij}^k)$. Let $F = (F_1, \dots, F_n)$. Then

$$\sum_{j=1}^n |t_{ij}| \frac{|z_j|}{1 + |z_j|} \leq |F_i(Z)| < \sum_{j=1}^n |t_{ij}| \frac{|z_j|}{1 - |z_j|}.$$

Both inequality are sharp.

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