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We generalize the Bubnov-Galerkin method to approximate the resolvent of the m-sectorial operator associated with a densely defined, closed, sectorial form in a Hilbert space. Some special cases of interest are also discussed.

1. Introduction. The Bubnov-Galerkin method [3] was originally devised to approximate the solutions of the equations of the form

$$(1) \qquad (z - A)f = g$$

where A is an operator in a Hilbert space, $\mathscr{H}$, g is a vector in $\mathscr{H}$ and z is a complex number. The method proceeds with solving the following set of equations:

$$(2) \qquad \sum_{j=1}^{n} \alpha_j(\phi_i \,|\, (z - A)\phi_j) = (\phi_i \,|\, g) \qquad i = 1, \cdots, n\,;$$

where $(.\,|\,.)$ denotes the scalar product in $\mathscr{H}$ and $\{\phi_i\} \subset \mathscr{D}(A)$ is some linearly independent (l.i.) set in $\mathscr{H}$. $\mathscr{D}(\cdot)$ denotes the domain. The questions of interest are the existence and the convergence of the solutions of equation (2). Until recently, the only cases that received a detailed treatment have been when A is compact, bounded or essentially self-adjoint [3, 6]. However, recently the following result was proven by Masson and Thewarapperuma [2]:

R.1. Let A be symmetric, bounded below by b, z be at a nonzero distance from $[b, \infty)$ and $\{\phi_i\}$ be the orthonormal set formed from $\{A^i h\}$ where h is in $\mathscr{D}(A^i)$ for each i. Then $\lim_{n\to\infty} \|\sum_{j=1}^{n} \alpha_j\phi_j - (z - A_F)^{-1}g\| = 0$, where $\|.\|$ denotes the norm in $\mathscr{H}$ and A_F is the Friedrichs extension of A.

Consider the following set of equations:

$$(3) \qquad \sum_{j=1}^{n} \alpha_j[z(\phi_i \,|\, \phi_j) - t(\phi_i, \phi_j)] = (\phi_i \,|\, g) \qquad i = 1, \cdots, n\,;$$

where t is a densely defined, closable, sectorial, sesquilinear form in $\mathscr{H}$. The sector of t will be denoted by S and since it causes no loss of generality, the vertex will be taken to be one. In the present note we determine the limit of $f_n = \sum_{j=1}^{n} \alpha_j\phi_j$ as n becomes large.

R.1. and some other generalizations of it, will follow from our main result (Theorem 1).

2. **Results.** Define a new scalar product $(.\,|\,.)_t$ on $\mathscr{D}(t)$ by $(u\,|\,v)_t = \mathrm{Re}.\, t(u, v)$, [1, pp. 309–10] and complete $\mathscr{D}(t)$ in the new metric to a Hilbert space $\mathscr{H}_t$. Let the closure of t be $\bar{t}$. We have that $\mathscr{D}(t) \subset \mathscr{D}(\bar{t}) = \mathscr{H}_t \subset \mathscr{H}$. The norm in $\mathscr{H}_t$ will be denoted by $\|.\|_t$. Also $\mathscr{B}(X, Y)$ will denote the space of bounded operators with $\mathscr{D}(\cdot) \subset X$ and range $\mathscr{R}(\cdot) \subset Y$, and $\mathscr{B}(X) = \mathscr{B}(X, X)$.

LEMMA 1. *Let t be as in equation* (3), $\{\phi_i\} \subset \mathscr{D}(t)$ and $g \in \mathscr{H}$. *Equation* (3) *is equivalent to*

$$(4) \qquad \sum_{j=1}^{n} \alpha_j(\phi_i \,|\, [1 - T(z)]\phi_j)_t = -(\phi_i \,|\, Bg)_t \qquad i = 1, \cdots, n\,;$$

where $B \in \mathscr{B}(\mathscr{H}, \mathscr{H}_t)$, $T(z) = (zB_t - C) \in \mathscr{B}(\mathscr{H}_t)$ and B_t is the restriction of B to $\mathscr{D}(t)$.

Proof. Since $t_1 = (t - \mathrm{Re}.\, t)$ is a bounded form on $\mathscr{H}_t$ [1, p. 314], there is a $C \in \mathscr{B}(\mathscr{H}_t)$ such that

$$t_1(u, v) = (u \,|\, Cv)_t;\ u, v \in \mathscr{D}(t)\,.$$

Also from Ref. [4] pp. 332–3, it follows that there is a unique $B \in \mathscr{B}(\mathscr{H}, \mathscr{H}_t)$ such that $\mathscr{D}(B) = \mathscr{H}$ and for $u \in \mathscr{H}_t$, $w \in \mathscr{H}$,

$$(5) \qquad (u \,|\, \omega) = (u \,|\, B\omega)_t\,.$$

In particular, in equation (3), $(\phi_i \,|\, g) = (\phi_i \,|\, Bg)_t$ and $(\phi_i \,|\, \phi_j) = (\phi_i \,|\, B\phi_j)_t = (\phi_i \,|\, B_t\phi_j)_t$.

The assertion now follows from direct substitution.

LEMMA 2. *In the notation of Lemma* 1, *we have that B_t, C are closable, B is closed and invertible and $B^{-1}(1 + \bar{C}) = A_t$ where A_t is the unique m-sectorial operator associated with $\bar{t}$.*

Proof. Since B_t and C are bounded and densely defined, they are closable. Since B is bounded and $\mathscr{D}(B) = \mathscr{H}$, it is closed. Invertibility of B has been proven in Reference [4] p. 333.

Now, $\mathscr{D}([B^{-1}(1 + \bar{C})]) \subset \mathscr{H}_t = \mathscr{D}(\bar{t})$ and for $u, v \in \mathscr{D}(t)$,

$$\begin{aligned}(u \,|\, B^{-1}(1 + \bar{C})v) &= (u \,|\, B^{-1}(1 + C)v)\\ &= (u \,|\, (1 + C)v)_t \qquad \text{(equation (5))}\\ &= t(u, v)\end{aligned}$$

From the closability of t, this result extends for $u, v \in \mathscr{H}_t$. The result now follows from Theorem 2.1, Chapter 6, Reference [1].

THEOREM 1. *In addition to the assumptions of Lemma* 1 *and* 2, *let* $\{\phi_i\}$ *be* l.i. *and complete in* $\mathscr{H}_t$, *and* z *be at a nonzero distance from* S. $f_n = \sum_{j=1}^{n} \alpha_j \phi_j$ *of equation* (3) *is then defined for each* n *and* $\lim_{n\to\infty} \| f_n - (z - A_t)^{-1} g \| = 0$.

Proof. From Lemma 1, equation (3) is equivalent to equation (4). Also without loss of generality, we may assume $\{\phi_i\}$ to be an orthonormal basis in $\mathscr{H}_t$. It is straightforward to check that (4) is equivalent to

$$(1 - T_n(z))f_n = -P_n B g$$

where $T_n(z) = P_n T(z) P_n$, and P_n is the ortho-projection on the n-dimentional subspace of $\mathscr{H}_t$ determined by $\{\phi_i\}$, $i = 1$ to n. It follows, for $h \in \mathscr{H}_t$, that

$$\lim_{n\to\infty} \| (T_n(z) - \bar{T}(z))h \|_t = 0 \, .$$

Also, since z is at a nonzero distance from S, dist. $(1, W(\bar{T}(z))) = d' > 0$, where $W(\cdot)$ denotes the numerical range. Further, since the spectrum of T_n, $\sigma(T_n) \subset (W(\bar{T}(z)) \cup \{0\})$, for each n, $(1 - T_n(z))^{-1} \in \mathscr{B}(\mathscr{H}_t)$ with $\| (1 - T_n(z))^{-1} \|_t \leqq 1/d$ where $d = \min.(1, d')$. Also $(1 - \bar{T}(z))^{-1} \in \mathscr{B}(\mathscr{H}_t)$.

Hence for $h \in \mathscr{H}_t$

$$\begin{aligned} &\| [(1 - T_n(z))^{-1} - (1 - \bar{T}(z))^{-1}]h \|_t \\ &\quad = \| (1 - T_n(z))^{-1}(T_n(z) - \bar{T}(z))(1 - \bar{T}(z))^{-1}h \|_t \\ &\quad \leqq \| (1 - T_n(z))^{-1} \|_t \, \| (T_n(z) - \bar{T}(z))(1 - \bar{T}(z))^{-1}h \|_t \\ &\quad \xrightarrow[n\to\infty]{} 0 \, . \end{aligned}$$

Further, for $g \in \mathscr{H}$,

$$\lim_{n\to\infty} \| (P_n B - B)g \|_t = 0$$

and hence

$$\lim_{n\to\infty} \| f_n - f \|_t = 0$$

where

$$\begin{aligned} f = -(1 - \bar{T}(z))^{-1}Bg &= -(1 - z\bar{B}_t + \bar{C})^{-1}Bg \\ &= (z - B^{-1}(1 + \bar{C}))^{-1}g \\ &= (z - A_t)^{-1}g \qquad \text{(Lemma 2)} \, . \end{aligned}$$

Assertion of the theorem follows by observing that $||\,.\,||_t \geqq ||\,.\,||$. For a symmetric t, $s = [b, \infty)$ with some $b > -\infty$, $\bar{C} = 0$ and $A_t = B^{-1}$ is self-adjoint.

In the following, f_n will stand for $\sum_{j=1}^{n} \alpha_j \phi_j$ as defined by equation (2).

COROLLARY 1. *Let A be densely defined sectorial operator and z be at a nonzero distance from its sector, $\{\phi_i\}$ be a* l.i. *basis in $\mathscr{D}(A)$. We have that* $\lim_{n\to\infty} ||f_n - (z - A_F)^{-1} g|| = 0$.

Proof. Define t of Theorem 1 by $t(u, v) = (u \mid Av)$, $u, v \in \mathscr{D}(A)$. t is closable from Theorem 1.27, Chapter 6 of [1]. Since $\{\phi_i\}$ is a l.i. basis in $\mathscr{D}(A)$ and $\mathscr{D}(A)$ is dense in $\mathscr{D}(\bar{t}) = \mathscr{H}_t$, it is a l.i. basis in $\mathscr{H}_t$. The result now follows from the fact that A_t of Theorem 1 now becomes A_F [1, pp. 325-6].

COROLLARY 2. *Let A be symmetric, bounded below by b, z be at a nonzero distance from $[b, \infty)$ and $\{\phi_i\}$ be a* l.i. *basis in $\mathscr{D}(A)$. Then* $\lim_{n\to\infty} ||f_n - (z - A_F)^{-1} g|| = 0$.

Proof. The result follows from Corollary 1, by noticing that the sector of A is $[b, \infty)$.

If the set $\{\phi_i\}$ is taken to be $\{A^i h\}$ for some $h \in \mathscr{D}(A^i)$ for $i = 0, 1, 2, \cdots$; the Bubnov-Galerkin method is called the method of moments [7]. Since $\{A^i h\}$ satisfies the conditions of Corollaries 1 and 2, the convergence of the method of moments also is established by these results. The result R.1 [2] thus is a special case of Corollary 2.

In Corollaries 1 and 2 we have considered the case of a densely defined A. In these results one can replace this condition by requiring that the form domain of A be dense. However since the Friedrichs extension is defined only for a densely defined A, the limit operator A_t may not be A_F. This situation is of a particular interest in Physics which we describe in brief.

Let A be given, formally, by $A = A_1 + A_2$, where A_1 and A_2 are symmetric but $\mathscr{D}(A) = \mathscr{D}(A_1) \cap \mathscr{D}(A_2)$ is not dense. However if the form domain of A is dense, the self-adjoint operator A_t associated with the form $t(u, v) = (u \mid (A_1 + A_2)v)$ is a legitimate operator to describe a physical system [5]. This construction enables one to include a larger class of interactions in the treatment than the requirement that A be densely defined [5]. It is obvious that the Bubnov-Galerkin method enables one to compute the resolvent of A_t in this case also, which is of prime importance in Physics.

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