NONSINGULAR DEFORMATIONS OF A DETERMINANTAL SCHEME

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We will be considering an affine algebraic scheme $X$ over a field $k$, which is determinantal, defined by the vanishing of the $l \times l$ minors of a matrix $R$.

We will show that deforming the constant and linear terms of the entries in the matrix $R$ gives an almost everywhere flat deformation of $X$, and that under certain simple conditions, and in particular if the dimension of $X$ is sufficiently low, this deformation has generically nonsingular fibers.

Essentially the same results were obtained simultaneously by D. Laksov [3] using more general theorems on transversality of mappings. He quotes a result of T. Svanes indicating that the codimension result, identical in both versions, is the best obtainable (see Example 3).

This article is a generalization of an earlier result about nonsingular deformations of Cohen-Macaulay schemes of codimension 2 (Schaps [4]). Moreover, since determinantal schemes were introduced by Macaulay as a generalization of complete intersection, the theorem proven in this paper can be regarded as a generalization of Bertini's theorem, that the generic deformation of a complete intersection is nonsingular.

The precise definition of a determinantal scheme is as follows:

DEFINITION. An affine scheme $X = \text{Spec}(k[Z_1, \ldots, Z_l]/J)$ is determinantal if $J$ is generated by all the $l \times l$ minors of an $m \times n$ matrix $R$ of polynomials, and $X$ is equidimensional of codimension $(m - l + 1)(n - l + 1)$.

On the course of the theorem, we will need to use the generic determinantal scheme, constructed as follows: Let $Y = (Y_{ij}), i = 1, \ldots, m, j = 1, \ldots, n$, be a set of indeterminates, and let $P^r_{Y_{ij}}$ be the ideal generated in $k[Y]$ by the $l \times l$ minors of the matrix $[Y_{ij}]$. Then it is known that $P^r_{Y}$ is a prime ideal of height $(m - l + 1)(n - l + 1)$. This number is thus the maximal codimension that can be obtained by a scheme generated by minors of this order in an $m \times n$ matrix. We will use a recent result by Hochster and Eagon [2], that every determinantal scheme is Cohen-Macaulay.
We will now proceed to the main theorem and its corollary. One example is included with the proof, a second reserved to the end.

**Theorem.** Let \( X = \text{Spec}(B) \) be a determinantal scheme, \( B \) being the quotient of \( k[Z] = k[Z_1, \ldots, Z_t] \) by an ideal generated by the \( l \times l \) minors of some \( m \times n \) matrix \( R = [r_{ij}] \). If \( l \) equals 1, or \( m = n = l \) or \( q < (m - l + 2)(n - l + 2) \), then \( X \) has a flat deformation whose generic fiber is nonsingular.

**Proof.** If \( m = n = l \), \( X \) is just a hypersurface, and we replace \( R \) by the \( 1 \times 1 \) matrix \([\det R]\). Let \( \bar{X} \) be the algebraic family of deformations of \( X \) defined by the \( l \times l \) minors of the deformed matrix \( \bar{R} = [\bar{r}_{ij}] \), where
\[
\bar{r}_{ij} = r_{ij} + u_{ij} + \sum_{i=1}^{q} v_{ij} z_i,
\]
the \( U \) and \( V \) being indeterminates. \( \bar{X} \) is itself determinantal, isomorphic to the product \( \text{Spec}(k[Y]/P_i) \times \text{Spec}(k[V, Z]) \), of the generic determinantal scheme of type \((m, n, l)\) with the affine space of dimension \( q(mn + 1) \). The isomorphism is induced by \( \phi: k[Y, V, Z] \to k[U, V, Z] \), with
\[
\phi: Y_{ij} \longrightarrow \bar{r}_{ij}(U, V, Z)
\]
\[
\phi, \phi^{-1}: V_{ij}^t \longrightarrow \bar{V}_{ij}^t
\]
\[
\phi, \phi^{-1}: Z_i \longrightarrow \bar{Z}_i
\]
\[
\phi^{-1}: U_{ij} \longrightarrow Y_{ij} - \bar{r}_{ij}(0, V, Z).
\]
This gives codim \( \bar{X} = (m - l + 1)(n - l + 1) \), and also allows us to determine the singular locus of \( \bar{X} \), which will induce singularities on the fibers. (The remaining fiber singularities come from tangencies between \( \bar{X} \) and the fiber of the ambient space.) The singular locus of the generic determinantal scheme is also determinantal, generated by the \((l - 1) \times (l - 1)\) minors. Thus the singular locus of \( \bar{X} \) is the set on which \( \text{rank } \bar{R} < l - 1 \). For \( l = 1, \bar{X} \) is a nonsingular complete intersection. For \( l > 1 \), the singular locus has codim \((m - l + 2)(n - l + 2) \). Since \( q < (m - l + 2)(n - l + 2) \), in that case, by hypothesis, the projection of this locus has positive codimension, so in either case there is an open subset of \( S = \text{Spec}(k[U, V]) \) over which \( \text{rank } \bar{R} \geq l - 1 \).

Before proceeding to the proof of the second part of the theorem, we will introduce some new notation. Let \( \mu \subset \{1, \ldots, m\} \) and \( \nu \subset \{1, \ldots, n\} \) designate sets of rows or columns, respectively, and let
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# prefixed to a set designate its cardinality. Then if \( \# \mu = \# \nu = l \), we let \( f_{\mu} \) be the subdeterminant of \( \tilde{R} \) with rows \( \mu \) and columns \( \nu \) (without any adjustment of sign). Similarly if \( \# \mu = \# \nu = l - 1 \), we let \( b_{\mu} \) be the subdeterminant with rows \( \mu \) and columns \( \nu \).

We now assume that rank \( \tilde{R} \geq l - 1 \) over some subset \( N \) of \( S \), open in the Zariski topology. By making a translation of coordinates if necessary, we may assume that the origin is in \( N \). Let \( \pi \) be the projection onto \( S \).

**Lemma 2.** Under these hypotheses, \( \pi^{-1}(N) \) is locally a complete intersection, generated in each open affine \( \tilde{X}_b \), \( b = b_{\mu_0} \), by \( (1/b)f_{\alpha \beta} \) for all \( \alpha, \beta \) such that \( \mu_0 \subset \alpha, \nu_0 \subset \beta \).

**Proof.** It is clear from our hypothesis on the rank of \( \tilde{R} \) that the sets \( \tilde{X}_b \), \( b = b_{\mu_0} \) with \( \# \mu_0 = \# \nu_0 = l - 1 \), do indeed cover \( \pi^{-1}(N) \). We will fix \( \mu_0 \) and \( \nu_0 \). Let \( I \) be the ideal in \( h[U, V, Z] \), generated by the \( (m - l + 1)(n - l + 1) \) functions

\[
h_{ij} = \pm b^{-1}f_{\alpha \beta},
\]

where \( \alpha = \mu_0 \cup \{i\}, \beta = \mu_0 \cup \{j\} \), and the sign is so adjusted that in the expansion of \( f_{\alpha \beta} \) along the row \( i \), \( \tilde{r}_{ij}b \) will have positive coefficient. If \( i \in \mu_0 \) or \( j \in \nu_0 \), let us write \( h_{ij} = 0 \).

Let \( f_{\mu} \) be any other \( l \times l \) subdeterminant. We will show that \( f_{\mu} \in I \). Decomposing \( f_{\alpha \beta} \) along the \( i \)th row, and dividing by \( b \), we have, after the adjustment of sign

\[
h_{ij} = \tilde{r}_{ij} + b^{-1}\sum_{t \in \nu_0} \pm b_{\mu_0} \tilde{r}_{it},
\]

where \( \sigma = (\nu_0 \cup \{j\}) - \{i\} \), and the sign of \( \tilde{r}_{it} \) is \( (-1)^{\alpha(i) - \alpha(j)} \), where \( \alpha(t) \) and \( \alpha(j) \) are the respective positions of these numbers in \( \alpha \), regarded as an ordered set. Let us denote by \( \tilde{r}_i \) the \( i \)th column of \( \tilde{R} \) and add to the \( j \)th column, for any \( j \in \nu_0 \), the partial sum

\[
b^{-1}\sum_{t \in \nu_0 \& i} \pm b_{\mu_0} \tilde{r}_t.
\]

The entry in row \( i \) of column \( j \) will then be

\[
h_{ij} - b^{-1}\sum_{t \in \nu \& i} \pm b_{\mu_0} \tilde{r}_{it}.
\]

Since we have added multiples of columns from the index set \( \nu \), and in fact from the set \( \nu \cap \nu_0 \) of unchanged columns, the value of the minor \( f_{\mu} \) will be unaffected by this operation which replaces \( \tilde{r}_{ij} \) by \( (*) \), applied to the \( d = (\nu - \nu_0) \) columns of \( \nu - \nu_0 \). We now use
the multilinearity of the determinant to decompose the sums in these $d$ columns. Since $(\nu_0 - \nu)$ is $d - 1$, $f_{\mu_0}$ is thus the sum of $d^3$ determinants, each of which either contains a column $h_j$, or contains $d$ columns which are multiples of the $d - 1$ columns $\bar{r}_t, t \in \nu_0 - \nu$. Since these latter determinants all vanish, we have $f_{\mu_0} \in I$.

**Example 1.** Consider the determinantal scheme generated by the $2 \times 2$ minors of the matrix

$$
\begin{bmatrix}
Z_1 & Z_2 & Z_3 \\
1 & Z_4 & Z_5 \\
1 & 1 & Z_6
\end{bmatrix}.
$$

We require the codimension to be $(3 - 2 + 1)(3 - 2 + 1) = 4$, and in fact we have a complete intersection generated by $Z_1 = 1, Z_i = Z_3, Z_5 = Z_6, Z_3 = Z_1 Z_6$. We construct the matrix $\bar{R} = [\bar{r}_{ij}]$, where, for example,

$$
\bar{r}_{11} = Z_1 + U_{11} + V_{11} Z_1 + \cdots + V_{11}^n Z_6.
$$

Let $\mu_0 = \{3\}$, and $\nu_0 = \{3\}$. Thus

$$
b = \bar{r}_{33} = Z_6 + U_{33} + \cdots
$$

and

$$
\begin{align*}
 h_{11} &= \bar{r}_{11} - \frac{\bar{r}_{11} \bar{r}_{13}}{b} \\
 h_{12} &= \bar{r}_{12} - \frac{\bar{r}_{12} \bar{r}_{13}}{b} \\
 h_{21} &= \bar{r}_{21} - \frac{\bar{r}_{21} \bar{r}_{23}}{b} \\
 h_{22} &= \bar{r}_{22} - \frac{\bar{r}_{22} \bar{r}_{23}}{b}.
\end{align*}
$$

Consider $f_{\mu_0}, \mu = \{1, 2\}, \nu = \{1, 2\}$

$$
f_{\mu_0} = 
\begin{vmatrix}
\bar{r}_{11} & \bar{r}_{12} \\
\bar{r}_{21} & \bar{r}_{22}
\end{vmatrix}
\begin{vmatrix}
h_{11} & \bar{r}_{13} \\
h_{21} & \bar{r}_{23}
\end{vmatrix}
= 
\begin{vmatrix} 
\bar{r}_{11} & h_{12} + \frac{\bar{r}_{12} \bar{r}_{13}}{b} \\
\bar{r}_{21} & h_{22} + \frac{\bar{r}_{22} \bar{r}_{23}}{b}
\end{vmatrix}
\begin{vmatrix}
\bar{r}_{13} & h_{12} \\
\bar{r}_{23} & h_{22}
\end{vmatrix}
= 
\begin{vmatrix} 
\bar{r}_{11} & h_{12} + \frac{\bar{r}_{12} \bar{r}_{13}}{b} \\
h_{21} & h_{22}
\end{vmatrix}
\begin{vmatrix}
\bar{r}_{13} & h_{12} \\
\bar{r}_{23} & h_{22}
\end{vmatrix}
+ 
\begin{vmatrix} 
\bar{r}_{11} & \bar{r}_{13} \\
h_{21} & \bar{r}_{23}
\end{vmatrix}
\begin{vmatrix}
\bar{r}_{13} & h_{12} \\
\bar{r}_{23} & h_{22}
\end{vmatrix}
+ 
\begin{vmatrix} 
\bar{r}_{11} & \bar{r}_{13} \\
h_{21} & \bar{r}_{23}
\end{vmatrix}
\begin{vmatrix}
\bar{r}_{13} & h_{12} \\
\bar{r}_{23} & h_{22}
\end{vmatrix}
+ 
\begin{vmatrix} 
\bar{r}_{11} & \bar{r}_{13} \\
h_{21} & \bar{r}_{23}
\end{vmatrix}
\begin{vmatrix}
\bar{r}_{13} & h_{12} \\
\bar{r}_{23} & h_{22}
\end{vmatrix}.
Therefore $f_{\mu \nu} \in I$.

We now continue with the proof of the main theorem. We let $c = (m - l + 1)(n - l + 1)$ be the codimension of $X$. We have $c \leq q$. Define a scheme $\tilde{V}$ to be the subscheme of $X_s$ defined by the vanishing of the $c \times c$ minors of the Jacobian matrix $[\partial h_{ij}/\partial Z_t]$ for $i \in \mu, j \in \nu$. By the Jacobian criterion, the singular scheme of any fiber of $X_s$ is supported on its intersection with $\tilde{V}$. (EGA IV, 0.20.5.14). If therefore we can show that $\text{codim } \tilde{V} \geq q + 1$, we will know that the fibers are nonsingular except over a proper subscheme of $S$, for if the closure of the projection of $\tilde{V}$ did not have positive codimension, there would be an open subset of the parameter space over which the fiber of $\tilde{V}$ would be nonempty, and thus of codimension less than or equal to $q$. This is possible only if $\tilde{V}$ has codimension less than or equal to $q$ over this set. Take indeterminants $W_{ij}^t, 1 \leq t \leq q, i \in \mu, j \in \nu$, corresponding to the entries in the $c \times q$ Jacobian matrix, and indeterminants $Y_{ij}$ corresponding to the $c$ generators $h_{ij}$ of $J_s$. Let

$$\mathcal{W} = \text{Spec } k[W, Y].$$

Now $c \leq q$, and thus $P_w$ is an ideal of height $q - c + 1$. Therefore $P_w^w + (Y)$ is an ideal of height $q + 1$. Thus

$$\text{codim}_{\mathcal{W}} \text{Spec } k[W, Y]/P_w^w + (Y)$$

is $q + 1$. Let $\hat{U}, \hat{V}$ be the subsets of $U, V$ consisting of all $U_{ij}, V_{ij}$ such that $i \in \mu$ or $j \in \nu$. We wish to construct an isomorphism

$$(k[W, Y]/(P_w^w + (Y)))[\hat{U}, \hat{V}, Z] \xrightarrow{} k[U, V, Z]/J_b.$$

Here $b = b_{s \nu_{q}}$ as always, and thus $b \in k[\hat{U}, \hat{V}, Z]$. We will map

$$Z \mapsto Z$$
$$\hat{U} \mapsto \hat{U}$$
$$\hat{V} \mapsto \hat{V}.$$

Hence, the invertibility of $b$ will be preserved.

As for the remaining indeterminates, we send

$$W_{ij}^t \mapsto \partial h_{ij}/\partial Z_t$$
$$Y_{ij} \mapsto h_{ij}.$$

To construct the inverse mapping $\phi$ we write

$$h_{ij} = \bar{r}_{ij} + g_{ij}(\hat{U}, \hat{V}, Z)$$
$$\partial h_{ij}/\partial Z_t = \partial r_{ij}/\partial Z_t + V_{ij} + \partial g_{ij}/\partial Z_t.$$

Therefore we set
\[ \Phi(V_{ij}) = W_{ij}t - \partial r_{ij}\partial Z_t - \partial g_{ij}/\partial Z_t \]
\[ \Phi(U_{ij}) = Y_{ij} - r_{ij} - \sum \Phi(V_{ij})Z_t - g_{ij}. \]

Since the mappings also establish an isomorphism between the ambient spaces
\[ \mathcal{X}_s = \text{Spec}(k[U, V, Z]) \]
and
\[ \mathcal{Y} = \text{Spec} [W, Y, \hat{U}, \hat{V}, Z], \]
it is clear that
\[ \text{codim}_{\mathcal{X}_s} V = \text{codim}_{\mathcal{Y}} \text{Spec} \left( k[W, Y, \hat{U}, \hat{V}, Z]/P^w_e + (Y) \right) \]
\[ = \text{codim}_{\mathcal{Y}} \text{Spec} \left( k[W, Y]/P^w_e + (Y) \right) \]
\[ = q + 1. \]

It remains to show that we can restrict \( \hat{X} \) to a flat deformation of \( \hat{X} \). We have proven above that the generic fiber of \( \hat{X} \) is locally the intersection of hypersurfaces. Since these are generically in general position, the generic fiber is nonempty and thus of codimension equal to the codimension of \( X \). (Shafarevich, Chap. 1. §6 [5]).

Thus if we let \( W \) be the constructible subset of \( S \) over which the fibers have this codimension \( c \), \( W \) will contain the origin, 0, and also a Zariski open subset of \( S \). If \( 0 \in \overline{S - W} \), let \( m \) be the maximum dimension of the components of \( \overline{S - W} \) containing 0, and let \( H \) be a regular subspace of \( S \) through 0 of codimension \( m \). By choosing \( H \) in general position, we can insure that any properties of the fibers over an open subset of \( S \) will also be true of the generic fiber of \( \hat{X} \) over \( H \), in particular, smoothness. If \( 0 \notin \overline{S - W} \), we will take \( H \) to be \( S \). The restriction of \( \hat{X} \) to \( H \cap W \) has equidimensional fibers, \( H \cap W \) is open since the intersection of \( H \) with \( \overline{S - W} \) consists of isolated points, and the restriction of \( X \) to this regular scheme is equidimensional, hence determinantal, hence Cohen-Macaulay. Since we may assume the generic fiber over \( H \) to be smooth, the theorem now follows from the lemma quoted below, a proof of which is included in Schaps [4]. The local version is in EGA IV, 6.1.5.

**Lemma.** Given a morphism of algebraic schemes \( f : X \to Y \) of finite type, \( Y \) regular, \( X \) Cohen-Macaulay, and the closed fibers of \( X \) over \( Y \) equidimensional, then the map \( f \) is flat.

**Example 2.** If \( k \) is an infinite field, there is a large and important class of reduced schemes which can be represented as determinantal
schemes, the union of all linear coordinate schemes of dimension $p$ in $q$ space, for $p < q$. One simply chooses a $q \times (p + 1)$ matrix $A = [a_{ij}]$ over $k$ such that all its maximal minors are nonzero, and lets $R = [a_{ij}Z_i]$. Let $s = p + 1$. The $\binom{q}{s}$ maximal minors are scalar multiples of the monomials $\Pi Z_i$ of degree $s$, and the scheme is thus supported on the union of the spaces.

$$Z_{i_1} = \cdots = Z_{i_{q-p}} = 0,$$

with distinct $i_j$.

The theorem tells us that this scheme has non-singular deformations for $q = p + 1$, a hypersurface, and for $q < 2(q - p + 1)$, that is, $q > 2(p - 1)$. D. Mumford conjectures that these are the only smoothable cases.

**Example 3.** A counter-example for the case $q = (m - l + 2)(n - l + 2), l > 1$, is the scheme generated by the minors of the matrix

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}$$

where $R = Y_{ij}, i \leq m - l + 2, j \leq n - l + 2$, and $I$ is the identity matrix of order $l - 2$. $X$ is actually the generic determinantal scheme of type $(m - l + 2, n - l + 2, 2)$, and therefore has an isolated singularity. By a result of T. Svanes (thesis, M.I.T., 1971), the generic member of any flat family deforming $X$ will also have an isolated singularity.

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