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**NONSINGULAR DEFORMATIONS OF A DETERMINANTAL
SCHEME**

MARY ELIZABETH SCHAPS

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We will be considering an affine algebraic scheme X over a field k , which is determinantal, defined by the vanishing of the $l \times l$ minors of a matrix R .

We will show that deforming the constant and linear terms of the entries in the matrix R gives an almost everywhere flat deformation of X , and that under certain simple conditions, and in particular if the dimension of X is sufficiently low, this deformation has generically nonsingular fibers.

Essentially the same results were obtained simultaneously by D. Laksov [3] using more general theorems on transversality of mappings. He quotes a result of T. Svanes indicating that the codimension result, identical in both versions, is the best obtainable (see Example 3).

This article is a generalization of an earlier result about nonsingular deformations of Cohen-Macaulay schemes of codimension 2 (Schaps [4]). Moreover, since determinantal schemes were introduced by Macaulay as a generalization of complete intersection, the theorem proven in this paper can be regarded as a generalization of Bertini's theorem, that the generic deformation of a complete intersection is nonsingular.

The precise definition of a determinantal scheme is as follows:

DEFINITION. An affine scheme $X = \text{Spec}(k[Z_1, \dots, Z_q]/J)$ is determinantal if J is generated by all the $l \times l$ minors of an $m \times n$ matrix R of polynomials, and X is equidimensional of codimension $(m - l + 1)(n - l + 1)$.

On the course of the theorem, we will need to use the generic determinantal scheme, constructed as follows: Let $Y = (Y_{ij})$, $i = 1, \dots, m$, $j = 1, \dots, n$, be a set of indeterminates, and let P_l^Y be the ideal generated in $k[Y]$ by the $l \times l$ minors of the matrix $[Y_{ij}]$. Then it is known that P_l^Y is a prime ideal of height $(m - l + 1)(n - l + 1)$. This number is thus the maximal codimension that can be obtained by a scheme generated by minors of this order in an $m \times n$ matrix. We will use a recent result by Hochster and Eagon [2], that every determinantal scheme is Cohen-Macaulay.

We will now proceed to the main theorem and its corollary. One example is included with the proof, a second reserved to the end.

THEOREM. *Let $X = \text{Spec}(B)$ be a determinantal scheme, B being the quotient of $k[Z] = k[Z_1, \dots, Z_q]$ by an ideal generated by the $l \times l$ minors of some $m \times n$ matrix $R = [r_{ij}]$. If l equals 1, or $m = n = l$ or $q < (m - l + 2)(n - l + 2)$, then X has a flat deformation whose generic fiber is nonsingular.*

Proof. If $m = n = l$, X is just a hypersurface, and we replace R by the 1×1 matrix $[\det R]$. Let \tilde{X} be the algebraic family of deformations of X defined by the $l \times l$ minors of the deformed matrix $\tilde{R} = [\tilde{r}_{ij}]$, where

$$\tilde{r}_{ij} = r_{ij} + U_{ij} + \sum_{t=1}^q V_{ij}^t Z_t,$$

the U and V being indeterminates. \tilde{X} is itself determinantal, isomorphic to the product $\text{Spec}(k[Y]/P_t^Y) \times \text{Spec}(k[V, Z])$, of the generic determinantal scheme of type (m, n, l) with the affine space of dimension $q(mn + 1)$. The isomorphism is induced by $\phi: k[Y, V, Z] \rightarrow k[U, V, Z]$, with

$$\begin{aligned} \phi: Y_{ij} &\longrightarrow \tilde{r}_{ij}(U, V, Z) \\ \phi, \phi^{-1}: V_{ij}^t &\longrightarrow V_{ij}^t \\ \phi, \phi^{-1}: Z_t &\longrightarrow Z_t \\ \phi^{-1}: U_{ij} &\longrightarrow Y_{ij} - \tilde{r}_{ij}(0, V, Z). \end{aligned}$$

This gives $\text{codim } \tilde{X} = (m - l + 1)(n - l + 1)$, and also allows us to determine the singular locus of \tilde{X} , which will induce singularities on the fibers. (The remaining fiber singularities come from tangencies between \tilde{X} and the fiber of the ambient space.) The singular locus of the generic determinantal scheme is also determinantal, generated by the $(l - 1) \times (l - 1)$ minors. Thus the singular locus of \tilde{X} is the set on which $\text{rank } \tilde{R} < l - 1$. For $l = 1$, \tilde{X} is a nonsingular complete intersection. For $l > 1$, the singular locus has $\text{codim } (m - l + 2)(n - l + 2)$. Since $q < (m - l + 2)(n - l + 2)$, in that case, by hypothesis, the projection of this locus has positive codimension, so in either case there is an open subset of $S = \text{Spec}(k[U, V])$ over which $\text{rank } \tilde{R} \geq l - 1$.

Before proceeding to the proof of the second part of the theorem, we will introduce some new notation. Let $\mu \subset \{1, \dots, m\}$ and $\nu \subset \{1, \dots, n\}$ designate sets of rows or columns, respectively, and let

prefixed to a set designate its cardinality. Then if $\#\mu = \#\nu = l$, we let $f_{\mu\nu}$ be the subdeterminant of \tilde{R} with rows μ and columns ν (without any adjustment of sign). Similarly if $\#\mu = \#\nu = l - 1$, we let $b_{\mu\nu}$ be the subdeterminant with rows μ and columns ν .

We now assume that $\text{rank } \tilde{R} \geq l - 1$ over some subset N of S , open in the Zariski topology. By making a translation of coordinates if necessary, we may assume that the origin is in N . Let π be the projection onto S .

LEMMA 2. *Under these hypotheses, $\pi^{-1}(N)$ is locally a complete intersection, generated in each open affine $\tilde{X}_b, b = b_{\mu_0\nu_0}$, by $(1/b)f_{\alpha\beta}$ for all α, β such that $\mu_0 \subset \alpha, \nu_0 \subset \beta$.*

Proof. It is clear from our hypothesis on the rank of \tilde{R} that the sets $\tilde{X}_b, b = b_{\mu_0\nu_0}$ with $\#\mu_0 = \#\nu_0 = l - 1$, do indeed cover $\pi^{-1}(N)$. We will fix μ_0 and ν_0 . Let I be the ideal in $k[U, V, Z]_b$ generated by the $(m - l + 1)(n - l + 1)$ functions

$$h_{ij} = \pm b^{-1}f_{\alpha\beta}$$

where $\alpha = \mu_0 \cup \{i\}, \beta = \nu_0 \cup \{j\}$, and the sign is so adjusted that in the expansion of $f_{\alpha\beta}$ along the row $i, \tilde{r}_{ij}b$ will have positive coefficient. If $i \in \mu_0$ or $j \in \nu_0$, let us write $h_{ij} = 0$.

Let $f_{\mu\nu}$ be any other $l \times l$ subdeterminant. We will show that $f_{\mu\nu} \in I$. Decomposing $f_{\alpha\beta}$ along the i th row, and dividing by b , we have, after the adjustment of sign

$$h_{ij} = \tilde{r}_{ij} + b^{-1} \sum_{t \in \nu_0} \pm b_{\mu_0\sigma} \tilde{r}_{it},$$

where $\sigma = (\nu_0 \cup \{j\}) - \{t\}$, and the sign of \tilde{r}_{it} is $(-1)^{\alpha(t) - \alpha(j)}$, where $\alpha(t)$ and $\alpha(j)$ are the respective positions of these numbers in α , regarded as an ordered set. Let us denote by \tilde{r}_t the t th column of \tilde{R} and add to the j th column, for any $j \notin \nu_0$, the partial sum

$$b^{-1} \sum_{t \in \nu_0 \cap \nu} \pm b_{\mu_0\sigma} \tilde{r}_t.$$

The entry in row i of column j will then be

$$(*) \quad h_{ij} - b^{-1} \sum_{t \in \nu_0 - \nu} \pm b_{\mu_0\sigma} \tilde{r}_{it}.$$

Since we have added multiples of columns from the index set ν , and in fact from the set $\nu \cap \nu_0$ of unchanged columns, the value of the minor $f_{\mu\nu}$ will be unaffected by this operation which replaces r_{ij} by $(*)$, applied to the $d = \#(\nu - \nu_0)$ columns of $\nu - \nu_0$. We now use

the multilinearity of the determinant to decompose the sums in these d columns. Since $\#(\nu_0 - \nu)$ is $d - 1$, $f_{\mu\nu}$ is thus the sum of d^2 determinants, each of which either contains a column h_j , or contains d columns which are multiples of the $d - 1$ columns $\tilde{r}_t, t \in \nu_0 - \nu$. Since these latter determinants all vanish, we have $f_{\mu\nu} \in I$.

EXAMPLE 1. Consider the determinantal scheme generated by the 2×2 minors of the matrix

$$\begin{bmatrix} Z_1 & Z_2 & Z_3 \\ 1 & Z_4 & Z_5 \\ 1 & 1 & Z_6 \end{bmatrix}.$$

We require the codimension to be $(3 - 2 + 1)(3 - 2 + 1) = 4$, and in fact we have a complete intersection generated by $Z_4 = 1, Z_1 = Z_2, Z_5 = Z_6, Z_3 = Z_1 Z_6$. We construct the matrix $\tilde{R} = [\tilde{r}_{ij}]$, where, for example,

$$\tilde{r}_{11} = Z_1 + U_{11} + V_{11}^1 Z_1 + \cdots + V_{11}^6 Z_6.$$

Let $\mu_0 = \{3\}$, and $\nu_0 = \{3\}$.

Thus

$$b = \tilde{r}_{33} = Z_6 + U_{33} + \cdots$$

and

$$\begin{aligned} h_{11} &= \tilde{r}_{11} - \frac{r_{31}}{b} \tilde{r}_{13} & h_{12} &= \tilde{r}_{12} - \frac{\tilde{r}_{32}}{b} \tilde{r}_{13} \\ h_{21} &= \tilde{r}_{21} - \frac{\tilde{r}_{31}}{b} \tilde{r}_{23} & h_{22} &= \tilde{r}_{22} - \frac{\tilde{r}_{32}}{b} \tilde{r}_{23}. \end{aligned}$$

Consider $f_{\mu\nu}, \mu = \{1, 2\}, \nu = \{1, 2\}$

$$\begin{aligned} f_{\mu\nu} &= \begin{vmatrix} \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{21} & \tilde{r}_{22} \end{vmatrix} \\ &= \begin{vmatrix} h_{11} + \frac{\tilde{r}_{31}}{b} \tilde{r}_{13} & h_{12} + \frac{\tilde{r}_{32}}{b} \tilde{r}_{13} \\ h_{21} + \frac{\tilde{r}_{31}}{b} \tilde{r}_{23} & h_{22} + \frac{\tilde{r}_{32}}{b} \tilde{r}_{23} \end{vmatrix} \\ &= \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} + \frac{\tilde{r}_{32}}{b} \begin{vmatrix} h_{11} & \tilde{r}_{13} \\ h_{21} & \tilde{r}_{23} \end{vmatrix} \\ &= \frac{\tilde{r}_{31}}{b} \begin{vmatrix} \tilde{r}_{13} & h_{12} \\ \tilde{r}_{23} & h_{22} \end{vmatrix} + \frac{\tilde{r}_{31} \tilde{r}_{32}}{b \cdot b} \begin{vmatrix} \tilde{r}_{13} & \tilde{r}_{13} \\ \tilde{r}_{23} & \tilde{r}_{23} \end{vmatrix} \end{aligned}$$

Therefore $f_{\mu_\nu} \in I$.

We now continue with the proof of the main theorem. We let $c = (m - l + 1)(n - l + 1)$ be the codimension of X . We have $c \leq q$. Define a scheme \tilde{V} to be the subscheme of \tilde{X}_b defined by the vanishing of the $c \times c$ minors of the Jacobian matrix $[\partial h_{ij}/\partial Z_t]$ for $i \in \mu_0, j \in \nu_0$. By the Jacobian criterion, the singular scheme of any fiber of \tilde{X}_b is supported on its intersection with \tilde{V} . (EGA IV, 0.20.5.14). If therefore we can show that $\text{codim } \tilde{V} \geq q + 1$, we will know that the fibers are nonsingular except over a proper subscheme of S , for if the closure of the projection of \tilde{V} did not have positive codimension, there would be an open subset of the parameter space over which the fiber of \tilde{V} would be nonempty, and thus of codimension less than or equal to q . This is possible only if \tilde{V} has codimension less than or equal to q over this set. Take indeterminants $W_{ij}^t, 1 \leq t \leq q, i \in \mu_0, j \in \nu_0$, corresponding to the entries in the $c \times q$ Jacobian matrix, and indeterminants Y_{ij} corresponding to the c generators h_{ij} of J_b . Let

$$\mathscr{W} = \text{Spec } k[W, Y].$$

Now $c \leq q$, and thus P_c^w is an ideal of height $q - c + 1$. Therefore $P_c^w + (Y)$ is an ideal of height $q + 1$. Thus

$$\text{codim}_{\mathscr{W}} \text{Spec } k[W, Y]/P_c^w + (Y)$$

is $q + 1$. Let \hat{U}, \hat{V} be the subsets of U, V consisting of all U_{ij}, V_{ij}^t such that $i \in \mu_0$ or $j \in \nu_0$. We wish to construct an isomorphism

$$(k[W, Y]/(P_c^w + (Y)))[\hat{U}, \hat{V}, Z]_b \longrightarrow k[U, V, Z]_b/J_b.$$

Here $b = b_{\mu_0\nu_0}$ as always, and thus $b \in k[\hat{U}, \hat{V}, Z]$. We will map

$$\begin{aligned} Z &\longrightarrow Z \\ \hat{U} &\longrightarrow \hat{U} \\ \hat{V} &\longrightarrow \hat{V}. \end{aligned}$$

Hence, the invertibility of b will be preserved.

As for the remaining indeterminates, we send

$$\begin{aligned} W_{ij}^t &\longrightarrow \partial h_{ij}/\partial Z_t \\ Y_{ij} &\longrightarrow h_{ij}. \end{aligned}$$

To construct the inverse mapping ϕ we write

$$\begin{aligned} h_{ij} &= \tilde{r}_{ij} + g_{ij}(\hat{U}, \hat{V}, Z) \\ \partial h_{ij}/\partial Z_t &= \partial r_{ij}/\partial Z_t + V_{ij}^t + \partial g_{ij}/\partial Z_t. \end{aligned}$$

Therefore we set

$$\begin{aligned}\phi(V_{ij}^t) &= W_{ij}^t - \partial r_{ij}/\partial Z_t - \partial g_{ij}/\partial Z_t \\ \phi(U_{ij}) &= Y_{ij} - r_{ij} - \sum \phi(V_{ij}^t)Z_t - g_{ij}.\end{aligned}$$

Since the mappings also establish an isomorphism between the ambient spaces

$$\mathcal{X}_b = \text{Spec}(k[U, V, Z]_b)$$

and

$$\mathcal{Y} = \text{Spec}[W, Y, \hat{U}, \hat{V}, Z]_b,$$

it is clear that

$$\begin{aligned}\text{codim}_{\mathcal{X}_b} V &= \text{codim}_{\mathcal{Y}} \text{Spec}(k[W, Y, \hat{U}, \hat{V}, Z]/P_c^W + (Y))_b \\ &= \text{codim}_{\mathcal{Y}} \text{Spec}(k[W, Y]/P_c^W + (Y)) \\ &= q + 1.\end{aligned}$$

It remains to show that we can restrict \tilde{X} to a flat deformation of \tilde{X} . We have proven above that the generic fiber of \tilde{X} is locally the intersection of hypersurfaces. Since these are generically in general position, the generic fiber is nonempty and thus of codimension equal to the codimension of X . (Shafarevich, Chap. 1. §6 [5]).

Thus if we let W be the constructible subset of S over which the fibers have this codimension c , W will contain the origin, 0 , and also a Zariski open subset of S . If $0 \in \overline{S - W}$, let m be the maximum dimension of the components of $\overline{S - W}$ containing 0 , and let H be a regular subspace of S through 0 of codimension m . By choosing H in general position, we can insure that any properties of the fibers over an open subset of S will also be true of the generic fiber of \tilde{X} over H , in particular, smoothness. If $0 \notin \overline{S - W}$, we will take H to be S . The restriction of \tilde{X} to $H \cap W$ has equidimensional fibers, $H \cap W$ is open since the intersection of H with $\overline{S - W}$ consists of isolated points, and the restriction of X to this regular scheme is equidimensional, hence determinantal, hence Cohen-Macaulay. Since we may assume the generic fiber over H to be smooth, the theorem now follows from the lemma quoted below, a proof of which is included in Schaps [4]. The local version is in EGA IV, 6.1.5.

LEMMA. *Given a morphism of algebraic schemes $f: X \rightarrow Y$ of finite type, Y regular, X Cohen-Macaulay, and the closed fibers of X over Y equidimensional, then the map f is flat.*

EXAMPLE 2. If k is an infinite field, there is a large and important class of reduced schemes which can be represented as determinantal

schemes, the union of all linear coordinate schemes of dimension p in q space, for $p < q$. One simply chooses a $q \times (p + 1)$ matrix $A = [a_{ij}]$ over k such that all its maximal minors are nonzero, and lets $R = [a_{ij}Z_i]$. Let $s = p + 1$. The $\binom{q}{s}$ maximal minors are scalar multiples of the monomials $\prod Z_i$ of degree s , and the scheme is thus supported on the union of the spaces.

$$Z_{i_1} = \dots = Z_{i_{q-p}} = 0,$$

with distinct i_j .

The theorem tells us that this scheme has non-singular deformations for $q = p + 1$, a hypersurface, and for $q < 2(q - p + 1)$, that is, $q > 2(p - 1)$. D. Mumford conjectures that these are the only smoothable cases.

EXAMPLE 3. A counter-example for the case $q = (m - l + 2)(n - l + 2)$, $l > 1$, is the scheme generated by the minors of the matrix

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}$$

where $R = Y_{ij}$, $i \leq m - l + 2$, $j \leq n - l + 2$, and I is the identity matrix of order $l - 2$. X is actually the generic determinantal scheme of type $(m - l + 2, n - l + 2, 2)$, and therefore has an isolated singularity. By a result of T. Svanes (thesis, M.I.T., 1971), the generic member of any flat family deforming X will also have an isolated singularity.

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