NONSINGULAR DEFORMATIONS OF A DETERMINANTAL SCHEME

MARY ELIZABETH SCHAPS
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We will be considering an affine algebraic scheme \( X \) over a field \( k \), which is determinantal, defined by the vanishing of the \( l \times l \) minors of a matrix \( R \).

We will show that deforming the constant and linear terms of the entries in the matrix \( R \) gives an almost everywhere flat deformation of \( X \), and that under certain simple conditions, and in particular if the dimension of \( X \) is sufficiently low, this deformation has generically nonsingular fibers.

Essentially the same results were obtained simultaneously by D. Laksov [3] using more general theorems on transversality of mappings. He quotes a result of T. Svanes indicating that the codimension result, identical in both versions, is the best obtainable (see Example 3).

This article is a generalization of an earlier result about nonsingular deformations of Cohen-Macaulay schemes of codimension 2 (Schaps [4]). Moreover, since determinantal schemes were introduced by Macaulay as a generalization of complete intersection, the theorem proven in this paper can be regarded as a generalization of Bertini’s theorem, that the generic deformation of a complete intersection is nonsingular.

The precise definition of a determinantal scheme is as follows:

**Definition.** An affine scheme \( X = \text{Spec}(k[Z_{ij}], \cdots, Z_{ij})/J \) is determinantal if \( J \) is generated by all the \( l \times l \) minors of an \( m \times n \) matrix \( R \) of polynomials, and \( X \) is equidimensional of codimension \( (m - l + 1)(n - l + 1) \).

On the course of the theorem, we will need to use the generic determinantal scheme, constructed as follows: Let \( Y = (Y_{ij}), i = 1, \cdots, m, j = 1, \cdots n, \) be a set of indeterminates, and let \( P^*_r \) be the ideal generated in \( k[Y] \) by the \( l \times l \) minors of the matrix \( [Y_{ij}] \). Then it is known that \( P^*_r \) is a prime ideal of height \( (m - l + 1)(n - l + 1) \). This number is thus the maximal codimension that can be obtained by a scheme generated by minors of this order in an \( m \times n \) matrix. We will use a recent result by Hochster and Eagon [2], that every determinantal scheme is Cohen-Macaulay.
We will now proceed to the main theorem and its corollary. One example is included with the proof, a second reserved to the end.

**Theorem.** Let \( X = \text{Spec} (B) \) be a determinantal scheme, \( B \) being the quotient of \( k[Z] = k[Z_1, \ldots, Z_q] \) by an ideal generated by the \( l \times l \) minors of some \( m \times n \) matrix \( R = [r_{ij}] \). If \( l \) equals 1, or \( m = n = l \) or \( q < (m - l + 2)(n - l + 2) \), then \( X \) has a flat deformation whose generic fiber is nonsingular.

**Proof.** If \( m = n = l \), \( X \) is just a hypersurface, and we replace \( R \) by the \( 1 \times 1 \) matrix \([\det R]\). Let \( \bar{X} \) be the algebraic family of deformations of \( X \) defined by the \( l \times l \) minors of the deformed matrix \( \bar{R} = [\bar{r}_{ij}] \), where

\[
\bar{r}_{ij} = r_{ij} + U_{ij} + \sum_{t=1}^q V_{ij}Z_t,
\]

the \( U \) and \( V \) being indeterminates. \( \bar{X} \) is itself determinantal, isomorphic to the product \( \text{Spec} (k[Y]/P_i) \times \text{Spec} (k[V, Z]) \), of the generic determinantal scheme of type \((m, n, l)\) with the affine space of dimension \( q(mn + 1) \). The isomorphism is induced by \( \phi: k[Y, V, Z] \to k[U, V, Z] \), with

\[
\phi: Y_{ij} \longrightarrow \bar{r}_{ij}(U, V, Z) \\
\phi, \phi^{-1}: V_{ij}' \longrightarrow V_{ij} \\
\phi, \phi^{-1}: Z_t \longrightarrow Z_t \\
\phi^{-1}: U_{ij} \longrightarrow Y_{ij} - \bar{r}_{ij}(0, V, Z).
\]

This gives codim \( \bar{X} = (m - l + 1)(n - l + 1) \), and also allows us to determine the singular locus of \( \bar{X} \), which will induce singularities on the fibers. (The remaining fiber singularities come from tangencies between \( \bar{X} \) and the fiber of the ambient space.) The singular locus of the generic determinantal scheme is also determinantal, generated by the \((l - 1) \times (l - 1)\) minors. Thus the singular locus of \( \bar{X} \) is the set on which rank \( \bar{R} < l - 1 \). For \( l = 1 \), \( \bar{X} \) is a nonsingular complete intersection. For \( l > 1 \), the singular locus has codim \((m - l + 2)(n - l + 2) \). Since \( q < (m - l + 2)(n - l + 2) \), in that case, by hypothesis, the projection of this locus has positive codimension, so in either case there is an open subset of \( S = \text{Spec} (k[U, V]) \) over which rank \( \bar{R} \geq l - 1 \).

Before proceeding to the proof of the second part of the theorem, we will introduce some new notation. Let \( \mu \subset\{1, \ldots, m\} \) and \( \nu \subset\{1, \ldots, n\} \) designate sets of rows or columns, respectively, and let
# prefixed to a set designate its cardinality. Then if $\#\mu = \#\nu = l$, we let $f_{\mu\nu}$ be the subdeterminant of $\tilde{R}$ with rows $\mu$ and columns $\nu$ (without any adjustment of sign). Similarly if $\#\mu = \#\nu = l - 1$, we let $b_{\mu\nu}$ be the subdeterminant with rows $\mu$ and columns $\nu$.

We now assume that rank $\tilde{R} \geq l - 1$ over some subset $N$ of $S$, open in the Zariski topology. By making a translation of coordinates if necessary, we may assume that the origin is in $N$. Let $\pi$ be the projection onto $S$.

**Lemma 2.** Under these hypotheses, $\pi^{-1}(N)$ is locally a complete intersection, generated in each open affine $X_b$, $b = b_{\mu\nu}$, by $(1/b)f_{\alpha\beta}$ for all $\alpha, \beta$ such that $\mu_0 \subseteq \alpha$, $\nu_0 \subseteq \beta$.

**Proof.** It is clear from our hypothesis on the rank of $\tilde{R}$ that the sets $\tilde{X}_b$, $b = b_{\mu\nu}$ with $\#\mu_0 = \#\nu_0 = l - 1$, do indeed cover $\pi^{-1}(N)$. We will fix $\mu_0$ and $\nu_0$. Let $I$ be the ideal in $k[U, V, Z]_b$ generated by the $(m - l + 1)(n - l + 1)$ functions

$$h_{ij} = \pm b^{-1}f_{\alpha\beta}$$

where $\alpha = \mu_0 \cup \{i\}$, $\beta = \nu_0 \cup \{j\}$, and the sign is so adjusted that in the expansion of $f_{\alpha\beta}$ along the row $i$, $\tilde{r}_{ij}b$ will have positive coefficient. If $i \in \mu_0$ or $j \in \nu_0$, let us write $h_{ij} = 0$.

Let $f_{\mu\nu}$ be any other $l \times l$ subdeterminant. We will show that $f_{\mu\nu} \in I$. Decomposing $f_{\alpha\beta}$ along the $i$th row, and dividing by $b$, we have, after the adjustment of sign

$$h_{ij} = \tilde{r}_{ij} + b^{-1}\sum_{t \in \nu_0} \pm b_{\nu_0} \tilde{r}_{it},$$

where $\sigma = (\nu_0 \cup \{j\}) - \{t\}$, and the sign of $\tilde{r}_{ij}$ is $(-1)^{\alpha(t) - \alpha(j)}$, where $\alpha(t)$ and $\alpha(j)$ are the respective positions of these numbers in $\alpha$, regarded as an ordered set. Let us denote by $\tilde{r}_t$ the $t$th column of $\tilde{R}$ and add to the $j$th column, for any $j \in \nu_0$, the partial sum

$$b^{-1}\sum_{t \in \nu_0} \pm b_{\nu_0} \tilde{r}_t.$$ 

The entry in row $i$ of column $j$ will then be

$$h_{ij} = b^{-1}\sum_{t \in \nu_0} \pm b_{\nu_0} \tilde{r}_{it}.\text{ (\*)}$$

Since we have added multiples of columns from the index set $\nu$, and in fact from the set $\nu \cap \nu_0$ of unchanged columns, the value of the minor $f_{\mu\nu}$ will be unaffected by this operation which replaces $r_{ij}$ by (\*), applied to the $d = \#(\nu - \nu_0)$ columns of $\nu - \nu_0$. We now use
the multilinearity of the determinant to decompose the sums in these $d$ columns. Since $^*(\nu_0 - \nu)$ is $d - 1$, $f_{\nu,\mu}$ is thus the sum of $d^2$ determinants, each of which either contains a column $h_j$, or contains $d$ columns which are multiples of the $d - 1$ columns $\tilde{r}_t$, $t \in \nu_0 - \nu$. Since these latter determinants all vanish, we have $f_{\nu,\mu} \in I$.

**Example 1.** Consider the determinantal scheme generated by the $2 \times 2$ minors of the matrix

\[
\begin{bmatrix}
Z_1 & Z_2 & Z_3 \\
1 & Z_4 & Z_5 \\
1 & 1 & Z_6
\end{bmatrix}.
\]

We require the codimension to be $(3 - 2 + 1)(3 - 2 + 1) = 4$, and in fact we have a complete intersection generated by $Z_1 = 1, Z_1 = Z_2, Z_5 = Z_6, Z_3 = Z_1Z_6$. We construct the matrix $R = [\tilde{r}_{ij}]$, where, for example,

\[
\tilde{r}_{11} = Z_1 + U_{11} + V_{11}Z_1 + \cdots + V_{16}Z_6.
\]

Let $\mu_0 = \{3\}$, and $\nu_0 = \{3\}$.

Thus

\[
b = \tilde{r}_{33} = Z_6 + U_{33} + \cdots
\]

and

\[
h_{11} = \tilde{r}_{11} - \frac{\tilde{r}_{31}\tilde{r}_{13}}{b} \quad h_{12} = \tilde{r}_{12} - \frac{\tilde{r}_{32}\tilde{r}_{13}}{b} \\

h_{21} = \tilde{r}_{21} - \frac{\tilde{r}_{31}\tilde{r}_{23}}{b} \quad h_{22} = \tilde{r}_{22} - \frac{\tilde{r}_{32}\tilde{r}_{23}}{b}.
\]

Consider $f_{\nu,\mu}$, $\mu = \{1, 2\}$, $\nu = \{1, 2\}$

\[
f_{\nu,\mu} = \begin{vmatrix}
\tilde{r}_{11} & \tilde{r}_{12} \\
\tilde{r}_{21} & \tilde{r}_{22}
\end{vmatrix} = \begin{vmatrix}
h_{11} + \frac{\tilde{r}_{31}\tilde{r}_{13}}{b} & h_{12} + \frac{\tilde{r}_{32}\tilde{r}_{13}}{b} \\
h_{21} + \frac{\tilde{r}_{31}\tilde{r}_{23}}{b} & h_{22} + \frac{\tilde{r}_{32}\tilde{r}_{23}}{b}
\end{vmatrix}
\]

\[
= \begin{vmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{vmatrix} + \frac{\tilde{r}_{32}}{b} \begin{vmatrix}
\tilde{r}_{13} & 1 \\
\tilde{r}_{23} & 1
\end{vmatrix}
\]

\[
= \begin{vmatrix}
\tilde{r}_{31} \\
\tilde{r}_{23}
\end{vmatrix} \frac{b}{b} + \begin{vmatrix}
\tilde{r}_{31} & \tilde{r}_{32} \\
\tilde{r}_{23} & \tilde{r}_{23}
\end{vmatrix} \frac{b}{b}
\]

\[
= \frac{\tilde{r}_{31}}{b} \begin{vmatrix}
1 & 1 \\
1 & 1
\end{vmatrix} + \frac{\tilde{r}_{31}\tilde{r}_{32}}{b} \begin{vmatrix}
1 & 1 \\
1 & 1
\end{vmatrix}.
\]
Therefore \( f_{\mu_0} \in I \).

We now continue with the proof of the main theorem. We let \( c = (m - l + 1)(n - l + 1) \) be the codimension of \( X \). We have \( c \leq q \). Define a scheme \( \tilde{V} \) to be the subscheme of \( \tilde{X}_b \) defined by the vanishing of the \( c \times c \) minors of the Jacobian matrix \([\partial h_{i,j}/\partial Z_t] \) for \( i \notin \mu_0 \), \( j \notin \nu_0 \). By the Jacobian criterion, the singular scheme of any fiber of \( \tilde{X}_b \) is supported on its intersection with \( \tilde{V} \). (EGA IV, 0.20.5.14). If therefore we can show that \( \text{codim} \tilde{V} \geq q + 1 \), we will know that the fibers are nonsingular except over a proper subscheme of \( S \), for if the closure of the projection of \( \tilde{V} \) did not have positive codimension, there would be an open subset of the parameter space over which the fiber of \( \tilde{V} \) would be nonempty, and thus of codimension less than or equal to \( q \). This is possible only if \( \tilde{V} \) has codimension less than or equal to \( q \) over this set. Take indeterminants \( W_{i,t} \), \( 1 \leq t \leq q \), \( i \notin \mu_0 \), \( j \notin \nu_0 \), corresponding to the entries in the \( c \times q \) Jacobian matrix, and indeterminants \( Y_{i,t} \) corresponding to the \( c \) generators \( h_{i,j} \) of \( J_b \). Let

\[ \mathcal{W} = \text{Spec} \, k[W, Y]. \]

Now \( c \leq q \), and thus \( P_c^w \) is an ideal of height \( q - c + 1 \). Therefore \( P_c^w + (Y) \) is an ideal of height \( q + 1 \). Thus

\[ \text{codim}_{\mathcal{W}} \text{Spec} \, k[W, Y]/P_c^w + (Y) \]

is \( q + 1 \). Let \( \tilde{U}, \tilde{V} \) be the subsets of \( U, V \) consisting of all \( U_{i,j}, V_{i,j} \) such that \( i \notin \mu_0 \) or \( j \notin \nu_0 \). We wish to construct an isomorphism

\[ (k[W, Y]/(P_c^w + (Y)))[\tilde{U}, \tilde{V}, Z]_b \rightarrow k[U, V, Z]_b/J_b. \]

Here \( b = b_{\nu_0 \nu_0} \) as always, and thus \( b \in k[\tilde{U}, \tilde{V}, Z] \). We will map

\[ Z \rightarrow Z \]
\[ \tilde{U} \rightarrow \tilde{U} \]
\[ \tilde{V} \rightarrow \tilde{V}. \]

Hence, the invertibility of \( b \) will be preserved.

As for the remaining indeterminants, we send

\[ W_{i,j} \rightarrow \partial h_{i,j}/\partial Z_t \]
\[ Y_{i,j} \rightarrow h_{i,j}. \]

To construct the inverse mapping \( \phi \) we write

\[ h_{i,j} = r_{i,j} + g_{i,j}(\tilde{U}, \tilde{V}, Z) \]
\[ \partial h_{i,j}/\partial Z_t = \partial r_{i,j}/\partial Z_t + V_{i,j} + \partial g_{i,j}/\partial Z_t. \]

Therefore we set
\( \phi(V_{ij}) = W_{ij} - \partial r_{ij}/\partial Z_t - \partial g_{ij}/\partial Z_t \)
\( \phi(U_{ij}) = Y_{ij} - r_{ij} - \sum \phi(V_{ij})Z_t - g_{ij}. \)

Since the mappings also establish an isomorphism between the ambient spaces

\[ \mathcal{X}_b = \text{Spec}(k[U, V, Z])_b \]

and

\[ \mathcal{Y} = \text{Spec}[W, Y, \hat{U}, \hat{V}, Z], \]

it is clear that

\[ \text{codim}_{\mathcal{X}_b} V = \text{codim}_{\mathcal{Y}} \text{Spec}(k[W, Y, \hat{U}, \hat{V}, Z]/P^w + (Y))_b \]
\[ = \text{codim}_{\mathcal{Y}} \text{Spec}(k[W, Y]/P^w + (Y)) \]
\[ = q + 1. \]

It remains to show that we can restrict \( \mathcal{X} \) to a flat deformation of \( \mathcal{X} \). We have proven above that the generic fiber of \( \mathcal{X} \) is locally the intersection of hypersurfaces. Since these are generically in general position, the generic fiber is nonempty and thus of codimension equal to the codimension of \( X \). (Shafarevich, Chap. 1. §6 [5]).

Thus if we let \( W \) be the constructible subset of \( S \) over which the fibers have this codimension \( c \), \( W \) will contain the origin, 0, and also a Zariski open subset of \( S \). If \( 0 \in S - W \), let \( m \) be the maximum dimension of the components of \( S - W \) containing 0, and let \( H \) be a regular subspace of \( S \) through 0 of codimension \( m \). By choosing \( H \) in general position, we can insure that any properties of the fibers over an open subset of \( S \) will also be true of the generic fiber of \( \mathcal{X} \) over \( H \), in particular, smoothness. If \( 0 \in S - W \), we will take \( H \) to be \( S \). The restriction of \( \mathcal{X} \) to \( H \cap W \) has equidimensional fibers, \( H \cap W \) is open since the intersection of \( H \) with \( S - W \) consists of isolated points, and the restriction of \( X \) to this regular scheme is equidimensional, hence determinantal, hence Cohen-Macaulay. Since we may assume the generic fiber over \( H \) to be smooth, the theorem now follows from the lemma quoted below, a proof of which is included in Schaps [4]. The local version is in EGA IV, 6.1.5.

**Lemma.** Given a morphism of algebraic schemes \( f: X \to Y \) of finite type, \( Y \) regular, \( X \) Cohen-Macaulay, and the closed fibers of \( X \) over \( Y \) equidimensional, then the map \( f \) is flat.

**Example 2.** If \( k \) is an infinite field, there is a large and important class of reduced schemes which can be represented as determinantal
schemes, the union of all linear coordinate schemes of dimension \( p \) in \( q \) space, for \( p < q \). One simply chooses a \( q \times (p + 1) \) matrix \( A = [a_{ij}] \) over \( k \) such that all its maximal minors are nonzero, and lets \( R = [a_{ij}Z_i] \). Let \( s = p + 1 \). The \( \binom{q}{s} \) maximal minors are scalar multiples of the monomials \( \Pi Z_i \) of degree \( s \), and the scheme is thus supported on the union of the spaces.

\[ Z_{i_1} = \cdots = Z_{i_{q-p}} = 0 , \]

with distinct \( i_j \).

The theorem tells us that this scheme has non-singular deformations for \( q = p + 1 \), a hypersurface, and for \( q < 2(q - p + 1) \), that is, \( q > 2(p - 1) \). D. Mumford conjectures that these are the only smoothable cases.

**Example 3.** A counter-example for the case \( q = (m - l + 2)(n - l + 2), l > 1 \), is the scheme generated by the minors of the matrix

\[
\begin{bmatrix}
R & 0 \\
0 & I
\end{bmatrix}
\]

where \( R = Y_{ij}, i \leq m - l + 2, j \leq n - l + 2 \), and \( I \) is the identity matrix of order \( l - 2 \). \( X \) is actually the generic determinantal scheme of type \((m - l + 2, n - l + 2, 2)\), and therefore has an isolated singularity. By a result of T. Svanes (thesis, M.I.T., 1971), the generic member of any flat family deforming \( X \) will also have an isolated singularity.

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