

Pacific Journal of Mathematics

**LEVEL CROSSING PROBABILITIES FOR A
MULTI-PARAMETER BROWNIAN PROCESS**

PEGGY STRAIT

LEVEL CROSSING PROBABILITIES FOR A MULTI-PARAMETER BROWNIAN PROCESS

PEGGY TANG STRAIT

Let $\{\xi_{s_1, s_2}; -\infty < s_1, < \infty, -\infty < s_2 < \infty\}$ be a Gaussian process with $\xi_{s_1, s_2} = 0$ if $s_1 = 0$ or $s_2 = 0$, mean values $E(\xi_{s_1, s_2}) = 0$, and covariances $E(\xi_{s_1, s_2} \xi_{s'_1, s'_2}) = 1/2 \min(s_1, s'_1) \min(s_2, s'_2)$. This is the two parameter Brownian process studied by J. D. Kuelbs, W. J. Park, P. T. Strait, and J. Yeh. In this paper, upper and lower bounds for level crossing probabilities of this process are derived.

More specifically, let (t_1, t_2) and (τ_1, τ_2) be two pairs of constants chosen so that $0 < t_1 < \infty, 0 < t_2 < \infty, 0 < \tau_1 < \infty$, and $0 < \tau_2 < \infty$. Let $\delta_1 = \tau_1/m, \delta_2 = \tau_2/n$ where m and n are integers, and define random variables $X_{h,k}$ for $h = 0, 1, 2, \dots, m$ and $k = 0, 1, 2, \dots, n$ as follows.

$$(1) \quad X_{h,k} = \begin{cases} 0 & \text{for } h = 0 \text{ or } k = 0 \\ \xi_{t_1+(h-1)\delta_1, t_2+(k-1)\delta_2} & \text{for } h = 1, 2, \dots, m; k = 1, 2, \dots, n. \end{cases}$$

For any given number a , define

$$(2) \quad \begin{aligned} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\ = P(X_{i,j} > a \text{ for } i = 1, 2, \dots, m; j = 1, 2, \dots, n). \end{aligned}$$

In this paper, upper and lower bounds (Theorems 1 and 2) for $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2)$ are derived.

2. Preliminary lemmas.

LEMMA 1. Let $\zeta_{h,k} = X_{h,k} - X_{h,k-1} - X_{h-1,k} + X_{h-1,k-1}$ for $h = 1, \dots, m$ and $k = 1, \dots, n$. Then,

(i) $\zeta_{h,k}, h = 1, \dots, m, k = 1, \dots, n$ are independent Gaussian random variables with means 0 and variances $\sigma_{h,k}^2$ given by

$$(3) \quad \begin{aligned} \sigma_{1,1}^2 &= \frac{1}{2} t_1 t_2 \\ \sigma_{1,k}^2 &= \frac{1}{2} \delta_1 t_2 \quad \text{for } k = 2, \dots, n \\ \sigma_{h,1}^2 &= \frac{1}{2} \delta_2 t_1 \quad \text{for } h = 2, \dots, m \\ \sigma_{h,k}^2 &= \frac{1}{2} \delta_1 \delta_2 \quad \text{for } h = 2, \dots, m; k = 2, \dots, m \end{aligned}$$

and

$$(4) \quad (ii) \quad X_{i,j} = \sum_{h=1}^i \sum_{k=1}^j \zeta_{h,k} \quad \text{for } i = 1, \dots, m; j = 1, \dots, n.$$

Proof of Lemma 1. To prove part (i) of the lemma, observe that

$$(5) \quad E(\zeta_{h,k}) = E(X_{h,k} - X_{h,k-1} - X_{h-1,k} + X_{h-1,k-1}) = 0 \\ \text{for } h = 1, \dots, m; k = 1, \dots, n.$$

$$(6) \quad E(\zeta_{h,k}^2) = E[(X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1})^2] \\ = \begin{cases} \frac{1}{2}t_1t_2 & \text{for } h = 1, k = 1 \\ \frac{1}{2}\delta_1t_2 & \text{for } h = 1; k = 2, \dots, n \\ \frac{1}{2}\delta_2t_1 & \text{for } h = 2, \dots, m; k = 1 \\ \frac{1}{2}\delta_1\delta_2 & \text{for } h = 2, \dots, m; k = 2, \dots, n. \end{cases}$$

$$(7) \quad E(\zeta_{h,k}\zeta_{p,q}) \\ = E[X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1}](X_{p,q} - X_{p-1,q} - X_{p,q-1} + X_{p-1,q-1}) \\ = 0 \quad \text{when } (h, k) \neq (p, q).$$

(In each of the equations (5), (6), and (7), the last term is derived by direct computation of the expression of the preceding term.) Thus, the random variables $\zeta_{h,k}$, $h = 1, \dots, m$, $k = 1, \dots, n$ are independent, Gaussian random variables with mean values 0 and variances given by equation (3).

To prove part (ii) of the lemma, observe that

$$(8) \quad \sum_{h=1}^i \sum_{k=1}^j \zeta_{h,k} = \sum_{h=1}^i \sum_{k=1}^j (X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1}) = X_{i,j}$$

For the remaining lemmas and theorems we add the following notations and definitions. Let $C = \{(i, j): i = 1, \dots, m; j = 1, \dots, n\}$, $C^* = C - \{(1, 1)\}$. Furthermore, let $P_{m,n}^-$ and $P_{m,n}^*$ be probabilities defined as follows.

$$(9) \quad P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \text{ for all } (i, j) \in C^* \text{ and } X_{1,1} \leq a)$$

$$(10) \quad P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \text{ for all } (i, j) \in C^*).$$

Then clearly,

$$(11) \quad P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) - P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2).$$

LEMMA 2. Let η_1, η_2, η_3 be normal random variables with $E(\eta_i) = 0$ and $\text{Var}(\eta_i) = (1/6)t_1 t_2$ for $i = 1, 2, 3$. Assume also that $\eta_1, \eta_2, \eta_3, \zeta_{1,1}, \zeta_{1,2}, \dots, \zeta_{m,n}$ form a set of independent random variables. Then

$$\begin{aligned}
 & P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) \\
 & \geq P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3} \quad \text{for all } (i, j) \in C^*\right) \\
 (12) \quad & \cdot P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3} \quad \text{for all } j = 1, 2, \dots, n\right) \\
 & \cdot P\left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > \frac{a}{3} \quad \text{for all } i = 1, 2, \dots, m\right).
 \end{aligned}$$

Proof of Lemma 2. For $i < 2$ or $j < 2$ define

$$\sum_{k=2}^i \sum_{k=2}^j \zeta_{h,k} = 0, \quad \sum_{k=2}^j \zeta_{1,k} = 0, \quad \text{and} \quad \sum_{h=2}^i \zeta_{h,1} = 0.$$

The proof of Lemma 2 follows.

$$\begin{aligned}
 & P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \quad \text{for } (i, j) \in C^*) \\
 & = P\left(\sum_{h=1}^i \sum_{k=1}^j \zeta_{h,k} > a \quad \text{for } (i, j) \in C^*\right) \\
 & = P\left(\sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} + \sum_{k=2}^j \zeta_{1,k} + \sum_{h=2}^i \zeta_{h,1} + \zeta_{1,1} > a \quad \text{for } (i, j) \in C^*\right) \\
 & = P\left(\sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} + \sum_{k=2}^j \zeta_{1,k} + \sum_{h=2}^i \zeta_{h,1} + \eta_1 + \eta_2 + \eta_3 > a \right. \\
 (13) \quad & \left. \text{for } (i, j) \in C^*\right) \\
 & \geq P\left[\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3}\right) \quad \text{and} \quad \left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3}\right) \right. \\
 & \quad \left. \text{and} \quad \left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > \frac{a}{3}\right) \quad \text{for } (i, j) \in C^*\right] \\
 & = P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3} \quad \text{for } (i, j) \in C^*\right) \\
 & \cdot P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3} \quad \text{for } j = 1, 2, \dots, n\right) \\
 & \cdot P\left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} < \frac{a}{3} \quad \text{for } i = 1, 2, \dots, m\right).
 \end{aligned}$$

LEMMA 3.

$$\begin{aligned}
 (14) \quad & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \quad \text{for } (i, j) \in C^*\right) \\
 & \geq \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4 \sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}, \quad \alpha = \frac{a}{\sqrt{3t_1 t_2}}.
 \end{aligned}$$

Proof of Lemma 3. Let $Y_{h,k}$, $h = 1, \dots, m$, $k = 1, \dots, n$ be independent standard normal random variables. Let

$$\begin{aligned}
 L_{m,n} &= \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\sum_{h=1}^i \sum_{k=1}^j Y_{h,k} \right) \\
 (15) \quad L_{m,n}^- &= \min(0, L_{m,n}) \\
 U_{m,n} &= \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\sum_{h=1}^i \sum_{k=1}^j Y_{h,k} \right), \quad U_{m,n}^+ = \max(0, U_{m,n})
 \end{aligned}$$

It is shown in [6] that

$$(16) \quad E(U_{m,n}^+) < 4\sqrt{mn}.$$

To prove Lemma 3, it is first shown that

$$\begin{aligned}
 (17) \quad &P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \text{ for } (i, j) \in C^*\right) \\
 &= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - E^-\left(\int_0^{-\sqrt{\delta_1 \delta_2 / t_1 t_2} L_{m-1, n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right)
 \end{aligned}$$

where E^- denotes that portion of the expectation obtained by integration over the negative range of $L_{m-1, n-1}$. (Later, E^+ shall also be used to denote expectation obtained by integration over the positive range of values.)

To prove equation (17), observe that

$$\begin{aligned}
 &P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \text{ for } (i, j) \in C^*\right) \\
 &= P\left(\eta_1 - a > - \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} \text{ for } (i, j) \in C^*\right) \\
 &= P\left(\eta_1 - a > \sqrt{\frac{\delta_1 \delta_2}{2}} \sum_{h=2}^i \sum_{k=2}^j Y_{h,k} \text{ for } (i, j) \in C^*\right) \\
 &= P\left(\eta_1 - a > - \sqrt{\frac{\delta_1 \delta_2}{2}} L_{m-1, n-1}^- \right) \\
 &= E\left(\int_{-\sqrt{\delta_1 \delta_2 / 2} L_{m-1, n-1}^-}^\infty \frac{1}{\sqrt{\pi} \frac{t_1 t_2}{3}} e^{-(w+a)^2 / (1/3) t_1 t_2} dw\right) \\
 &= E\left(\int_{-\sqrt{3\delta_1 \delta_2 / t_1 t_2} L_{m-1, n-1}^-}^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 (18) \quad &= E^+\left(\int_{-\sqrt{3\delta_1 \delta_2 / t_1 t_2} L_{m-1, n-1}^-}^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &\quad + E^-\left(\int_{-\sqrt{3\delta_1 \delta_2 / t_1 t_2} L_{m-1, n-1}^-}^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right)
 \end{aligned}$$

$$\begin{aligned}
 &= E^+\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &\quad + E^-\left(\int_{-\sqrt{3\hat{\delta}_1\hat{\delta}_2/t_1t_2}L_{m-1,n-1}}^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &= E^+\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) + E^-\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &\quad - E^-\left(\int_0^{-\sqrt{3\hat{\delta}_1\hat{\delta}_2/t_1t_2}L_{m-1,n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
 &= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - E^-\left(\int_0^{-\sqrt{3\hat{\delta}_1\hat{\delta}_2/t_1t_2}L_{m-1,n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right).
 \end{aligned}$$

Next, apply the inequality

$$(19) \quad \int_0^x e^{-1/2(u+\alpha)^2} du < x \quad \text{for } x > 0$$

to obtain

$$(20) \quad E^-\left(\int_0^{-\sqrt{3\hat{\delta}_1\hat{\delta}_2/t_1t_2}L_{m-1,n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) < -\sqrt{\frac{2}{\pi}} \sqrt{\frac{3\hat{\delta}_1\hat{\delta}_2}{t_1t_2}} E^-(L_{m-1,n-1}).$$

Now, observe that

$$(21) \quad E^-(L_{m-1,n-1}) = E(L_{m-1,n-1}^-) = -E(U_{m-1,n-1}^+)$$

then combine this with equation (16), to obtain

$$(22) \quad E^-(L_{m-1,n-1}) > -4\sqrt{(m-1)(n-1)}.$$

Finally, apply equations (17), (20), and (22) to obtain

$$\begin{aligned}
 (23) \quad &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > a \quad \text{for } (i,j) \in C^*\right) \\
 &\geq \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \left\{ \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du + \sqrt{\frac{2}{\pi}} \sqrt{\frac{3\hat{\delta}_1\hat{\delta}_2}{t_1t_2}} E^-(L_{m-1,n-1}) \right\} \\
 &\geq \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \left\{ \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du + \sqrt{\frac{2}{\pi}} \sqrt{\frac{3\hat{\delta}_1\hat{\delta}_2}{t_1t_2}} (-4\sqrt{(m-1)(n-1)}) \right\} \\
 &= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\hat{\delta}_1\hat{\delta}_2}{\pi t_1t_2}}.
 \end{aligned}$$

LEMMA 4.

$$\begin{aligned}
 (a) \quad &\lim_{n \rightarrow \infty} P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > 0 \quad \text{for } j = 1, 2, \dots, n\right) \\
 &= \frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2}\right]
 \end{aligned}$$

$$\begin{aligned}
 & \text{(b) } \lim_{m \rightarrow \infty} P\left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > 0 \text{ for } i = 1, 2, \dots, m\right) \\
 & \quad = \frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_2}{t_2}\right)^{-1/2}\right] \\
 & \text{(c) For small } \frac{\tau_1}{t_1}, \\
 & \text{(24) } \lim_{n \rightarrow \infty} P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > a \text{ for } j = 1, 2, \dots, n\right) \\
 & \quad = \Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-3a^2/t_1 t_2} + O\left(\frac{3\tau_1}{t_1}\right) \\
 & \text{(d) For small } \frac{\tau_2}{t_2}, \\
 & \lim_{m \rightarrow \infty} P\left(\eta_3 + \sum_{k=2}^i \zeta_{k,1} a \text{ for } i = 1, 2, \dots, m\right) \\
 & \quad = \Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-\frac{3a^2}{t_1 t_2}} + O\left(\frac{3\tau_2}{t_2}\right)
 \end{aligned}$$

where $\Phi(\)$ is the standardized normal distribution function

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du .$$

Proof of Lemma 4. Let

$$\text{(25) } Y_k = \sqrt{\frac{2}{\delta_1 t_2}} \zeta_{1,k} \text{ for } k = 2, \dots, n$$

so that Y_2, Y_3, \dots, Y_n are independent standard normal random variables. Let

$$\text{(26) } \delta = \frac{\delta_1 t_2}{2}$$

$$\begin{aligned}
 \text{(27) } X_1 &= \eta_2 \\
 X_2 &= \eta_2 + \zeta_{1,2} = \eta_2 + \sqrt{\frac{\delta_1 t_2}{2}} Y_2 = X_1 + \delta^{1/2} Y_2 \\
 &\vdots \\
 X_n &= \eta_2 + \sum_{k=2}^n \zeta_{1,k} = \eta_2 + \sqrt{\frac{\delta_1 t_2}{2}} \sum_{k=2}^n Y_k = X_1 + \delta^{1/2} \sum_{k=1}^n Y_k
 \end{aligned}$$

where X_1 is normal, $E(X_1) = 0$, $\text{Var}(X_1) = (1/6)t_1 t_2$. Let

$$\text{(28) } t = \frac{1}{6} t_1 t_2, \quad \tau = \frac{n \delta_1 t_2}{2} = \frac{\tau_1 t_2}{2} .$$

For random variables X_1, X_2, \dots, X_n satisfying the conditions given

above, J. A. McFadden and J. L. Lewis proved in [3] (equation (34) on page 310) that

$$(29) \quad \lim_{n \rightarrow \infty} P(X_i > 0 \text{ for all } i = 1, \dots, n) = \lim_{n \rightarrow \infty} P_{n+1}(0, t) \\ = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{\tau}{t} \right)^{-1/2} \right].$$

Hence in this case

$$(30) \quad \lim_{n \rightarrow \infty} P \left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > 0 \text{ for } j = 1, 2, \dots, n \right) \\ = \lim_{n \rightarrow \infty} P(X_i > 0 \text{ for all } i = 1, \dots, n) \\ = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{\tau}{t} \right)^{-1/2} \right] = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1} \right)^{-1/2} \right].$$

Similarly,

$$(31) \quad \lim_{m \rightarrow \infty} P \left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > 0 \text{ for } i = 1, \dots, m \right) \\ = \frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2} \right)^{-1/2} \right].$$

For the case $a \neq 0$ and $\tau_1/t_1, \tau_2/t_2$ small, use equation (37) of McFadden and Lewis [3] which states that

$$(32) \quad \lim_{n \rightarrow \infty} P(X_i > a \text{ for all } i = 1, \dots, n) \\ = \lim_{n \rightarrow \infty} P_{n+1}(a, t) = \Phi \left(\frac{-a}{t^{1/2}} \right) - \frac{1}{\pi} \left(\frac{\tau}{t} \right)^{1/2} e^{-a^2/2t} + O(\tau/t)$$

to obtain

$$(33) \quad \lim_{n \rightarrow \infty} P \left(\eta_2 + \sum_{k=2}^j \zeta_{i,k} > a \text{ for } i = 1, 2, \dots, n \right) \\ = \Phi \left(-\frac{a}{\sqrt{\frac{1}{6}t_1t_2}} \right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1} \right)^{1/2} e^{-3a^2/t_1t_2} + O \left(\frac{3\tau_1}{t_1} \right).$$

Similarly,

$$(34) \quad \lim_{m \rightarrow \infty} P \left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > 0 \text{ for } i = 1, 2, \dots, m \right) \\ = \Phi \left(-\frac{\sqrt{6a}}{\sqrt{t_1t_2}} \right) - \frac{1}{\pi} \left(\frac{3t_2}{t_2} \right)^{1/2} e^{-3a^2/t_1t_2} + O \left(\frac{3\tau_2}{t_1} \right).$$

LEMMA 5.

$$(35) \quad \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2).$$

Proof of Lemma 5. Consider the term $P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2)$ of equation (11).

$$(36) \quad \begin{aligned} P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2) &= P(X_{i,j} > a \text{ for } (i, j) \in C^* \text{ and } X_{1,1} \leq a) \\ &= P(X_{i,j} > a \text{ for } (i, j) \in C^* \text{ and } X_{1,1} < a). \end{aligned}$$

Therefore, by continuity of the sample paths (see [4] for proof of this fact)

$$(37) \quad \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2) = 0 \quad \text{for } a \neq 0.$$

But this implies that equation (11) is equivalent to

$$(38) \quad P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P_{m,n}^*(a, t_1, t_2, \tau_1, \tau_2) - P_{m,n}^-(a, t_1, t_2, \tau_1, \tau_2)$$

Upon taking limits on both sides of (38), equation (35) of Lemma 5 is obtained.

3. The main theorems.

THEOREM 1. *A lower bound.*

$$(39) \quad \begin{aligned} &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\ &\geq \left(1 - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \cdot \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2} \right]\right) \\ &\quad \cdot \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2}\right)^{1/2} \right]\right) \end{aligned}$$

and for $a \neq 0$, τ_1/t_1 small, τ_2/t_2 small,

$$(40) \quad \begin{aligned} \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) &\geq \left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \\ &\quad \cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-a^2/3t_1 t_2} O\left(\frac{3\tau_1}{t_1}\right)\right] \\ &\quad \cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right], \end{aligned}$$

where $\alpha = \frac{a}{\sqrt{3t_1 t_2}}$

Proof of Theorem 1. Apply Lemma 2 to Lemma 5 to obtain

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\
 (41) \quad & \geq P\left(\eta_1 + \sum_{h=2}^i \sum_{k=2}^j \zeta_{h,k} > \frac{a}{3} \text{ for } (i, j) \in C^*\right) \\
 & \cdot P\left(\eta_2 + \sum_{k=2}^j \zeta_{1,k} > \frac{a}{3} \text{ for } j = 1, 2, \dots, n\right) \\
 & \cdot P\left(\eta_3 + \sum_{h=2}^i \zeta_{h,1} > \frac{a}{3} \text{ for } i = 1, 2, \dots, m\right).
 \end{aligned}$$

Then use Lemmas 3 and 4 to obtain

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\
 (42) \quad & \geq \left(1 - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \left(\frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_1}{t_1}\right)^{1/2}\right]\right) \left(\frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_2}{t_2}\right)^{-1/2}\right]\right)
 \end{aligned}$$

and for $a \neq 0$,

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\
 (43) \quad & \geq \left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \\
 & \cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_1}{t_1}\right)\right] \\
 & \cdot \left[\Phi\left(-\frac{6a}{t_1 t_2}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right]
 \end{aligned}$$

where $\alpha = \frac{a}{\sqrt{3t_1 t_2}}$.

THEOREM 2. *An upper bound.*

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\
 (44) \quad & \leq \min \left\{ \left(\frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2}\right]\right), \right. \\
 & \left. \left(\frac{1}{\pi} \sin^{-1}\left[\left(1 + \frac{3\tau_2}{t_2}\right)^{-1/2}\right]\right) \right\}
 \end{aligned}$$

and for $a \neq 0$, τ_1/t_1 small, and τ^2/t_2 small,

$$\begin{aligned}
 & \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\
 (45) \quad & \leq \min \left\{ \left[\Phi\left(-a\sqrt{\frac{6}{t_1 t_2}}\right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_1}{t_1}\right)\right], \right. \\
 & \left. \left[\Phi\left(-a\sqrt{\frac{6}{t_1 t_2}}\right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-a^2/3t_1 t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right] \right\}.
 \end{aligned}$$

Proof of Theorem 2. Observe that

$$(46) \quad \begin{aligned} &P(X_{i,j} > a \text{ for } (i, j) \in C) \\ &\leq \min \{P(X_{1,j} > a, j = 1, \dots, n), P(X_{i,1} > a, i = 1, \dots, m)\}. \end{aligned}$$

For the case $a = 0$, we apply equation (29) to obtain

$$(47) \quad \begin{aligned} &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P(X_{i,j} > 0 \text{ for } (i, j) \in C) \\ &\leq \min \left\{ \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_1}{t_1} \right)^{-1/2} \right] \right), \left(\frac{1}{\pi} \sin^{-1} \left[\left(1 + \frac{3\tau_2}{t_2} \right)^{-1/2} \right] \right) \right\}. \end{aligned}$$

For the case $a \neq 0$, apply equation (32) to obtain

$$(48) \quad \begin{aligned} &\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} P(X_{i,j} > a \text{ for } (i, j) \in C) \\ &\leq \min \left[\left\{ \Phi \left(-a \sqrt{\frac{6}{t_1 t_2}} \right) - \frac{1}{\pi} \left(\frac{3\tau_1}{t_1} \right)^{1/2} e^{-3a^2/t_1 t_2} + O \left(\frac{3\tau_1}{t_1} \right) \right\}, \right. \\ &\quad \left. \left\{ \Phi \left(-a \sqrt{\frac{6}{t_1 t_2}} \right) - \frac{1}{\pi} \left(\frac{3\tau_2}{t_2} \right)^{1/2} e^{-3a^2/t_1 t_2} + O \left(\frac{3\tau_2}{t_2} \right) \right\} \right] \end{aligned}$$

for small $\frac{\tau_1}{t_1}, \frac{\tau_2}{t_2}$.

4. Remark. The method used here may be generalized to apply to the N -parameter Brownian process.

REFERENCES

1. J. D. Kuelbs, Integration on space of continuous functions, Ph. D. thesis, Univ. of Minnesota, 1965.
2. Won Joon Park, *A multi-parameter Gaussian process*, The Annals of Math. Stat., (1970), 1582-1595.
3. J. A. McFadden and James L. Lewis, *Multivariate normal integrals for highly correlated samples from a Wiener process*, J. Applied Probability, **4** (1967), 303-312.
4. P. T. Strait, *Sample function regularity for Gaussian processes with the parameter in a Hilbert space*, Pacific J. Math., **19** (1966), 159-173.
5. ———, *On Kitagawa's functional integral*, Tôhoku Math. J., the second series, **19** (1967), 75-78.
6. M. J. Wichura, *Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters*, The Annals of Math. Stat., **40** (1969), 681-687.
7. J. Yeh, *Wiener measure in a space of functions of two variables*, Amer. Math. Soc. Trans., **95** (1960), 433-450.
8. ———, *Cameron-Martin translation theorems in the Wiener space of functions of two variables*, Amer. Math. Soc. Trans., **107** (1963), 409-420.

Received August 19, 1975.

QUEENS COLLEGE OF THE CITY UNIVERSITY OF NEW YORK

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Pacific Journal of Mathematics

Vol. 65, No. 1

September, 1976

David Lee Armacost, <i>Compactly cogenerated LCA groups</i>	1
Sun Man Chang, <i>On continuous image averaging of probability measures</i>	13
J. Chidambaraswamy, <i>Generalized Dedekind ψ-functions with respect to a polynomial. II</i>	19
Freddy Delbaen, <i>The Dunford-Pettis property for certain uniform algebras</i>	29
Robert Benjamin Feinberg, <i>Faithful distributive modules over incidence algebras</i>	35
Paul Froeschl, <i>Chained rings</i>	47
John Brady Garnett and Anthony G. O'Farrell, <i>Sobolev approximation by a sum of subalgebras on the circle</i>	55
Hugh M. Hilden, José M. Montesinos and Thomas Lusk Thickstun, <i>Closed oriented 3-manifolds as 3-fold branched coverings of S^3 of special type</i>	65
Atsushi Inoue, <i>On a class of unbounded operator algebras</i>	77
Peter Kleinschmidt, <i>On facets with non-arbitrary shapes</i>	97
Narendrakumar Ramanlal Ladhawala, <i>Absolute summability of Walsh-Fourier series</i>	103
Howard Wilson Lambert, <i>Links which are unknottable by maps</i>	109
Kyung Bai Lee, <i>On certain g-first countable spaces</i>	113
Richard Ira Loeb, <i>A Hahn decomposition for linear maps</i>	119
Moshe Marcus and Victor Julius Mizel, <i>A characterization of non-linear functionals on W_1^p possessing autonomous kernels. I</i>	135
James Miller, <i>Subordinating factor sequences and convex functions of several variables</i>	159
Keith Pierce, <i>Amalgamated sums of abelian l-groups</i>	167
Jonathan Rosenberg, <i>The C^*-algebras of some real and p-adic solvable groups</i>	175
Hugo Rossi and Michele Vergne, <i>Group representations on Hilbert spaces defined in terms of ∂_b-cohomology on the Silov boundary of a Siegel domain</i>	193
Mary Elizabeth Schaps, <i>Nonsingular deformations of a determinantal scheme</i>	209
S. R. Singh, <i>Some convergence properties of the Bubnov-Galerkin method</i>	217
Peggy Strait, <i>Level crossing probabilities for a multi-parameter Brownian process</i>	223
Robert M. Tardiff, <i>Topologies for probabilistic metric spaces</i>	233
Benjamin Baxter Wells, Jr., <i>Rearrangements of functions on the ring of integers of a p-series field</i>	253
Robert Francis Wheeler, <i>Well-behaved and totally bounded approximate identities for $C_0(X)$</i>	261
Delores Arletta Williams, <i>Gauss sums and integral quadratic forms over local fields of characteristic 2</i>	271
John Yuan, <i>On the construction of one-parameter semigroups in topological semigroups</i>	285