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**SOME CONVERGENCE PROPERTIES OF THE  
BUBNOV-GALERKIN METHOD**

S. R. SINGH

## SOME CONVERGENCE PROPERTIES OF THE BUBNOV-GALERKIN METHOD

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**We generalize the Bubnov-Galerkin method to approximate the resolvent of the  $m$ -sectorial operator associated with a densely defined, closed, sectorial form in a Hilbert space. Some special cases of interest are also discussed.**

1. Introduction. The Bubnov-Galerkin method [3] was originally devised to approximate the solutions of the equations of the form

$$(1) \quad (z - A)f = g$$

where  $A$  is an operator in a Hilbert space,  $\mathcal{H}$ ,  $g$  is a vector in  $\mathcal{H}$  and  $z$  is a complex number. The method proceeds with solving the following set of equations:

$$(2) \quad \sum_{j=1}^n \alpha_j (\phi_i | (z - A)\phi_j) = (\phi_i | g) \quad i = 1, \dots, n;$$

where  $(\cdot | \cdot)$  denotes the scalar product in  $\mathcal{H}$  and  $\{\phi_i\} \subset \mathcal{D}(A)$  is some linearly independent (l.i.) set in  $\mathcal{H}$ .  $\mathcal{D}(\cdot)$  denotes the domain. The questions of interest are the existence and the convergence of the solutions of equation (2). Until recently, the only cases that received a detailed treatment have been when  $A$  is compact, bounded or essentially self-adjoint [3, 6]. However, recently the following result was proven by Masson and Thewarapperuma [2]:

R.1. Let  $A$  be symmetric, bounded below by  $b$ ,  $z$  be at a non-zero distance from  $[b, \infty)$  and  $\{\phi_i\}$  be the orthonormal set formed from  $\{A^i h\}$  where  $h$  is in  $\mathcal{D}(A^i)$  for each  $i$ . Then  $\lim_{n \rightarrow \infty} \|\sum_{j=1}^n \alpha_j \phi_j - (z - A_p)^{-1} g\| = 0$ , where  $\|\cdot\|$  denotes the norm in  $\mathcal{H}$  and  $A_p$  is the Friedrichs extension of  $A$ .

Consider the following set of equations:

$$(3) \quad \sum_{j=1}^n \alpha_j [z(\phi_i | \phi_j) - t(\phi_i, \phi_j)] = (\phi_i | g) \quad i = 1, \dots, n;$$

where  $t$  is a densely defined, closable, sectorial, sesquilinear form in  $\mathcal{H}$ . The sector of  $t$  will be denoted by  $S$  and since it causes no loss of generality, the vertex will be taken to be one. In the present note we determine the limit of  $f_n = \sum_{j=1}^n \alpha_j \phi_j$  as  $n$  becomes large.

R.1. and some other generalizations of it, will follow from our main result (Theorem 1).

2. Results. Define a new scalar product  $(\cdot | \cdot)_t$  on  $\mathcal{D}(t)$  by  $(u | v)_t = \text{Re. } t(u, v)$ , [1, pp. 309-10] and complete  $\mathcal{D}(t)$  in the new metric to a Hilbert space  $\mathcal{H}_t$ . Let the closure of  $t$  be  $\bar{t}$ . We have that  $\mathcal{D}(t) \subset \mathcal{D}(\bar{t}) = \mathcal{H}_t \subset \mathcal{H}$ . The norm in  $\mathcal{H}_t$  will be denoted by  $\|\cdot\|_t$ . Also  $\mathcal{B}(X, Y)$  will denote the space of bounded operators with  $\mathcal{D}(\cdot) \subset X$  and range  $\mathcal{R}(\cdot) \subset Y$ , and  $\mathcal{B}(X) = \mathcal{B}(X, X)$ .

LEMMA 1. Let  $t$  be as in equation (3),  $\{\phi_i\} \subset \mathcal{D}(t)$  and  $g \in \mathcal{H}$ . Equation (3) is equivalent to

$$(4) \quad \sum_{j=1}^n \alpha_j (\phi_i | [1 - T(z)] \phi_j)_t = -(\phi_i | Bg)_t \quad i = 1, \dots, n;$$

where  $B \in \mathcal{B}(\mathcal{H}, \mathcal{H}_t)$ ,  $T(z) = (zB_t - C) \in \mathcal{B}(\mathcal{H}_t)$  and  $B_t$  is the restriction of  $B$  to  $\mathcal{D}(t)$ .

*Proof.* Since  $t_t = (t - \text{Re. } t)$  is a bounded form on  $\mathcal{H}_t$  [1, p. 314], there is a  $C \in \mathcal{B}(\mathcal{H}_t)$  such that

$$t_t(u, v) = (u | Cv)_t; \quad u, v \in \mathcal{D}(t).$$

Also from Ref. [4] pp. 332-3, it follows that there is a unique  $B \in \mathcal{B}(\mathcal{H}, \mathcal{H}_t)$  such that  $\mathcal{D}(B) = \mathcal{H}$  and for  $u \in \mathcal{H}_t$ ,  $w \in \mathcal{H}$ ,

$$(5) \quad (u | \omega) = (u | B\omega)_t.$$

In particular, in equation (3),  $(\phi_i | g) = (\phi_i | Bg)_t$  and  $(\phi_i | \phi_j) = (\phi_i | B\phi_j)_t = (\phi_i | B_t \phi_j)_t$ .

The assertion now follows from direct substitution.

LEMMA 2. In the notation of Lemma 1, we have that  $B_t, C$  are closable,  $B$  is closed and invertible and  $B^{-1}(1 + \bar{C}) = A_t$  where  $A_t$  is the unique  $m$ -sectorial operator associated with  $\bar{t}$ .

*Proof.* Since  $B_t$  and  $C$  are bounded and densely defined, they are closable. Since  $B$  is bounded and  $\mathcal{D}(B) = \mathcal{H}$ , it is closed. Invertibility of  $B$  has been proven in Reference [4] p. 333.

Now,  $\mathcal{D}([B^{-1}(1 + \bar{C})]) \subset \mathcal{H}_t = \mathcal{D}(\bar{t})$  and for  $u, v \in \mathcal{D}(t)$ ,

$$\begin{aligned} (u | B^{-1}(1 + \bar{C})v) &= (u | B^{-1}(1 + C)v) \\ &= (u | (1 + C)v)_t \quad (\text{equation (5)}) \\ &= t(u, v) \end{aligned}$$

From the closability of  $t$ , this result extends for  $u, v \in \mathcal{H}_t$ . The

result now follows from Theorem 2.1, Chapter 6, Reference [1].

**THEOREM 1.** *In addition to the assumptions of Lemma 1 and 2, let  $\{\phi_i\}$  be l.i. and complete in  $\mathcal{H}_t$ , and  $z$  be at a nonzero distance from  $S$ .  $f_n = \sum_{j=1}^n \alpha_j \phi_j$  of equation (3) is then defined for each  $n$  and  $\lim_{n \rightarrow \infty} \|f_n - (z - A_t)^{-1}g\| = 0$ .*

*Proof.* From Lemma 1, equation (3) is equivalent to equation (4). Also without loss of generality, we may assume  $\{\phi_i\}$  to be an orthonormal basis in  $\mathcal{H}_t$ . It is straightforward to check that (4) is equivalent to

$$(1 - T_n(z))f_n = -P_n Bg$$

where  $T_n(z) = P_n T(z) P_n$ , and  $P_n$  is the ortho-projection on the  $n$ -dimensional subspace of  $\mathcal{H}_t$  determined by  $\{\phi_i\}$ ,  $i = 1$  to  $n$ . It follows, for  $h \in \mathcal{H}_t$ , that

$$\lim_{n \rightarrow \infty} \|(T_n(z) - \bar{T}(z))h\|_t = 0.$$

Also, since  $z$  is at a nonzero distance from  $S$ ,  $\text{dist.}(1, W(\bar{T}(z))) = d' > 0$ , where  $W(\cdot)$  denotes the numerical range. Further, since the spectrum of  $T_n$ ,  $\sigma(T_n) \subset (W(\bar{T}(z)) \cup \{0\})$ , for each  $n$ ,  $(1 - T_n(z))^{-1} \in \mathcal{B}(\mathcal{H}_t)$  with  $\|(1 - T_n(z))^{-1}\|_t \leq 1/d$  where  $d = \min.(1, d')$ . Also  $(1 - \bar{T}(z))^{-1} \in \mathcal{B}(\mathcal{H}_t)$ .

Hence for  $h \in \mathcal{H}_t$

$$\begin{aligned} & \|[(1 - T_n(z))^{-1} - (1 - \bar{T}(z))^{-1}]h\|_t \\ &= \|(1 - T_n(z))^{-1}(T_n(z) - \bar{T}(z))(1 - \bar{T}(z))^{-1}h\|_t \\ &\leq \|(1 - T_n(z))^{-1}\|_t \|(T_n(z) - \bar{T}(z))(1 - \bar{T}(z))^{-1}h\|_t \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Further, for  $g \in \mathcal{H}$ ,

$$\lim_{n \rightarrow \infty} \|(P_n B - B)g\|_t = 0$$

and hence

$$\lim_{n \rightarrow \infty} \|f_n - f\|_t = 0$$

where

$$\begin{aligned} f &= -(1 - \bar{T}(z))^{-1}Bg = -(1 - z\bar{B}_t + \bar{C})^{-1}Bg \\ &= (z - B^{-1}(1 + \bar{C}))^{-1}g \\ &= (z - A_t)^{-1}g \quad (\text{Lemma 2}). \end{aligned}$$

Assertion of the theorem follows by observing that  $\|\cdot\|_t \geq \|\cdot\|$ . For a symmetric  $t$ ,  $s = [b, \infty)$  with some  $b > -\infty$ ,  $\bar{C} = 0$  and  $A_t = B^{-1}$  is self-adjoint.

In the following,  $f_n$  will stand for  $\sum_{j=1}^n \alpha_j \phi_j$  as defined by equation (2).

**COROLLARY 1.** *Let  $A$  be densely defined sectorial operator and  $z$  be at a nonzero distance from its sector,  $\{\phi_i\}$  be a l.i. basis in  $\mathcal{D}(A)$ . We have that  $\lim_{n \rightarrow \infty} \|f_n - (z - A_F)^{-1}g\| = 0$ .*

*Proof.* Define  $t$  of Theorem 1 by  $t(u, v) = (u | Av)$ ,  $u, v \in \mathcal{D}(A)$ .  $t$  is closable from Theorem 1.27, Chapter 6 of [1]. Since  $\{\phi_i\}$  is a l.i. basis in  $\mathcal{D}(A)$  and  $\mathcal{D}(A)$  is dense in  $\mathcal{D}(\bar{t}) = \mathcal{H}_t$ , it is a l.i. basis in  $\mathcal{H}_t$ . The result now follows from the fact that  $A_t$  of Theorem 1 now becomes  $A_F$  [1, pp. 325-6].

**COROLLARY 2.** *Let  $A$  be symmetric, bounded below by  $b$ ,  $z$  be at a nonzero distance from  $[b, \infty)$  and  $\{\phi_i\}$  be a l.i. basis in  $\mathcal{D}(A)$ . Then  $\lim_{n \rightarrow \infty} \|f_n - (z - A_F)^{-1}g\| = 0$ .*

*Proof.* The result follows from Corollary 1, by noticing that the sector of  $A$  is  $[b, \infty)$ .

If the set  $\{\phi_i\}$  is taken to be  $\{A^i h\}$  for some  $h \in \mathcal{D}(A^i)$  for  $i = 0, 1, 2, \dots$ ; the Bubnov-Galerkin method is called the method of moments [7]. Since  $\{A^i h\}$  satisfies the conditions of Corollaries 1 and 2, the convergence of the method of moments also is established by these results. The result R.1 [2] thus is a special case of Corollary 2.

In Corollaries 1 and 2 we have considered the case of a densely defined  $A$ . In these results one can replace this condition by requiring that the form domain of  $A$  be dense. However since the Friedrichs extension is defined only for a densely defined  $A$ , the limit operator  $A_t$  may not be  $A_F$ . This situation is of a particular interest in Physics which we describe in brief.

Let  $A$  be given, formally, by  $A = A_1 + A_2$ , where  $A_1$  and  $A_2$  are symmetric but  $\mathcal{D}(A) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$  is not dense. However if the form domain of  $A$  is dense, the self-adjoint operator  $A_t$  associated with the form  $t(u, v) = (u | (A_1 + A_2)v)$  is a legitimate operator to describe a physical system [5]. This construction enables one to include a larger class of interactions in the treatment than the requirement that  $A$  be densely defined [5]. It is obvious that the Bubnov-Galerkin method enables one to compute the resolvent of  $A_t$  in this case also, which is of prime importance in Physics.

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