GROUP REPRESENTATIONS ON HILBERT SPACES DEFINED IN TERMS OF $\bar{\partial}_b$-COHOMOLOGY ON THE SILOV BOUNDARY OF A SIEGEL DOMAIN

HUGO ROSSI AND MICHELE VERGNE
GROUP REPRESENTATIONS ON HILBERT SPACES
DEFINED IN TERMS OF $\partial_b$-COHOMOLOGY
ON THE SILOV BOUNDARY OF A
SIEGEL DOMAIN

H. Rossi and M. Vergne

Let $Q$ be a $C^\alpha$-valued quadratic form on $C^m$. Let $N(Q)$ be the 2-step nilpotent group defined on $R^n \times C^m$ by the group law

$$(x, u) \cdot (x', u') = (x + x' + 2 \text{Im} Q(u, u'), u + u') .$$

Then $N(Q)$ has a faithful representation as a group of complex affine transformations of $C^{n+m}$ as follows:

$$g \cdot (z, u) = (z + x_0) + i(2Q(u, u_0) + Q(u, u_0), u_0 + u_0) ,$$

where $g = (x_0, u_0)$. The orbit of the origin is the surface

$$\Sigma = \{(z, u) \in C^{n+m}; \text{Im} z = Q(u, u)\} .$$

This surface is of the type introduced in [11], and has an induced $\delta_b$-complex (as described in that paper) which is, roughly speaking, the residual part (along $\Sigma$) of the $\partial$-complex on $C^{n+m}$. Since the action of $N(Q)$ is complex analytic, it lifts to an action on the spaces $E^q$ of this complex which commutes with $\delta_b$. Since the action of $N(Q)$ is by translations, the ordinary Euclidean inner product on $C^m$ is $N(Q)$-invariant, and thus $N(Q)$ acts unitarily in the $L^2$-metrics on $C^\alpha(E^q)$ defined by

$$|| \Sigma a_I d\bar{u}_I ||^2 = \int_\Sigma |a_I|^2 dV$$

where $dV$ is ordinary Lebesgue surface measure. In this way we obtain unitary representations $\rho_q$ of $N(Q)$ on the square-integrable cohomology spaces $H^q(E)$ of the induced $\delta_b$-complex.

These are generalizations of the so-called Fock or Segal-Bargmann representations [2, 4, 10, 13], and the representations studied by Carmona [3]. In this paper, we explicitly determine these representations and exhibit operators which intertwine the $\rho_q$ with certain direct integrals of the Fock representations.

This is accomplished by means of a generalized Paley-Wiener theorem arising out of Fourier-Laplace transformation in the $x$ (Re $z$) variable. Let us describe this result. For $\xi \in R^n$, let $Q_\xi(u, v) = \langle \xi, Q(u, v) \rangle$. Let $H^q(\xi)$ be the square-integrable cohomology of the $\partial$-complex on $C^m$ relative to the norm

193
Let $U_q = \{ \xi \in \mathbb{R}^n \ast \text{ the quadratic form } Q_\xi \text{ has } q \text{ negative and } n-q \text{ positive eigenvalues} \}$. Let $U = \bigcup U_q$.

**Theorem.** For $\xi \in U$, $H^q(\xi) \neq \{0\}$ if and only if $\xi \in U_q$. In particular the fibration $H^q(\xi) \to \xi$ is a (locally trivial) Hilbert fibration on $U_q$ and the following result holds!

**Theorem.** Let $H^q(F)$ be the space of square-integrable sections of the fibration $H^q(\xi) \to \xi$ over $U_q$. Then the Fourier-Laplace transform, defined for functions by

$$\tilde{\alpha}(\xi, u) = \int a_t(x + iQ(u, u))e^{-i\langle\xi, x + iQ(u, u)\rangle}dx$$

induces an isometry of $H^q(E)$ with $H^q(F)$.

Furthermore, this transform followed by a suitable variable change (in $C^m$, dependent on $\xi$) is the sought-for intertwining operator.

2. A Paley-Wiener theorem for $\tilde{\delta}$-cohomology on certain homogeneous surfaces. Let $Q$ be a nondegenerate $C^\ast$-valued hermitian form defined on $C^m$. That $Q$ is nondegenerate means that the only solution of

$$Q(u, v) = 0 \text{ for all } u \in C^m$$

is $v = 0$. Equivalently, there is a $\xi \in R^n$ such that the $C$-valued form

(2.1) $$Q_\xi(u, v) = \langle \xi, Q(u, v) \rangle$$

is nondegenerate. Given such a $Q$ we introduce the real submanifold of $C^{n+m}$:

(2.2) $$\Sigma = \Sigma(Q) = \{(z, u) \in C^{n+m}; \text{Im } z = Q(u, u)\}.$$ 

Let $N(Q)$ be the 2-step nilpotent group defined on $R^n \times C^m$ by the group law

(2.3) $$(x, u) \cdot (x', u') = (x + x' + 2 \text{Im } Q(u, u'), u + u').$$

Then $N(Q)$ has a faithful realization in the group of complex affine transformations of $C^{n+m}$ as follows

(2.4) $$(z, u) \xrightarrow{(x_0, u_0)} (z + x_0 + i(2Q(u, u_0) + Q(u_0, u_0)), u + u_0),$$

so that $\Sigma$ is the orbit of 0. The correspondence $N(Q) \to \Sigma$ given by
g \rightarrow g \cdot 0, (x, u) \rightarrow (x + iQ(u, v), u), is a diffeomorphism, and in certain contexts we may identify \( N(Q) \) with \( \Sigma \) under this correspondence. If we let \( dx, du \) represent Lebesgue measure in \( \mathbb{R}^n, C^m \), then \( dx du \) is the Haar measure of \( N(Q) \). We shall return, in §4, to the study of representations of \( N(Q) \) connected with its realization as \( \Sigma \); in this and the next section we shall carry out the relevant analysis.

\( \Sigma \) is a surface of the type studied in [11], Chapter I, (with \( V = \{0\} \)). Here we shall summarize the relevant results in that paper.

Let \( A \rightarrow \Sigma \) be the complex vector bundle of antiholomorphic tangent vectors along \( \Sigma \), and \( E^q = A^q A^* \) the bundle of \( q \)-forms on \( A \). For \( V \rightarrow \Sigma \) any vector bundle we shall let \( C^\infty(V) \) represent the sheaf of \( C^\infty \) sections of \( V \). Let \( \partial \delta_b : C^\infty(E^q) \rightarrow C^\infty(E^{q+1}) \) be the differential operator induced (as in [10]) by exterior differentiation. The complex \( (E^q, \partial \delta_b) \) is referred to as the \( \partial \delta_b \)-complex on \( \Sigma \).

We can make this complex explicit as follows. Let \( z_1, \ldots, z_k, \ldots, z_n, u_1, \ldots, u_a, \ldots, u_m \) be coordinates for \( C^n \times C^m \). Then, the (restrictions of the) forms \( d\bar{u}_a, 1 \leq a \leq m \) form a basis for \( E^1 \). The dual vectors \( U_a, 1 \leq a \leq m \) giving a basis for \( A \) are as follows:

\[
U_a = \frac{\partial}{\partial \bar{u}_a} + i \sum_k Q_k(u, E_a) \frac{\partial}{\partial x_k}
\]

where \( Q_k = z_k \circ Q \) and \( \{E_a\} \) is the basis of \( C^m \) dual to the coordinates \( u_a \).

Then \( E^q \) has as basis the forms \( \{d\bar{u}_I; I = (i_1, \ldots, i_q) \) with \( i_1 < \cdots < i_q \}. \) Any \( q \)-form is written

\[
\omega = \sum_{|I|=q} a_I d\bar{u}_I,
\]

where \( \Sigma' \) refers to summation only over those \( q \)-tuples in increasing order. If \( J \) is an arbitrary \( q \)-tuple, \( [J] \) will refer to the same \( q \)-tuple written in increasing order, and \( \varepsilon_J \) is the sign of the permutation \( J \rightarrow [J] \). We define the coefficients \( a_J \) of \( \omega \) for unordered \( q \)-tuples by \( a_J = \varepsilon_J a_{[J]} \). Now, in this notation we have

\[
\partial \delta_b \omega = \sum_{|I|=q} \sum_{a=1}^{m} U_a(a_I) d\bar{u}_a \wedge d\bar{u}_I
\]

where \( \varepsilon_J^a = 0 \) if \( \alpha I \neq J \) set theoretically, and \( \varepsilon_J^a = \varepsilon_J \) otherwise.

Now, we turn to \( R^{n*} \times C^m \). We shall refer to the coordinate of \( R^{n*} \) by \( \xi \). Let \( A_u \) be the vector bundle on \( R^{n*} \times C^m \) of antiholomorphic vector fields along the \( C^m \)-leaves: the leaves \( \xi = \text{constant} \). Let \( F^q \) be the vector bundle of \( q \)-forms on \( A_u \), and \( \partial \delta_u : C^\infty(F^q) \rightarrow C^\infty(F^{q+1}) \) the differential operator induced by exterior differentiation.
We make this complex explicit as follows. Let \( \xi_1, \ldots, \xi_n, u_1, \ldots, u_m \) be coordinates in \( \mathbb{R}^{*n} \times C^m \). Then, with the same conventions as above, \( F^q \) has the basis \{\( du_I; I = (i_1, \ldots, i_q), i_1 < \cdots < i_q \)\} and any \( \omega \in C^\infty(F^q) \) has the form

\[
\omega = \sum' \phi_I d\bar{u}_I .
\]

We have

\[
\bar{\partial}_u \omega = \sum'_{|I|=q} \sum_{\alpha=1}^m \frac{\partial \phi_I}{\partial \bar{u}_\alpha} d\bar{u}_\alpha \wedge d\bar{u}_I .
\]

We now bring in Lemma I.3.2 of [11] which relates these two complexes.

2.10. DEFINITION. Let \( \pi: \mathbb{R}^{*n} \times C^m \to \mathbb{R}^{*n} (\pi: \mathbb{R}^{*n} \times C^m \to \mathbb{R}^{*n}) \) be the projection on the first factor. Let \( C_0^\infty(E^q)(C_0^\infty(F^q)) \) be the set of \( \omega \in C^\infty(E^q)(C^\infty(F^q)) \) such that \( \pi(\text{support of } \omega) \) is relatively compact. For \( \omega = \Sigma a_I d\bar{u}_I \in C_0^\infty(E^q) \), define \( \hat{\omega} \in C^\infty(F^q) \) by \( \Sigma' \partial_I d\bar{u}_I \), where, for functions

\[
\hat{\omega}(\xi, u) = \int_{\mathbb{R}^{*n}} a(x + iQ(u, u), u) e^{-i\xi x - iQ(u, u)} dx
\]

where \( \mathcal{F}_x \) is the partial (in the \( x \)-variables) Fourier transform.

2.12. LEMMA (See I.3.2 of [11].) \( (\bar{\partial}_u \omega)^\wedge = \bar{\partial}_u \hat{\omega} \).

Here we shall introduce inner products of the spaces \( C^\infty(E^q), C^\infty(F^q) \). (Although the expressions we use to define norms could be infinite, by completion we shall mean in the following, the completion of the space of norm-finite forms.) First, we consider \( C^m \) as endowed with the standard hermitian inner product in which the set of vectors \( \{(0, \ldots, 1, \ldots, 0)\} \) is orthonormal. Let \( u_1, \ldots, u_m \) be an orthonormal basis of \( C^m \); we shall call \( \{u_1, \ldots, u_m\} \) an orthonormal coordinate set. The following definitions are independent of such a choice of orthonormal coordinate set.

2.13. DEFINITION. For \( \omega = \Sigma' a_I d\bar{u}_I \) in \( C^\infty(E^q) \), define

\[
|| \omega ||_x^2 = \sum'_{I} \int_x |a_I|^2 dx du .
\]

For \( \omega = \Sigma' \phi_I d\bar{u}_I \) in \( C^\infty(F^q) \), define

\[
|| \omega ||_u^2 = \sum_{I} \int_{\mathbb{R}^{*n} \times C^m} |\phi_I|^2 e^{-2Q(u, u)} d\xi du .
\]
2.14. **Lemma.** If \( \omega \in C_\delta^\infty(E^q) \), we have \( \hat{\omega} \in C_\delta^\infty(F^q) \) and \( \| \omega \|_\delta = \| \hat{\omega} \|_\delta \).

**Proof.** This is an immediate consequence of the Plancherel formula.

The following formalism (which is fairly standard; see [5, 8]) developing the \( L^2 \)-cohomology associated to the complex applies equally well to either complex. We shall make our definitions for a complex \((G^q, \delta)\) which refers to either one of the given complexes. In the sequel we shall distinguish between them by a subscript \((b\) or \(u\).

2.15. **Definition.** The formal adjoint \( \partial: C_\delta^\infty(G^q) \rightarrow C_\delta^\infty(G^{q-1}) \) is that differential operator defined by the equation

\[
(\delta^* \alpha, \omega) = (\alpha, \partial \omega) \quad \text{(for all } \alpha \text{ of compact support)}.
\]

We can find the expression for \( \partial \) by integrating by parts. For example, on \( E^q \) it is given by

\[
(\partial (\sum' a_j d\bar{u}_j), \omega) = \sum' \left( \sum_{j=1}^m \bar{U}_a(a_n,j) \right) d\bar{u}_j.
\]

2.17. **Definition.** Let \( L^q \) be the Hilbert space completion of \((\text{the norm finite } \omega \text{ in}) C_\delta^\infty(G^q)\). Define the \( W \)-norm on \( C_\delta^\infty(G^q) \) by

\[
W^q(\omega) = W(\omega, \omega) = \| \omega \|^2 + \| \delta \omega \|^2 + \| \partial \omega \|^2.
\]

Let \( W^q \) be the Hilbert space completion of \( C_\delta^\infty(G^q) \) in the \( W \)-norm.

Notice that \( \bar{\partial}: C_\delta^\infty(G^q) \rightarrow L^{q+1}, \partial: C_\delta^\infty(G^q) \rightarrow L^{q-1} \) extend continuously to \( W^q \). We shall denote their extensions by the same symbols.

2.18. **Lemma.** If \( \omega \in C_\delta^\infty(G^q) \) and \( W^q(\omega) < \infty \), then \( \omega \in W^q \).

**Proof.** We must show that \( \omega \) is approximable in the \( W \)-norm by elements in \( C_\delta^\infty(G^q) \). Let \( h \in C^\infty(R) \) be such that

(1) \( 0 \leq h(t) \leq 1 \) for all \( t \)

(ii) \( h(t) = 1 \) if \( t \leq 1/2 \)

(iii) \( h(t) = 0 \) if \( t \geq 1 \).

Define \( h_\nu \) on \( R^n(R^n) \) by

\[
h_\nu(t) = h(|t|/2), \quad t \in R^n(R^n).
\]

For \( \omega \in C_\delta^\infty(G^q) \), let \( \omega_\nu = h_\nu \omega \). Since \( h_\nu \rightarrow 1 \) boundedly, so long as \( \omega \in L^q, \omega \rightarrow \omega \) in \( L^q \), by dominated convergence. Since \( \bar{\partial}, \partial \) involve no differentiations in \( \xi, \bar{\partial} \omega_\nu = h_\nu \bar{\partial} \omega, \partial \omega_\nu = h_\nu \partial \omega \). Thus \( \omega_\nu \rightarrow \omega, \bar{\partial} \omega_\nu \rightarrow \bar{\partial} \omega, \partial \omega_\nu \rightarrow \partial \omega \) in \( L^q \) or, what is the same \( \omega_\nu \rightarrow \omega \) in \( W^q \).
2.19. DEFINITION. The qth $L^2$-cohomology space of the complex $(G^q, \tilde{\partial})$ is

$$H^q(G) = \{\omega \in W^q; \tilde{\partial}\omega = \partial\omega = 0\}.$$ 

2.20 THEOREM. The correspondence $\omega \rightarrow \hat{\omega}$ induces an isometry $H^q(E) \cong H^q(F)$.

Proof. (i) We first observe that, by Fourier inversion, the Lemma 2.12 can be worked from $F$ to $E$. More precisely, let $\phi = \Sigma' \phi_j d\bar{u}_j \in C_\infty^0(F^q)$. Define

$$\tilde{\phi} = \Sigma' \tilde{\phi}_j d\bar{u}_j$$

where, for a function $\phi$,

$$\tilde{\phi}(z, u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi(\xi, u) e^{i(z, \xi)} d\xi .$$

Then, just as in the proof of Lemma 2.12 (see [11]) we can verify

$$\langle \partial_u \phi, \gamma \rangle = \partial_u \tilde{\phi} .$$

(ii) Using the above, we can verify that

$$\langle \partial_\alpha \omega, \gamma \rangle = \partial_u \tilde{\partial}_u \omega, \omega \in C_\infty^0(E^q) .$$

For, let us take $\alpha \in C_\infty^0(F^q)$, and let $\beta = \partial_\alpha$. Then, by the Plancherel formula

$$\langle \partial_\alpha \omega, \gamma \rangle = \langle \partial_\alpha \omega, \beta \rangle = \langle \omega, \tilde{\partial}_u \beta \rangle = \langle \omega, \tilde{\partial}_u \alpha \rangle ;$$

this for all $\alpha \in C_\infty^0(E^q)$, so we must have $\langle \partial_\alpha \omega, \gamma \rangle = \partial_u \tilde{\omega}$.

(iii) Let $\omega \in C_\infty^0(E^q)$. Then, by (2.23) and Lemma 2.18, $\tilde{\omega} \in W^q(F)$, and $W^q(\omega) = W^q(\tilde{\omega})$. Thus the map $\omega \rightarrow \tilde{\omega}$ extends to an isometry of $W^q(E)$ into $W^q(F)$. Since this isometry transports $\partial_u$ and $\tilde{\partial}_u$ to $\partial_u$ and $\partial_u$, it takes $H^q(E)$ into $H^q(F)$.

(iv) this map is surjective. Let $\omega \in H^q(F)$. Then $\omega = \lim \omega_\nu$, $\omega_\nu \in C_\infty^0(F^q)$, with $\tilde{\partial}_u \omega_\nu \rightarrow 0, \partial_u \omega_\nu \rightarrow 0$. By (i), $\omega_\nu = \tilde{\alpha}_\nu$ with $\langle \partial_\alpha \omega_\nu, \gamma \rangle = \partial_u \tilde{\partial}_u \omega_\nu$. Since the correspondence $\omega \rightarrow \alpha$ is isometric in the $W$-norm, the $\{\alpha_\nu\}$ are also Cauchy, so $\alpha_\nu \rightarrow \alpha$ for some $\alpha$, and $\tilde{\partial}_\alpha \omega_\nu \rightarrow 0, \partial_\alpha \omega_\nu \rightarrow 0$. Thus $\alpha \in H^q(E)$, and $\tilde{\alpha} = \omega$.

For the remainder of this and the next section we shall be concerned with an explicit determination of the spaces $H^q(F)$. First, we introduced the $L^2$-cohomology along the $\xi$-fibers of $R^*_n \times C^m, \xi \in R^*_n$.

Let $C_{0,q}$ represent the space of $C^\infty(0, q)$-forms on $C^m$. For $\xi \in R^*_n$, introduce the $\xi$-norm
\[ || \Sigma' a_I d \bar{u}_I ||_e^2 = \sum_I \int_{C^m} |a_I(u)|^2 e^{-2q_I(u, u)} du. \]

Now, we can apply the definitions 2.15-2.19 to the \( \delta \)-complex \((C^{0, q}, \delta)\) together with the \( \tilde{\delta} \)-norm. We shall let \( H^q(\xi) \) refer to the associated \( L^2 \)-cohomology space:

\begin{equation}
H^q(\xi) = \{ \omega \in W^q(\xi); \delta \omega = \partial_i \omega = 0 \}
\end{equation}

where \( W^q_\xi \) is the completion of \( C^{0, q} \) in the norm

\[ W^q_\xi(\omega) = || \omega ||^2 + || \tilde{\delta} \omega ||^2 + || \partial_\xi \omega ||^2. \]

For \( \omega \in L^q(F), \omega = \Sigma' a_I d \bar{u}_I \), define \( \omega_\xi \) by fixing \( \xi \):

\[ \omega_\xi(u) = \Sigma' a_I(\xi, u) d \bar{u}_I. \]

Then \( \omega_\xi \) is defined and in \( L^q(\xi) \) for almost all \( \xi \).

2.25. Proposition. For \( \omega \in H^q(F), \omega_\xi \in H^q(\xi) \) for almost all \( \xi \).

Proof. The following facts, for \( \omega \in C^{0}(F^n) \), are easily verified:

\begin{equation}
\begin{align*}
|| \omega ||^2_u &= \int_{R^n} || \omega_\xi ||^2 d \xi, \\
\tilde{\delta} \omega_\xi &= (\tilde{\delta}_u \omega)_\xi, \partial_\xi \omega_\xi = (\partial_u \omega)_\xi.
\end{align*}
\end{equation}

Since \( \omega \in H^q(F) \), we can find a sequence \( \omega_v \in C^{0}(F^n) \) such that \( \omega_v \to \omega, \tilde{\delta}_u \omega_v \to 0, \partial_u \omega_v \to 0 \) in \( L^q(F) \). Replace \{\( \omega_v \)\} by a subsequence converging so fast that

\[ \sum_v || \omega_v - \omega_{v-1} ||^2_u < \infty \]

\[ \sum_v || \tilde{\delta}_u \omega_v ||^2 < \infty \]

\[ \sum_v || \partial_u \omega_v ||^2_u < \infty. \]

Then, for almost all \( \xi \), the series being integrated on the right are all finite. For such a \( \xi \), we will have the first series telescoping and the general term of the other series tending to zero. Thus \( \{\omega_{v, \xi}\} \) converges with \( \tilde{\delta}_\xi \omega_{v, \xi} \to 0, \partial_\xi \omega_{v, \xi} \to 0 \) in \( L^q(\xi) \). Thus \( \lim \omega_{v, \xi} \) is in \( H^q(\xi) \), but for almost all \( \xi, \lim \omega_{v, \xi} = \omega_\xi \).

3. Computation of \( H^q(\xi) \). First, we summarize the situation of the preceding section. \( Q \) is a nondegenerate \( C^n \)-valued hermitian form on \( C^m \). For \( \xi \in R^n \), we introduce the scalar hermitian form

\[ Q_\xi(u, v) = \langle \xi, Q(u, v) \rangle. \]
3.1. DEFINITION. Let \( U = \{ \xi \in \mathbb{R}^n; Q_\xi \text{ is nondegenerate} \} \).

Our basic hypothesis is that \( U = \emptyset \); in this case \( \mathbb{R}^n - U \) has measure zero. Let \( \langle | \rangle \) represent the Euclidean inner product on \( C^n \). For \( \xi \in U \), define the operator \( A_\xi \) by

\[
\langle A_\xi u | v \rangle = Q_\xi(u, v).
\]

Since \( Q_\xi \) is hermitian, \( A_\xi \) is self-adjoint, so \( C^n \) has an orthonormal basis of eigenvectors of \( A_\xi \). If \( u_1 = u_1(\xi), \ldots, u_m = u_m(\xi) \) are linear forms dual to such a basis and \( \lambda_1, \ldots, \lambda_m \) are the corresponding eigenvalues, we compute that

\[
Q_\xi(u, v) = \sum \lambda_i u_i \overline{v}_i.
\]

Now the \( \lambda_i \) are real and since \( Q \) is nondegenerate no \( \lambda_i \) is zero. Reordering, we can find positive numbers \( \mu_1, \ldots, \mu_m \) such that

\[
Q_\xi(u, v) = \sum_{i=1}^{q} \mu_i^2 u_i \overline{v}_i - \sum_{i=q+1}^{m} \mu_i^2 u_i \overline{v}_i.
\]

The number \( q \) is determined by \( Q_\xi \), it is the dimension of a maximal space to which \( Q_\xi \) restricts as an inner product.

3.3. DEFINITION. \( U_q = \{ \xi \in U; Q_\xi \text{ has the form (3.2)} \} \).

3.4. PROPOSITION. For each \( \xi \in U_q \), we can find an orthonormal coordinate set for \( C^n, u_1, \ldots, u_m \), so that (3.2) holds. The correspondence \( \xi \to (u_1, \ldots, u_m) \) can be chosen (locally) so as to depend smoothly on \( \xi \).

The proposition is clear. Now, we shall fix a \( \xi \in U_q \), and, to keep the notation clear we shall suppress reference to this \( \xi \), denoting

\[
\phi(u) = Q_\xi(u, u) = \sum_{i=1}^{q} \mu_i^2 |u_i|^2 - \sum_{i=q+1}^{m} \mu_i^2 |u_i|^2.
\]

We will now compute the cohomology spaces \( H^q(\xi) \) following the notation and ideas of Hörmander [7].

As in \( \S 2 \), \( C^{0,q}_\xi \) is the space of smooth \( q \)-forms defined on \( C^n; C^{0,q}_\xi \), those of compact support. We consider the Hilbert space norm on \( C^{0,q}_\xi \), for \( \omega = \sum a_I d\overline{u}_I \)

\[
||\omega||^2 = \sum_{i} \int |a_I|^2 \overline{\omega}(d\overline{u}_I).
\]

This expression is valid for \( \omega \) so represented in terms of any orthonormal coordinate set \( u_1, \ldots, u_m \). Let, for \( f \) a smooth function
\[ \partial_j f = \frac{\partial f}{\partial u_j}, \bar{\partial}_j f = \frac{\partial f}{\partial \bar{u}_j}, \]
(3.6)
\[ \partial_j f = e^{-\phi} \partial_j (e^{\phi} f) = \partial_j \phi \cdot f + \partial_j f, \]
\[ \bar{\partial}_j f = e^{-\bar{\phi}} \bar{\partial}_j (e^{\bar{\phi}} f) = \bar{\partial}_j \bar{\phi} \cdot f + \bar{\partial}_j f. \]

Thus,
(3.7)
\[ [\bar{\partial}_j, \partial_b] = \bar{\partial}_j \partial_b - \partial_b \bar{\partial}_j = \partial_j \lambda_j. \]

Furthermore, if either \( f \) or \( g \) is compactly supported
(8.3)
\[ \int_{\mathcal{D}} (\partial_j f) g e^{\phi} du = -\int_{\mathcal{D}} f (\partial_j g) e^{\phi} du \]
and similarly for the barred operators. Now, for \( \omega = \Sigma' a_I d\bar{u}_I \) a \( q \)-form we have
(3.9)
\[ \bar{\partial} \omega = \sum'_I \sum_{j=1}^m \bar{\partial}_j a_I d\bar{u}_j \wedge d\bar{u}_I, \]
(3.10)
\[ \partial \omega = \sum'_I \sum_{j=1}^m \partial_j (a_I) d\bar{u}_I \]
where \( \partial \) is the formal adjoint of \( \bar{\partial} \). (Here the ' refers to the summation convention introduced in the preceding section.) Finally, we shall need two fundamental identities. First, if \( f \) is smooth and compactly supported,
(3.11)
\[ \int_{\mathcal{D}} |\bar{\partial}_j f|^2 e^{\phi} du - \int_{\mathcal{D}} |\partial_j f|^2 e^{\phi} du + \lambda_j \int_{\mathcal{D}} |f|^2 e^{\phi} du = 0. \]

This follows from applying (3.8) to (3.7) in its integrated form:
\[ \lambda_j \int |f|^2 e^{\phi} du = \int \left[ \bar{\partial}_j, \partial_j \right] f \cdot \bar{f} e^{\phi} du. \]

By direct computation we obtain, for \( \omega = \Sigma' a_I d\bar{u}_I \in C_0^p \),
\[ ||\bar{\partial} \omega||^2 + ||\partial \omega||^2 = \sum'_{I,J} \sum_{j=1}^m \int_{\mathcal{D}} (\partial_j a_{JK} \bar{\partial}_j a_{IK} - \bar{\partial}_j a_{JK} \bar{\partial}_I a_{IK}) e^{\phi} du \]
\[ + \sum'_{I,J} \int_{\mathcal{D}} |\partial_j a_I|^2 e^{\phi} du. \]

Using the above integration-by-parts formula on the first term on the right, this becomes
(3.12)
\[ ||\bar{\partial} \omega||^2 + ||\partial \omega||^2 = \sum'_{I,J} \int |\bar{\partial}_j a_I|^2 e^{\phi} du - \sum'_{K} \sum_{j=1}^m \lambda_j \int |a_{JK}|^2 e^{\phi} du \]
(These are respectively the analogues of (2.1.8)' and (2.1.13) of [7].)
Let \( c = \min |\lambda_i| > 0 \).

### 3.13. Lemma

Let \( N \) be the multi index \((1, 2, \cdots, q)\). Then, for \( \omega = \sum a_I d\bar{u}_I \in C_0^{0,p} \), we have

\[
||\tilde{\omega}||^2 + ||\partial \omega||^2 \geq \sum_{I \in N} c \int |a_I|^2 e^\phi du + \sum_{j=1}^{q} \int |\partial_j a_I|^2 e^\phi du + \sum_{j=q+1}^{m} \int |\tilde{\partial}_j a_I|^2 e^\phi du.
\]

**Proof.** Let us adopt the notation \( \lambda_I = \sum_{i \in I} \lambda_i \). Note that for \( I \neq N, \lambda_N - \lambda_I \geq c > 0 \). We rewrite (3.12) as

\[
(3.14) \quad ||\tilde{\omega}||^2 + ||\partial \omega||^2 \geq \sum_{j=1}^{q} \left( \sum_{j=1}^{q} \left( \int |\tilde{\partial}_j a_I|^2 e^\phi du - \lambda_I \int |a_I|^2 e^\phi du \right) \right).
\]

We treat each term individually.

\[
\sum_{j=1}^{q} \left( \int |\tilde{\partial}_j a_I|^2 e^\phi du - \lambda_I \int |a_I|^2 e^\phi du \right) = \sum_{j=1}^{q} \left( \int |\tilde{\partial}_j a_I|^2 e^\phi du - \lambda_N \int |a_I|^2 e^\phi du + (\lambda_N - \lambda_I) \int |a_I|^2 e^\phi du \right).
\]

Applying (3.11) to the second term (note \( \lambda_N = \lambda_1 + \cdots + \lambda_q \)), we obtain

\[
= \sum_{j=1}^{q} \left( \int |\tilde{\partial}_j a_I|^2 e^\phi du + \sum_{j=1}^{q} \left( \int |\partial_j f|^2 e^\phi du - \int |\tilde{\partial}_j a_I|^2 e^\phi du \right) + (\lambda_N - \lambda_I) \right) \int |a_I|^2 e^\phi du.
\]

If \( I = N \), the first term drops out; otherwise it dominates \( c \int |a_I|^2 e^\phi du \).

The lemma is proven.

Now, we recall that \( W^p \) is defined as the Hilbert space completion of those \( \omega \in C_0^{0,p} \) such that

\[
W^2(\omega) = ||\omega||^2 + ||\tilde{\omega}||^2 + ||\partial \omega||^2 < \infty
\]

in this \( W \)-norm. \( H^p = H^p(\xi) = \ker \tilde{\partial} \cap \ker \partial \). The relevance of the above estimate is that it holds on \( W^p \), because \( C_0^{0,p} \) is dense in \( W^p \) as we now prove.

### 3.15. Lemma

\( C_0^{0,p} \) is dense in \( W^p \) in the \( W \)-norm.

**Proof.** Let \( h \) be as introduced in Lemma 2.18, and let \( h_v(u) = h(|u|/2^v) \). Suppose \( \omega \in C_0^{0,p} \) has finite \( W \)-norm. Let \( \omega_v = h_v \cdot \omega \). We shall show that \( \omega_v \rightarrow \omega \) in the \( W \)-norm, or, what is the same,
First of all, since \( h_u \to 1 \) boundedly we can conclude that
\[
\omega_v \to \omega, \quad d\omega_v \to d\omega, \quad \partial\omega_v \to \partial\omega.
\]

It remains only to show that the last terms in (3.17) tend to zero as \( \nu \to \infty \). Each term is a fixed linear combination of terms of the form \((D \cdot h_j)\alpha\), where \( D \) is a constant coefficient first order operator, and \( \alpha \) is a typical coefficient of \( \omega \). Now, the \((D \cdot h_j)\) are uniformly bounded and have disjoint supports, so \( \Sigma (D \cdot h_j)^2 \) is bounded. Thus \( (\sum (D \cdot h_j)^2) \alpha^2 \) is integrable, so the general term tends to zero in \( L^1 \). Thus the last term in (3.17) tends to zero in \( L^2 \), so the lemma is proven.

3.18. THEOREM. (1) For \( \xi \in U_q \), we have \( H^p(\xi) = \{0\} \) for \( p \neq q \).
(2) Let \( u_1, \ldots, u_m \) be the basis of \( C^m \) found in Proposition 3.4, and let \( v_i = \mu_i \bar{u}_i, \ldots, v_q = \mu_q \bar{u}_q, v_{q+1} = \mu_{q+1} u_{q+1}, \ldots, v_m = \mu_m u_m \). Then

\[
H^q(\xi) = \left\{ \omega = f(v) \exp \left( - \sum_{i=1}^q |v_i|^2 \right) d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_q \right\},
\]

where \( f \) is holomorphic and

\[
||\omega||^2 = \frac{1}{(\mu_1 \cdots \mu_m)^2} \left| f \right|^2 |v|^{2-|v|} dv < \infty.
\]

Proof. Let \( \omega \in H^p(\xi) \), \( \omega = \Sigma^i a_i d\bar{u}_i \). By the preceding lemma there is a sequence \( \{\omega_i\} \subset C^0_\nu \) such that \( \omega_i \to \omega \) in \( L^p \) and \( \tilde{\partial}\omega_i \to 0 \), \( \partial\omega_i \to 0 \) in \( L^p \). By the estimate in Lemma 3.13 we conclude that, for \( \omega_v = \Sigma a_{i,v} d\bar{u}_i, a_{i,v} \to a_i \), and

(a) \quad for \( I \neq N = \{1, \ldots, q\}, a_{i,v} \to 0 \),

(b) \quad for \( j > q \), \( \frac{\partial a_{N,v}}{\partial \bar{u}_j} \to 0 \) in \( L^1_{loc} \),

(c) \quad for \( j \leq q \), \( \frac{\partial}{\partial u_j} (e^\nu a_{N,v}) \to 0 \) in \( L^1_{loc} \).

From (a) we conclude that \( a_i = 0 \) for \( I \neq N \). Thus (1) is proven, and for \( \nu = q \), we have \( \omega = ad\bar{u}_1 \wedge \cdots \wedge d\bar{u}_q \) where \( a = \lim a_v \) with
in \( L_{\text{loc}} \). Thus \( f(u) = a(u) \exp(\sum_{i=1}^{q} \mu_i |u_i|^2) \) is a weak solution of
\[
\frac{\partial f}{\partial u_j} = 0, \quad j > q, \quad \frac{\partial \bar{\omega}}{\partial u_j} = 0, \quad j \leq q
\]
By the regularity theorem for the Cauchy-Riemann equations, it follows that \( f \) is holomorphic in \( \bar{u}_1, \ldots, \bar{u}_q, u_{q+1}, \ldots, u_m \)
and
\[
\int |f(u)|^2 \exp\left( - \sum_{i=1}^{m} \mu_i^2 |u_i|^2 \right) du = \int |a|^2 e^\theta du = ||\omega||^2.
\]
This is, up to the desired change of variable, what was to be proved.

The preceding results tell us that the fibration \( H^q(\xi) \to \xi \) is a
locally trivial bundle of Hilbert spaces, with generic fiber naturally
isomorphic to
\[
(3.20) \quad H_0 = \left\{ f \in \mathcal{C}(C^m); \int_{C^m} |f(v)|^2 e^{-|\omega|^2} dv < \infty \right\}.
\]
We want to observe that \( H^q(F) \) is a space of square integrable
sections on \( U_q \) of this bundle.

3.21. Theorem. Let \( S^q(F) \) be the space of \( C^\infty \) sections of \( F^q \)
over \( U_q \) such that, for all \( \xi \in U_q \), \( \omega_\xi \in H^q(\xi) \) and
\[
(3.22) \quad ||\omega||^2 = \int_{U_q} ||\omega_\xi||^2 d\xi < \infty.
\]
Then \( H^q(F) \) is the completion of \( S^q(F) \) in this norm.

Proof. By (2.26), for such \( \omega \in S^q(F) \) we have \( ||\omega||^2 = ||\omega||^2 \),
\( \bar{\omega}_\xi \omega = \partial_\xi \omega = 0 \), and so \( S^q(F) \) is isometric to a subspace of \( H^q(F) \).
We have to show that \( S^q(F) \) is dense.

Let \( \omega \in H^q(F) \). By Proposition 2.25, \( \omega_\xi \in H^q(\xi) \) for almost all \( \xi \in U_q \), so \( \omega \) is supported in \( U_q \).
Fix \( \xi_0 \in U_q \), and let \( N \) be a neighborhood of \( \xi_0 \) such that we can find smooth functions \( u_\xi(\xi, u), \ldots, u_m(\xi, u) \)
defined on \( N \times C^m \) such that
(a) for all \( \xi, u_\xi(\xi, u), \ldots, u_m(\xi, u) \) form an orthonormal coordinate
set for \( C^m \),
(b) \( Q_\xi(u, u) = \sum_{i=1}^{q} \mu_i^2 |u_i(\xi, u)|^2 - \sum_{i=q+1}^{m} \mu_i^2 |u_i(\xi, u)|^2 \).
Let
\[
\Omega_\xi = \exp\left( - \sum_{i=1}^{q} \mu_i^2 |u_i|^2 \right) d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_q.
\]
Let \( d(\xi) = [\mu_1(\xi) \cdots \mu_m(\xi)]^{-2}, \)
\( \nu_1 = \mu_1 \bar{u}_1, \ldots, \nu_q = \mu_q \bar{u}_q, \nu_{q+1} = \mu_{q+1} u_{q+1}, \ldots, \nu_m = \mu_m u_m \).
Then, for almost all \( \xi \in N \),
\[
\omega(\xi, u) = f(\xi, v) \Omega_\xi,
\]
and
The proof of Theorem 2.26 of [10] applies on the right, to show that $f$ can be approximated by functions of the form $\sum_{k=1}^{\infty} l_k(\xi) P_k(u)$, where $l_k \in C_c^\infty(N)$ and $P_k$ is a polynomial.

For such an $f$, $f\Omega_\xi$ is in $S^q(F)$. Thus $\omega|_N$ is the closure of $S^q(F)$. Now, if we cover $U_q$ by a locally finite collection of open sets $\{N_i\}$ of this type, then for any $\omega \in H^q(F)$ supported in $N_i$, $\omega$ is in the closure of $S^q(F)$. Let $\{\rho_i\}$ be a partition of unity subordinate to the cover $\{N_i\}$. It is easy to verify that, for $\omega \in H^q(F)$, $\rho_i \omega \in H^q(F)$ and $\omega = \sum_i \rho_i \omega$ in $W^q(F)$. Since each $\rho_i \omega$ is in the closure of $S^q(F)$, so also is $\omega$.

4. Representations of $N(Q)$ on $H^q(\Sigma)$. Recall the group $N(Q)$ introduced at the beginning of §2 and its action by complex affine transformations on $C^{n+m}$, as given by (2.4). Since $\Sigma$ is an orbit of $N(Q)$, and $N(Q)$ preserves the complex structure of $C^{n+m}$, it preserves the induced CR-structure on $\Sigma$. That is, for $n \in N(Q)$, the differential $dn$ preserves the bundle $A$ of holomorphic tangent vectors tangent to $\Sigma$. Since $E^q = A^q(A^*)$, there is induced an action of $N(Q)$ on $C^\infty(E^q)$ given by

$$(n \cdot \omega)(v_p) = \omega(dn^{-1}(v_p)) .$$

We can make this explicit, referring to the coordinates of §2:

$$n \cdot \omega = \{n \cdot \omega \}_t, \quad (n \cdot \omega)(p) = \Sigma a_t(n^{-1} \cdot p) d\tilde{u}_t$$

(the reason this is so simple is that the action of $N(Q)$ is by pure translation). Since $N(Q)$ preserves the measure $dxdudu$ on $\Sigma$, this correspondence $\omega \rightarrow n \cdot \omega$ defines an isometry of $L^q(\Sigma)$, as defined in (2.13). Clearly $\tilde{\partial}_b(n \cdot \omega) = n \cdot \tilde{\partial}_b \omega$, so we also have, since $\partial_b$ is the formal adjoint of $\tilde{\partial}_b$, $\partial_b(n \cdot \omega) = n \cdot \partial_b \omega$. Thus the action (4.1) induces an isometry of $W^q$ preserving $H^q(\Sigma)$.

4.2. Definition. Let $\rho_q$ denote the unitary representation of $N(Q)$ on $H^q(\Sigma)$ induced by the action (4.1).

Now, we summarize the content of Theorem 3.19 as it applies to the representation $\rho_q$. First of all, the correspondence $\omega \rightarrow \hat{\omega}$ (as defined by (2.10) induced an isometry of $H^q(\Sigma)$ with $H^q(F)$ (Theorem 2.20), defined in terms of the $\tilde{\partial}_u$-complex on $R^{n-r} \times C^m$. We shall let $\hat{\rho}_q$ represent the transport of $\rho_q$ to $H^q(F)$ via this correspondence. Explicitly, $\hat{\rho}_q$ is induced by this action of $N(Q)$ on $C^\infty(F^q)$:

$$n \cdot \hat{\omega} = (n \cdot \omega)^-, \quad n \in N(Q) .$$
Let us explicitly compute \( p \). For \( a \in C^\infty(E^q) \), and \( n = (x_0, u_0) \) in \( N(Q) \), we have

\[
\begin{align*}
n \cdot a(x, u) &= a((-x_0, -u_0)(x, u)) = a(x - x_0 - 2 \Im Q(u_0, u), u - u_0), \\
n \cdot \tilde{a}(\xi, u) &= (n \cdot a)^\tau(\xi, u) = (\mathcal{F}_x(n \cdot a))(\xi, u)e^{Q_x(u, u)} \\
&= e^{-\xi \cdot (x_0, \cdot, 0)}e^{\Im Q(u_0, u)}(\mathcal{F}_x a)(\xi, u - u_0)e^{Q_x(u, u)} \\
&= e^{-\xi \cdot (x_0)}e^{-\xi \cdot (u_0, u_0)}e^{2Q_x(u, u_0)}\tilde{a}(\xi, u - u_0).
\end{align*}
\]

Thus

\[
(4.3) \quad n \cdot \omega(\xi, u) = e^{-\xi \cdot (x_0)}e^{-\xi \cdot (u_0, u_0)}e^{2Q_x(u, u_0)}\omega(\xi, u - u_0)
\]

for \( n = (x_0, u_0) \) and \( \omega \in C^\infty(F^q) \).

The content of Theorem 3.19 is that \( H^q(F) \) can be realized as the space of square-integrable sections of the Hilbert fibration \( H^q(\xi) \rightarrow \xi \) over \( U_v \). From (4.3) we see that the action of \( N(Q) \) is fiber-preserving. More precisely, we can freeze \( \xi \) in (4.3) and let it define an action on the space \( C^{n,q} \) of \( q \)-forms on \( C^m \):

\[
(\rho(\xi)n)\omega(u) = e^{-\xi \cdot (x_0, \cdot)}e^{-\xi \cdot (u_0, u_0)}e^{2Q_x(u, u_0)}\omega(u - u_0).
\]

Since \( Q_\xi(u, u_0) \) is holomorphic in \( u \), this action commutes with \( \bar{\partial} \). This action is isometric in the norm (3.5) (where \( \phi(u) = Q_\xi(u, u) \)), so there is induced an unitary representation \( \rho(\xi) \) of \( N(Q) \) on \( H^q(\xi) \). Now Theorem 3.19 reads as follows.

**4.4. Theorem.** \( \rho_\xi \sim \bar{\rho}_\xi \sim \int_{U_v} \oplus \rho(\xi)d\xi \).

Finally, we would like to point out that the representations \( \rho(\xi) \) are those (in the case \( n = 1 \)) found by Carmona [3]. They are irreducible, and we use Theorem 3.16 to see that. The coordinates \( v_1(\xi), \ldots, v_m(\xi) \) found in that theorem are the coordinates produced by Ogden and Vagi [9] in their description of the Plancherel formula for the groups \( N(Q) \). Theorem 3.16 describes the intertwining operator which intertwines \( \rho(\xi) \) with their representation \( \pi_\xi \). We can generalize their theorem.

**4.5. Theorem.** The representation \( \bigoplus \rho_\xi \) of \( N(Q) \) on \( \bigoplus H^q(E) \) is isometric to a subrepresentation of the left regular representation on \( L^2(N(Q)) \) in which every irreducible (except for a set of Plancherel measure zero) occurs with multiplicity one.

In the language of Auslander and Kostant, the vector bundle \( A \) of holomorphic tangent vectors tangent to \( \Sigma \), arises from a Lie subalgebra \( \mathfrak{h} \) of \( \mathfrak{m}(Q)^* \). If \( \mathfrak{z} \) is the center of \( \mathfrak{m}(Q) \), then \( \mathfrak{z}^\mathfrak{c} \oplus \mathfrak{h} \) is a
polarization at $\xi$, for all $\xi \in \mathfrak{g}^*(\subset \mu(Q)^*)$ which is positive if and only if $\xi \in U_\circ$. If $\xi \in U_\circ, q \neq 0$, then the new coordinates of Theorem 3.16 relate to a positive polarization at $\xi$, and Theorem 3.16 exhibits the intertwining operator between the representations corresponding to these polarizations.

REFERENCES

3. J. Carmona, Représentations du groupe de Heisenberg dans les espaces de $(0, q)$-formes, Math. Annalen, 205 (1973) 89–112.

Received January 8, 1976.

THE UNIVERSITY OF UTAH
AND
C.N.R.S. PARIS
Pacific Journal of Mathematics
Vol. 65, No. 1 September, 1976

David Lee Armacost, *Compactly cogenerated LCA groups* ........................................ 1
Sun Man Chang, *On continuous image averaging of probability measures* ........... 13
J. Chidambaramswamy, *Generalized Dedekind ψ-functions with respect to a polynomial. II* ................................................................. 19
Freddy Delbaen, *The Dunford-Pettis property for certain uniform algebras* ...... 29
Robert Benjamin Feinberg, *Faithful distributive modules over incidence algebras* .................................................................................. 35
Paul Froeschl, *Chained rings* .................................................................................. 47
John Brady Garnett and Anthony G. O’Farrell, *Sobolev approximation by a sum of subalgebras on the circle* ........................................... 55
Hugh M. Hilden, José M. Montesinos and Thomas Lusk Thickstun, *Closed oriented 3-manifolds as 3-fold branched coverings of S^3 of special type* 65
Atsushi Inoue, *On a class of unbounded operator algebras* .................................. 77
Peter Kleinschmidt, *On facets with non-arbitrary shapes* .................................... 97
Narendrakumar Ramanlal Ladhawala, *Absolute summability of Walsh-Fourier series* ....................................................................................... 103
Howard Wilson Lambert, *Links which are unknottable by maps* ......................... 109
Kyung Bai Lee, *On certain g-first countable spaces* ............................................. 113
Richard Ira Loebl, *A Hahn decomposition for linear maps* ................................ 119
James Miller, *Subordinating factor sequences and convex functions of several variables* .................................................................................. 159
Keith Pierce, *Amalgamated sums of abelian l-groups* ......................................... 167
Jonathan Rosenberg, *The C*-algebras of some real and p-adic solvable groups* .................................................................................. 175
Hugo Rossi and Michele Vergne, *Group representations on Hilbert spaces defined in terms of ∂_b-cohomology on the Silov boundary of a Siegel domain* .... 193
Mary Elizabeth Schaps, *Nonsingular deformations of a determinantal scheme* .......................................................... 209
S. R. Singh, *Some convergence properties of the Bubnov-Galerkin method* ....... 217
Peggy Strait, *Level crossing probabilities for a multi-parameter Brownian process* .................................................................................. 223
Robert M. Tardiff, *Topologies for probabilistic metric spaces* ............................ 233
Benjamin Baxter Wells, Jr., *Rearrangements of functions on the ring of integers of a p-series field* ................................................................ 253
Robert Francis Wheeler, *Well-behaved and totally bounded approximate identities for C_0(X)* ................................................................. 261
Delores Arletta Williams, *Gauss sums and integral quadratic forms over local fields of characteristic 2* .................................................. 271
John Yuan, *On the construction of one-parameter semigroups in topological semigroups* .............................................................................. 285