SOME CONVERGENCE PROPERTIES OF THE BUBNOV-GALERKIN METHOD

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We generalize the Bubnov-Galerkin method to approximate the resolvent of the \( m \)-sectorial operator associated with a densely defined, closed, sectorial form in a Hilbert space. Some special cases of interest are also discussed.

1. Introduction. The Bubnov-Galerkin method [3] was originally devised to approximate the solutions of the equations of the form

\[(z - A)f = g\]

where \( A \) is an operator in a Hilbert space, \( \mathcal{H} \), \( g \) is a vector in \( \mathcal{H} \) and \( z \) is a complex number. The method proceeds with solving the following set of equations:

\[
\sum_{j=1}^{n} \alpha_j (\phi_i | (z - A)\phi_j) = (\phi_i | g) \quad i = 1, \ldots, n ;
\]

where \((. | .)\) denotes the scalar product in \( \mathcal{H} \) and \( \{\phi_i\} \subset \mathcal{D}(A) \) is some linearly independent (l.i.) set in \( \mathcal{H} \). \( \mathcal{D}(\cdot) \) denotes the domain. The questions of interest are the existence and the convergence of the solutions of equation (2). Until recently, the only cases that received a detailed treatment have been when \( A \) is compact, bounded or essentially self-adjoint [3, 6]. However, recently the following result was proven by Masson and Thewarapperuma [2]:

R.I. Let \( A \) be symmetric, bounded below by \( b \), \( z \) be at a non-zero distance from \([b, \infty)\) and \( \{\phi_i\} \) be the orthonormal set formed from \( \{A^i h\} \) where \( h \) is in \( \mathcal{D}(A^i) \) for each \( i \). Then \( \lim_{n \to \infty} || \sum_{j=1}^{n} \alpha_j \phi_j - (z - A_F)^{-1}g || = 0 \), where \( ||.|| \) denotes the norm in \( \mathcal{H} \) and \( A_F \) is the Friedrichs extension of \( A \).

Consider the following set of equations:

\[
\sum_{j=1}^{n} \alpha_j [z(\phi_i | \phi_j) - t(\phi_i, \phi_j)] = (\phi_i | g) \quad i = 1, \ldots, n ;
\]

where \( t \) is a densely defined, closable, sectorial, sesquilinear form in \( \mathcal{H} \). The sector of \( t \) will be denoted by \( S \) and since it causes no loss of generality, the vertex will be taken to be one. In the present note we determine the limit of \( f_n = \sum_{j=1}^{n} \alpha_j \phi_j \) as \( n \) becomes large.
R.I. and some other generalizations of it, will follow from our main result (Theorem 1).

2. Results. Define a new scalar product \( (\cdot | \cdot) \) on \( \mathcal{D}(t) \) by
\[
(u | v)_t = \text{Re. } t(u, v),
\]
[1, pp. 309-10] and complete \( \mathcal{D}(t) \) in the new metric to a Hilbert space \( \mathcal{H}_t \). Let the closure of \( t \) be \( \bar{t} \). We have that \( \mathcal{D}(t) \subset \mathcal{D}(\bar{t}) = \mathcal{H}_t \subset \mathcal{H} \). The norm in \( \mathcal{H}_t \) will be denoted by \( ||\cdot||_t \). Also \( \mathcal{B}(X, Y) \) will denote the space of bounded operators with \( \mathcal{D}(\cdot) \subset X \) and range \( \mathcal{R}(\cdot) \subset Y \), and \( \mathcal{B}(X) = \mathcal{B}(X, X) \).

**Lemma 1.** Let \( t \) be as in equation (3), \( \{\phi_i\} \subset \mathcal{D}(t) \) and \( g \in \mathcal{H} \). Equation (3) is equivalent to
\[
\sum_{j=1}^{n} \alpha_j (\phi_j | [1 - T(z)]\phi_j)_t = - (\phi_i | Bg)_t, \quad i = 1, \ldots, n;
\]
where \( B \in \mathcal{B}(\mathcal{H}, \mathcal{H}_t) \), \( T(z) = (zB_i - C) \in \mathcal{B}(\mathcal{H}_t) \) and \( B_i \) is the restriction of \( B \) to \( \mathcal{D}(t) \).

**Proof.** Since \( t_i = (t - \text{Re. } t) \) is a bounded form on \( \mathcal{H}_t \) [1, p. 314], there is a \( C \in \mathcal{B}(\mathcal{H}_t) \) such that
\[
t_i(u, v) = (u | Cv)_t; \quad u, v \in \mathcal{D}(t).
\]
Also from Ref. [4] pp. 332-3, it follows that there is a unique \( B \in \mathcal{B}(\mathcal{H}, \mathcal{H}_t) \) such that \( \mathcal{D}(B) = \mathcal{H} \) and for \( u \in \mathcal{H}_t \), \( w \in \mathcal{H} \),
\[
(u | \omega) = (u | B\omega)_t.
\]
In particular, in equation (3), \( (\phi_i | g) = (\phi_i | Bg)_t \) and \( (\phi_i | \phi_j)_t = (\phi_i | B\phi_j)_t = (\phi_i | B(\phi_j))_t \).

The assertion now follows from direct substitution.

**Lemma 2.** In the notation of Lemma 1, we have that \( B_i, C \) are closable, \( B \) is closed and invertible and \( B^{-1}(1 + C) = A_i \), where \( A_i \) is the unique \( m \)-sectorial operator associated with \( \bar{t} \).

**Proof.** Since \( B_i \) and \( C \) are bounded and densely defined, they are closable. Since \( B \) is bounded and \( \mathcal{D}(B) = \mathcal{H} \), it is closed. Invertibility of \( B \) has been proven in Reference [4] p. 333.

Now, \( \mathcal{D}([B^{-1}(1 + C)]) \subset \mathcal{H}_i = \mathcal{D}(\bar{t}) \) and for \( u, v \in \mathcal{D}(t) \),
\[
(u | B^{-1}(1 + C)v) = (u | B^{-1}(1 + C)v) = (u | (1 + C)v)_t = t(u, v) \quad \text{(equation (5))}
\]
From the closability of \( t \), this result extends for \( u, v \in \mathcal{H}_i \). The
result now follows from Theorem 2.1, Chapter 6, Reference [1].

**Theorem 1.** In addition to the assumptions of Lemma 1 and 2, let \( \{ \phi_i \} \) be l.i. and complete in \( \mathcal{H}_i \), and \( z \) be at a nonzero distance from \( S \). \( f_n = \sum_{i=1}^{\infty} \alpha_i \phi_i \) of equation (3) is then defined for each \( n \) and \( \lim_{n \to \infty} \| f_n - (z - A_i)^{-1}g \| = 0. \)

**Proof.** From Lemma 1, equation (3) is equivalent to equation (4). Also without loss of generality, we may assume \( \{ \phi_i \} \) to be an orthonormal basis in \( \mathcal{H}_i \). It is straightforward to check that (4) is equivalent to

\[
(1 - T_n(z))f_n = -P_nBg
\]

where \( T_n(z) = P_nT(z)P_n \), and \( P_n \) is the ortho-projection on the \( n \)-dimensional subspace of \( \mathcal{H}_i \) determined by \( \{ \phi_i \} \), \( i = 1 \) to \( n \). It follows, for \( h \in \mathcal{H}_i \), that

\[
\lim_{n \to \infty} \| (T_n(z) - T(z))h \|_i = 0.
\]

Also, since \( z \) is at a nonzero distance from \( S \), dist. \( 1, W(T(z)) = d' > 0 \), where \( W(\cdot) \) denotes the numerical range. Furthermore, since the spectrum of \( T_n, \sigma(T_n) \subset (W(T(z)) \cup \{0\}) \), for each \( n \), \( (1 - T_n(z))^{-1} \in \mathcal{B}(\mathcal{H}_i) \) with \( \| (1 - T_n(z))^{-1} \|_i \leq 1/d \) where \( d = \min(1, d') \). Also \( (1 - T(z))^{-1} \in \mathcal{B}(\mathcal{H}_i) \).

Hence for \( h \in \mathcal{H}_i \)

\[
\|[(1 - T_n(z))^{-1} - (1 - T(z))^{-1}]h\|_i
= \| (1 - T_n(z))^{-1}(T_n(z) - T(z))(1 - T(z))^{-1}h \|_i
\leq \| (1 - T_n(z))^{-1} \|_i \| (T_n(z) - T(z))(1 - T(z))^{-1}h \|_i\]

\[
\longrightarrow 0.
\]

Further, for \( g \in \mathcal{H} \),

\[
\lim_{n \to \infty} \| (P_nB - B)g \|_i = 0
\]

and hence

\[
\lim_{n \to \infty} \| f_n - f \|_i = 0
\]

where

\[
f = -(1 - T(z))^{-1}Bg = -(1 - zB_i + \tilde{C})^{-1}Bg
\]

\[
= (z - B^{-1}(1 + \tilde{C}))^{-1}g
\]

\[
= (z - A_i)^{-1}g \quad \text{(Lemma 2)}.
\]
Assertion of the theorem follows by observing that \( \| \cdot \|_t \geq \| \cdot \| \). For a symmetric \( t, s = [b, \infty) \) with some \( b > -\infty \), \( C = 0 \) and \( A_i = B^{-1} \) is self-adjoint.

In the following, \( f_n \) will stand for \( \sum_{i=1}^n \alpha_i \phi_i \) as defined by equation (2).

**COROLLARY 1.** Let \( A \) be densely defined sectorial operator and \( z \) be at a nonzero distance from its sector, \( \{ \phi_i \} \) be a l.i. basis in \( \mathcal{D}(A) \). We have that \( \lim_{n \to \infty} \| f_n - (z - A)^{-1}g \| = 0 \).

**Proof.** Define \( t \) of Theorem 1 by \( t(u, v) = (u | Av) \), \( u, v \in \mathcal{D}(A) \). \( t \) is closable from Theorem 1.27, Chapter 6 of [1]. Since \( \{ \phi_i \} \) is a l.i. basis in \( \mathcal{D}(A) \) and \( \mathcal{D}(A) \) is dense in \( \mathcal{D}(\tilde{t}) = \mathcal{H} \), it is a l.i. basis in \( \mathcal{H} \). The result now follows from the fact that \( A_i \) of Theorem 1 now becomes \( A_F \) [1, pp. 325-6].

**COROLLARY 2.** Let \( A \) be symmetric, bounded below by \( b \), \( z \) be at a nonzero distance from \([b, \infty)\) and \( \{ \phi_i \} \) be a l.i. basis in \( \mathcal{D}(A) \). Then \( \lim_{n \to \infty} \| f_n - (z - A_F)^{-1}g \| = 0 \).

**Proof.** The result follows from Corollary 1, by noticing that the sector of \( A \) is \([b, \infty)\).

If the set \( \{ \phi_i \} \) is taken to be \( \{ A^i h \} \) for some \( h \in \mathcal{D}(A^i) \) for \( i = 0, 1, 2, \cdots \); the Bubnov-Galerkin method is called the method of moments [7]. Since \( \{ A^i h \} \) satisfies the conditions of Corollaries 1 and 2, the convergence of the method of moments also is established by these results. The result R.1 [2] thus is a special case of Corollary 2.

In Corollaries 1 and 2 we have considered the case of a densely defined \( A \). In these results one can replace this condition by requiring that the form domain of \( A \) be dense. However since the Friedrichs extension is defined only for a densely defined \( A \), the limit operator \( A_i \) may not be \( A_F \). This situation is of a particular interest in Physics which we describe in brief.

Let \( A \) be given, formally, by \( A = A_1 + A_2 \), where \( A_1 \) and \( A_2 \) are symmetric but \( \mathcal{D}(A) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2) \) is not dense. However if the form domain of \( A \) is dense, the self-adjoint operator \( A_i \) associated with the form \( t(u, v) = (u | (A_1 + A_2)v) \) is a legitimate operator to describe a physical system [5]. This construction enables one to include a larger class of interactions in the treatment than the requirement that \( A \) be densely defined [5]. It is obvious that the Bubnov-Galerkin method enables one to compute the resolvent of \( A_i \) in this case also, which is of prime importance in Physics.
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REFERENCES


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UNIVERSITY OF WESTERN ONTARIO
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